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Finite representability and super-ideals of operators

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Introduction

In this paper we define a new notion of finite representability of operators between Banach spaces. For a given quasi-normed operator ideal $[A, \mathbf{A}]$, we introduce the class of super-A-operators: T is a *super-A-operator* iff each operator T_0 which is finitely representable in T belongs to the ideal A. If $[A, \mathbf{A}]$ is regular, then the class of super-A-operators equipped with a natural quasi-norm becomes a quasi-normed operator ideal. Our notion of finite representability of operators differs from those of Beauzamy [1], [2]. This has the following reason: The notions introduced in [1], [2] are useful for the study of uniformly convexifying operators, but they are not easy to handle in the case of general operator ideals. Namely, if we start with those notions to define super-A-operators for a given ideal $[A, \mathbf{A}]$, then the class of super-A-operators does not constitute a quasi-normed operator ideal, in general, even if $[A, \mathbf{A}]$ is regular (see the example in Section 2). Secondly, it seems to be unknown whether the ultraproduct $(T)_U$ of I copies of T is finitely representable in T in the sense of [1], [2]. Both lacks are removed by our definition. Let us still mention that all results of [1], [2], and [3] remain valid if we replace those notions of finite representability by our one.

Section 1 is devoted to a criterion of finite representability of operators in terms of ultraproducts. The methods developed here allow us to solve a problem of A. Pietsch [16], concerning ultrastable ideals.

In Section 2 we prove that the class of super-A-operators equipped with a suitable quasi-norm defined there constitutes a quasi-normed operator ideal (denoted by $[A, \mathbf{A}]^{\text{super}}$) if $[A, \mathbf{A}]$ is regular. It turns out that the assumption on regularity is essential.

There is a natural connection between operator ideals and classes of Banach spaces (or, as it is usually said, properties of Banach spaces). Each operator ideal A generates a class of Banach spaces (more precisely, a space ideal), denoted by Space(A), namely, the class of all spaces E for which the identity I_E belongs to A. In Section 3 we give a characterization of Space(A^{super}). There arises a new notion of finite representability of Banach spaces, called *finite dual-representability*, which is stronger than the usual one. This notion can be applied to properties of Banach spaces that carry across to complemented subspaces only (but not to subspaces, in general). On the other hand, if we start with a property which is hereditary with respect to subspaces, then the usual finite representability and the finite dual-representability yield the same super-property.

In Section 4 we study the action of some procedures (injective hull, maximal hull, etc.), which are usually applied in ideal theory, on the super-ideals. These results can be used to determine super-ideals of given operator ideals. They also yield information on basic properties of the super-ideals (e.g., injectivity, surjectivity).

The final Section 5 is devoted to some applications and examples. With the help of the preceeding results we characterize the super-ideals of classical operator ideals (weakly compact, unconditionally summing operators, etc.). In this way we obtain also some new characterizations of the uniformly convexifying operators, introduced by Beauzamy [1]. The uniform approximation property, defined by Pełczyński and Rosenthal [14], appears as the super-property of the bounded approximation property (in the sense of the introduced finite dual-representability of Banach spaces). This result enables us to answer a question of Lindenstrauss and Tzafriri [13]. Finally, it turns out that the operator ideals whose definitions are, in a certain sense, already of finite-dimensional type constitute, in fact, super-ideals.

The methods of operator ideal theory, developed by Pietsch [16], [17], as well as the ultraproduct techniques, introduced by Dacunha-Castelle and Krivine [5], are the main tools in this paper.

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The author would like to express his gratitude to Professor A. Pietsch for many helpful conversations and suggestions.

Notation

Throughout this paper we consider bounded linear operators between Banach spaces only. Let E and F be Banach spaces. E' denotes the dual of E, $L(E, F)$ the space of all bounded linear operators from E to F, I_E the identity map of E and B_E the unit ball of E. The canonical embedding of a subspace $M \subset E$ is denoted by J_M, the canonical quotient map from E to E/M is denoted by Q_M. The dimension of M is denoted by $\dim M$, the codimension by $\operatorname{codim} M$. The symbol $\operatorname{Dim}(E)$ stands for the collection of all finite-dimensional subspaces of E, $\operatorname{Cod}(E)$ for the collection of all finite-codimensional subspaces of E. M^0 is the polar of M with respect to the duality (E, E'), unless stated otherwise.

Let $T \in L(E, F)$. We denote by T' the dual operator. K_E is the canonical injection of E into E''. An operator $T \in L(E, F)$ is called a *metric* *injection* whenever T is an isometry of E onto $T(E)$. The operator T is called a *metric surjection* if the induced operator $T\colon E/\mathrm{Ker}\,T \to F$ is an isometry. Let $\varepsilon > 0$. $T \in L(E, F)$ is called a $(1+\varepsilon)$-*isomorphism* if T is an isomorphism of E onto F which satisfies $\|T\| \leqslant 1+\varepsilon$ and $\|T^{-1}\| \leqslant 1+\varepsilon$.

Let U be an ultrafilter on an index set I, and let E_i $(i \in I)$ be a family of Banach spaces. $(E_i)_U$ denotes the ultraproduct of the family E_i with respect to the ultrafilter U. If $E_i = E$ for all $i \in I$, then we write $(E)_U$ instead of $(E_i)_U$. If $E_i \subset E$ and $x \in E$, then we denote by $(x)_U$ the element $(x_i)_U \in (E_i)_U$, where $x_i = x$ if $x \in E_i$, $x_i = 0$ if $x \notin E_i$. For given subsets $M_i \subset E_i$, the symbol $(M_i)_U$ denotes the set of all elements of $(E_i)_U$ of the form $(x_i)_U$, $x_i \in M_i$ $(i \in I)$. The ultraproduct of operators $T_i \in L(E_i, F_i)$ is denoted by $(T_i)_U$, analogously we write $(T)_U$ if $T_i = T$ for all $i \in I$.

A quasi-normed operator ideal is denoted by $[A, A]$, where the first letter stands for the operator ideal, the second stands for the quasi-norm. The components of the ideal are denoted by $A(E, F)$.

All necessary facts on operator ideals can be found in the monograph of A. Pietsch [16], [17] and in papers [15] and [20]. For the basic facts on ultraproducts of Banach spaces and operators we refer to [5], [18], [23], and [24].

1. Finite representability of operators

To introduce the notion of finite representability of operators, we need some preparations. Let E_1, E_2, F_1, F_2 be finite-dimensional Banach spaces and let $T_1 \in L(E_1, F_1)$, $T_2 \in L(E_2, F_2)$. The tripel (E_1, F_1, T_1) is called $(1+\varepsilon)$-*isomorphic* to the tripel (E_2, F_2, T_2) whenever $\dim E_1 = \dim E_2$, $\dim F_1 = \dim F_2$ and there exist $(1+\varepsilon)$-isomorphisms $V\colon E_1 \to E_2$ and $W\colon F_2 \to F_1$ such that

$$\|WT_2V - T_1\| \leqslant \varepsilon.$$

It follows that (E_2, F_2, T_2) is $(1+\delta)$-isomorphic to (E_1, F_1, T_1) for $\delta = \varepsilon(1+\varepsilon)^2$.

Let E and F be arbitrary Banach spaces and let $T \in L(E, F)$. Each tripel of the form $(M, F/N, Q_N T J_M)$ with $M \in \mathrm{Dim}(E)$, $N \in \mathrm{Cod}(F)$ is called an *elementary part* of T.

DEFINITION 1.1. An operator $T_0 \in L(E_0, F_0)$ is called *finitely representable in* $T \in L(E, F)$ (T_0 f.r. T, in short) if, for each $\varepsilon > 0$, each elementary part of T_0 is $(1+\varepsilon)$-isomorphic to an elementary part of T.

In other words, given $\varepsilon > 0$, $M_0 \in \mathrm{Dim}(E_0)$, $N_0 \in \mathrm{Cod}(F_0)$, there exist $M \in \mathrm{Dim}(E)$, $N \in \mathrm{Cod}(F)$, and $(1+\varepsilon)$-isomorphisms $V\colon M_0 \to M$

and $W: F/N \to F_0/N_0$ such that the following diagram commutes "exactly up to ε":

$$\begin{array}{ccccccc} M & \xrightarrow{J_M} & E & \xrightarrow{T} & F & \xrightarrow{Q_N} & F/N \\ {\scriptstyle V}\uparrow & & & & & & \downarrow{\scriptstyle W} \\ M_0 & \xrightarrow[J_{M_0}]{} & E_0 & \xrightarrow[T_0]{} & F_0 & \xrightarrow[Q_{N_0}]{} & F_0/N_0 \end{array}$$

i.e.

$$\|WQ_N T J_M V - Q_{N_0} T_0 J_{M_0}\| \leqslant \varepsilon.$$

Remark. We do not know whether commutativity up to ε can be replaced by full commutativity. This is the case when T_0 is an injection (see Section 3).

We now state the main theorem of this section.

THEOREM 1.2. *$T_0 \in L(E_0, F_0)$ is finitely representable in $T \in L(E, F)$ iff there exist an ultrafilter U, a metric injection $J: E_0 \to (E)_U$ and a metric surjection $Q: (F)_U \to F_0''$ such that*

$$K_{F_0} T_0 = Q(T)_U J.$$

In particular, $(T)_U$ is finitely representable in T.

For the proof we need the following result, which is due to Kürsten [10], [11] and, in a somewhat sharper form, to Stern (cf. [23]). We give here a short proof of Kürsten's variant, using methods of ideal theory.

LEMMA 1.3. *Let U be an ultrafilter on an index set I, let E_i $(i \in I)$ be a family of Banach spaces and $M \subset (E_i)_U'$ a finite-dimensional subspace of the dual of the ultraproduct $(E_i)_U$. Furthermore, let $\varepsilon > 0$ and let $\{u_k\}_{k=1}^n$ be a finite subset of $(E_i)_U$. Then there exists a map $S: M \to (E_i')_U$ which is a $(1+\varepsilon)$-isomorphism onto the image $S(M)$ and satisfies*

$$\langle Sv, u_k\rangle = \langle v, u_k\rangle \quad (v \in M,\ k = 1, \ldots, n).$$

Proof. It follows immediately from the definition of the ultraproduct of operators and the finite dimension of M that we can identify canonically

$$L(M, (E_i')_U) = (L(M, E_i'))_U.$$

Furthermore,

$$N(M', E_i) = I(M', E_i)$$

and

$$N(M', (E_i)_U) = I(M', (E_i)_U),$$

where N denotes the ideal of nuclear, I the ideal of integral operators (cf. [15], [16]). The ideal of integral operators is ultrastable (cf. [18] for the definition). This can easily be deduced from the well-known theorems on factorization of integral operators [15] and on integral operators with values in $L_1(\mu)$ spaces ([8], Th. 11, p. 141). It follows that we can identify

$$N(M', (E_i)_U) = (N(M', E_i))_U.$$

Now let

$$X = \operatorname{span}\{v \otimes u_k \colon v \in M,\ k = 1, \ldots, m\} \subset N(M', (E_i)_U),$$

where u_k, $k = n+1, \ldots, m$, are chosen such that

$$\|v\| \geqslant (1-\varepsilon) \sup_k \langle v, u_k \rangle \qquad (v \in M).$$

Since X is finite dimensional, the usual ultraproduct methods yield the existence of subspaces of the same dimension $X_i \subset N(M', E_i)$ such that $(X_i)_U = X$ and for $i \in D_0$, $D_0 \in U$ there are $(1+\varepsilon)$-isomorphisms S_i: $X_i \to X$ with $(S_i)_U = I_X$. Since

$$L(M, (E_i)'_U) = N(M', (E_i)_U)'$$

(cf. [15]), the canonical embedding J_M: $M \to (E_i)'_U$ defines a linear functional Φ of norm 1 on $N(M', E_i)_U$. Now $\Phi \circ S_i$ is a functional of norm $\leqslant 1+\varepsilon$ on X_i. Let Φ_i be the Hahn–Banach extension of $\Phi \circ S_i$ to the whole space $N(M', E_i)$ and put $\Phi_0 = (\Phi_i)_U$. We have

$$\Phi_0 \in (N(M', E_i)')_U = L(M, (E'_i)_U),$$

$\|\Phi_0\| \leqslant 1+\varepsilon$ and $\Phi_0|_X = \Phi|_X$. Φ_0 defines an operator S which is easily seen to satisfy the conclusion. This completes the proof.

Let us remark that Stern's variant can be proved in a similar way.

Proof of Theorem 1.2. *Part* 1: Suppose T_0 is finitely representable in T. Let I be the set of all triples $i = (M_0, N_0, \varepsilon)$ with $M_0 \in \operatorname{Dim}(E_0)$, $N_0 \in \operatorname{Cod}(F_0)$ and $\varepsilon > 0$. It will be convenient to denote the components of the tripel i by M_{0i}, $N_{>0i}$, ε_i. The natural order on I is defined by the relation

$$(M_{01}, N_{01}, \varepsilon_1) \prec (M_{02}, N_{02}, \varepsilon_2)$$

iff

$$M_{01} \subset M_{02}, \qquad N_{01} \supset N_{02}, \qquad \varepsilon_1 < \varepsilon_2.$$

Let U be an ultrafilter on I dominating the order filter. Since T_0 f.r. T, we can find, for each $i \in I$, spaces $M_i \in \operatorname{Dim}(E)$, $N_i \in \operatorname{Cod}(F)$ and $(1+\varepsilon_i)$-isomorphisms V_i and W_i such that the following diagram commutes exactly up to ε_i:

$$\begin{array}{ccccccc}
M_i & \xrightarrow{J_{M_i}} & E & \xrightarrow{T} & F & \xrightarrow{Q_{N_i}} & F/N_i \\
\uparrow{\scriptstyle V_i} & & & & & & \downarrow{\scriptstyle W_i} \\
M_{0i} & \xrightarrow[J_{M_{0i}}]{} & E_0 & \xrightarrow[T_0]{} & F_0 & \xrightarrow[Q_{N_{0i}}]{} & F_0/N_{0i}
\end{array}$$

Now put $J_1 x = (x)_U$ for $x \in E_0$. Then J_1 is a metric injection of E_0 into $(M_{0i})_U$ (cf. [24]). Since the maps V_i are $(1+\varepsilon_i)$-isomorphisms, it follows that

$$(V_i)_U\colon (M_{0i})_U \to (M_i)_U$$

is an isometry. Finally,

$$(J_{M_i})_U\colon (M_i)_U \to (E)_U$$

is a metric injection, and we put

$$J = (J_{M_i})_U (V_i)_U J_1 .$$

In order to define Q, remark that the polars N_{0i}^0 of N_{0i} are finite-dimensional subspaces of F_0'. Moreover, it is easily verified that all finite-dimensional subspaces of F_0' can be generated in this way. Thus, there is a canonical (metric) injection

$$J_2\colon F_0' \to (N_{0i}^0)_U$$

which is defined similarly to J_1. Furthermore,

$$(W_i')_U\colon (N_{0i}^0)_U \to (N_i^0)_U$$

is an isometry, and

$$(Q_{N_i}')_U\colon (N_i^0)_U \to (F')_U$$

is a metric injection. Hence the composition

$$J_3 = (Q_{N_i}')_U (W_i')_U J_2$$

is a metric embedding of F_0' into $(F')_U$. Consequently,

$$J_3'\colon (F')_U' \to F_0''$$

is a metric surjection. Since $(F)_U$ can be identified with a subspace of $(F')_U'$, we put

$$Q = J_3'|_{(F)_U}\colon (F)_U \to F_0'' .$$

It follows from the definition of Q that, for $(y_i)_U \in (F)_U$,

$$Q(y_i)_U = F_0'\text{-}\lim_U y_{0i},$$

where $y_{0i} \in F_0$ are chosen such that

$$Q_{N_{0i}} y_{0i} = W_i Q_{N_i} y_{0i}$$

and

$$\|y_{0i}\| \leqslant \|W_i Q_{N_i} y_i\| + \varepsilon_i \qquad (i \in I).$$

We can conclude that Q is a metric surjection. In fact, let $y_0'' \in F_0''$; find elements $z_i \in F_0/N_{0i}$ by restriction of y_0'' to N_{0i}^0. Next choose $y_i \in F$ such that

$$z_{0i} = W_i Q_{N_i} y_i, \qquad \|y_i\| \leqslant \|y_0''\| + 2\varepsilon_i.$$

We get $y_0'' = Q(y_i)_U$, i.e. Q is the desired surjection. Now let $x \in E_0$, $y' \in F_0'$. Then

$$(T)_U Jx = (TV_i x)_U,$$

where the right-hand side is defined for $i \succ i_0$. Furthermore, it follows that

$$Q(TV_i x)_U = F_0'\text{-}\lim_U y_{0i},$$

where

$$Q_{N_{0i}} y_{0i} = W_i Q_{N_i} TV_i x.$$

Therefore,

$$\langle Q(T)_U Jx, y'\rangle = \lim_U \langle y_{0i}, y'\rangle = \lim_U \langle W_i Q_{N_i} TJ_{N_i} V_i x, y'\rangle.$$

Again, the right-hand side is defined for $i \succ i_1$. The definition of finite representability yields

$$\lim_U \langle W_i Q_{N_i} TJ_{M_i} V_i x, y'\rangle = \lim_U \langle Q_{N_{0i}} T_0 J_{M_{0i}} x, y'\rangle$$

and thus,

$$Q(T)_U J = K_{F_0} T_0.$$

This concludes the proof of Part 1.

Part 2. Suppose there is a metric injection $J\colon E_0 \to (E)_U$ and a metric surjection $Q\colon (F)_U \to F_0''$ such that

$$K_{F_0} T_0 = Q(T)_U J.$$

It follows immediately from Definition 1.1 that $K_{F_0} T$ f.r. T implies T_0 f.r. T. Hence it is sufficient to prove $(T)_U$ f.r. T. Let $M_0 \in \operatorname{Dim}((E)_U)$, $N_0 \in \operatorname{Cod}((F)_U)$ and $\varepsilon > 0$. Choose a basis $\{u_j\}_{j=1}^m$ in M_0, and let $u_j = (x_{ji})_U$ for $j = 1, \ldots, m$. The properties of ultraproducts yield that there exists a $D_0 \in U$ such that the map $u_j \to x_{ji}$ defines a $(1+\varepsilon)$-isomorphism V_i of M_0 onto $M_i = \operatorname{span}\{x_{ji}\}_{j=1}^m$ for each $i \in D_0$. Denote the polar N_0^0 by M_1, $M_1 \in \operatorname{Dim}((F)_U')$. By Lemma 1.3, there is a $(1+\varepsilon)$-isomorphism S of M_1 onto a subspace M_2 of $(F')_U$ satisfying

$$\langle Sv, (T)_U u_j\rangle = \langle v, (T)_U u_j\rangle \qquad (v \in M_1,\ j = 1, \ldots, m).$$

Let $\{v_k\}_{k=1}^n$ be a basis of M_1. Then $w_k = Sv_k$ $(k = 1, \ldots, n)$ constitutes a basis of M_2 if ε is chosen sufficiently small. Put $w_k = (y_{ki}')_U$. There is a D_1 such that for $i \in D_1$ the map $w_k \to y_{ki}'$ defines a $(1+\varepsilon)$-isomorphism $\tilde{W}_i$

of M_2 onto $M_{2i} = \operatorname{span}\{y'_{ki}\}_{k=1}^n$. Hence, $\tilde{W}_i S$ is a $(1+\varepsilon)^2$-isomorphism of M_1 onto M_{2i}. Now put

$$N_i = M_{2i}^0 = \{y \in F\colon \langle y, y'\rangle = 0,\ y' \in M_{2i}\}.$$

Then we have $F/N_i = M'_{2i}$ and, on the other hand, $(F)_U/N_0 = M'_1$. Therefore we define

$$W_i = (\tilde{W}_i S)'\colon F/N_i \to (F)_U/N_0.$$

W_i is a $(1+\varepsilon)^2$-isomorphism.

Finally, we prove that, for a certain i, the following diagram commutes exactly up to ε:

$$\begin{array}{ccccccc} M_i & \xrightarrow{J_{M_i}} & E & \xrightarrow{T} & F & \xrightarrow{Q_{N_i}} & F/N_i \\ {\scriptstyle V_i}\uparrow & & & & & & \downarrow{\scriptstyle W_i} \\ M_0 & \xrightarrow[J_{M_0}]{} & (E)_U & \xrightarrow[(T)_U]{} & (F)_U & \xrightarrow[Q_{N_0}]{} & (F)_U/N_0 \end{array}$$

We have for $i \in D_0 \cap D_1$

$$\begin{aligned}\langle W_i Q_{N_i} T J_{M_i} V_i u_j, v_k\rangle &= \langle Q_{N_i} T J_{M_i} V_i u_j, y'_{ki}\rangle \\ &= \langle Q_{N_i} T J_{M_i} x_{ji}, y'_{ki}\rangle = \langle T x_{ji}, y'_{ki}\rangle.\end{aligned}$$

On the other hand, we get

$$\langle Q_{N_0}(T)_U J_{M_0} u_j, v_k\rangle = \langle (T)_U u_j, v_k\rangle = \langle (T)_U u_j, w_k\rangle = \lim_U \langle T x_{ij}, y'_{ki}\rangle.$$

Consequently,

$$\lim_U \langle W_i Q_{N_i} T J_{M_i} V_i u_j, v_k\rangle = \langle Q_{N_0}(T)_U J_{M_0} u_j, v_k\rangle$$

and therefore, by the finite dimension of M_0 and M_1,

$$\lim_U \|W_i Q_{N_i} T J_{M_i} V_i - Q_{N_0}(T)_U J_{M_0}\| = 0.$$

This proves Theorem 1.2.

Remark. The construction of Q in Part 1 also yields the following lemma which we need in Section 2:

LEMMA 1.4. *If $T_0 \in L(E_0, F_0)$ is finitely representable in $T \in L(E, F)$, then there is an ultrafilter U such that*

$$K_{F_0} T_0 = QJ(T)_U J_1,$$

where $J_1\colon E_0 \to (E)_U$ is a metric injection, $J\colon (F)_U \to (F')'_U$ the canonical embedding, and $Q\colon (F')'_U \to F''_0$ is a metric surjection.

Let us recall (cf. [16], [18]) that a quasi-normed operator ideal is said to be *ultrastable* if

$$(T_i)_U \in A\big((E_i)_U, (F_i)_U\big), \qquad A\big((T_i)_U\big) \leqslant \lim_U A(T_i)$$

for each ultrafilter U on an index set I, and every family of operators $T_i \in A(E_i, F_i)$ with $\sup A(T_i) < \infty$. For a given operator ideal $[A, A]$, the dual ideal $[A^{\mathrm{dual}}, A^{\mathrm{dual}}]$ is defined in the following way:

$$T \in A^{\mathrm{dual}}(E, F) \quad \text{iff} \quad T' \in A(F', E').$$

We put $A^{\mathrm{dual}}(T) = A(T')$. In [16] A. Pietsch posed the following problem: Is ultrastability carried across from the ideal to the dual ideal? The methods of finite representability can be applied to solve this problem in the affirmative:

THEOREM 1.5. *If $[A, A]$ is an ultrastable quasi-normed operator ideal, then $[A, A]^{\mathrm{dual}}$ is ultrastable, as well.*

First we prove a lemma, which is an extension of Lemma 1.3.

LEMMA 1.6. *Let $T_i \in L(E_i, F_i)$ $(i \in I)$ be a bounded family of operators and let U be an ultrafilter on I. Then $(T_i)'_U$ is finitely representable in $(T'_i)_U$.*

Proof. Let $M_0 \subset (F_i)'_U$, $N_0 \subset (E_i)'_U$, $\dim M_0 < \infty$, $\operatorname{codim} N_0 < \infty$, and let $\varepsilon > 0$. $N_0^0 \subset (E_i)''_U$ is finite dimensional, hence, by the principle of local reflexivity (cf. [12], p. 196), we can find a $(1+\varepsilon)$-isomorphism R from N_0^0 onto a subspace $M_1 \subset (E_i)_U$ such that

$$\langle (T_i)'_U u, Rv\rangle = \langle (T_i)'_U u, v\rangle \quad (u \in M_0,\ v \in N_0^0).$$

On the other hand, by Lemma 1.3, there is a $(1+\varepsilon)$-isomorphism S from M_0 onto $M \subset (F'_i)_U$ with

$$\langle Su, (T_i)_U Rv\rangle = \langle u, (T_i)_U Rv\rangle \quad (u \in M_0,\ v \in N_0^0).$$

We put

$$V = S\colon M_0 \to M \quad \text{and} \quad W = R'\colon (E'_i)_U/N \to (E_i)'/N_0,$$

where

$$N = N_0 \cap (E'_i)_U = \{z \in (E'_i)_U\colon z|_{M_1} = 0\}.$$

For $u \in M_0$, $v \in N_0^0$, we get

$$\begin{aligned}\langle Q_{N_0}(T_i)'_U J_{M_0} u, v\rangle &= \langle (T_i)'_U u, v\rangle = \langle (T_i)'_U u, Rv\rangle \\ &= \langle u, (T_i)_U Rv\rangle = \langle Su, (T_i)_U Rv\rangle = \langle R' Q_N (T'_i)_U Vu, v\rangle \\ &= \langle W Q_N (T'_i)_U J_M Vu, v\rangle.\end{aligned}$$

Thus,

$$Q_{N_0}(T_i)'_U J_{M_0} = W Q_N (T'_i)_U J_M V,$$

i.e. $(T_i)'_U$ is finitely representable in $(T'_i)_U$.

Proof of Theorem 1.5. Let I be an index set, U an ultrafilter on I and $T_i \in A^{\mathrm{dual}}(E_i, F_i)$ $(i \in I)$ a family of operators with $\sup A^{\mathrm{dual}}(T_i) < \infty$. By the definition of $[A, A]^{\mathrm{dual}}$, we have $T'_i \in A(F'_i, E'_i)$ and $\sup A(T'_i) < \infty$. The ultrastability is proved if we show

$$(T_i)'_U \in A\big((F_i)'_U, (E_i)'_U\big), \quad A\big((T_i)'_U\big) \leqslant \lim_U A(T'_i).$$

It follows from the ultrastability of $[A, A]$ that

$$(T_i')_U \in A\big((F_i')_U, (E_i')_U\big), \qquad A\big((T_i')_U\big) \leqslant \lim_U A(T_i').$$

By Lemma 1.6, $(T_i)_U'$ f.r. $(T_i')_U$. Now we apply Theorem 1.2 to find an ultrafilter $\tilde{U}$ and operators Q and J such that

$$K_{(E_i)_U'}(T_i)_U' = Q\big((T_i')_U\big)_{\tilde{U}} J,$$

consequently,

$$(T_i)_U' = PQ\big((T_i')_U\big)_{\tilde{U}} J,$$

where P is the canonical projection of $(E_i)_U'''$ onto $(E_i)_U'$. We obtain

$$(T_i)_U' \in A\big((F_i)_U', (E_i)_U'\big)$$

and

$$A\big((T_i)_U'\big) \leqslant A\big[\big((T_i')_U\big)_{\tilde{U}}\big] \leqslant \lim_{\tilde{U}} A\big((T_i')_U\big) = A\big((T_i')_U\big) \leqslant \lim_U A(T_i').$$

This completes the proof.

2. Super-ideals of operators

DEFINITION 2.1. Let $[A, A]$ be a quasi-normed operator ideal. An operator $T \in L(E, F)$ *belongs to* $A^{\mathrm{super}}(E, F)$ (the set of super-A-operators) iff $T_0 \in L(E_0, F_0)$ and T_0 f.r. T imply $T_0 \in A(E_0, F_0)$. For $T \in A^{\mathrm{super}}(E, F)$ we put

$$A^{\mathrm{super}}(T) = \sup_{T_0 \text{ f.r. } T} A(T_0).$$

The pair $[A^{\mathrm{super}}, A^{\mathrm{super}}]$ is called the *super-ideal* of $[A, A]$. We also write $[A, A]^{\mathrm{super}}$ instead of $[A^{\mathrm{super}}, A^{\mathrm{super}}]$. The quasi-normed operator ideal $[A, A]$ itself is called *super-ideal* if $[A, A]^{\mathrm{super}} = [A, A]$.

We now prove the central result of this section. Let us still mention that a quasi-normed operator ideal $[A, A]$ is *regular* if $T \in L(E, F)$ and $K_F T \in A(E, F'')$ imply $T \in A(E, F)$ and $A(T) = A(K_F T)$ (cf. [16], [17]).

THEOREM 2.2. *Let* $[A, A]$ *be a quasi-normed regular operator ideal. Then* $[A, A]^{\mathrm{super}}$ *is a quasi-normed (and regular) operator ideal as well. Furthermore, if* $[A, A]$ *is* p*-normed (normed, respectively) then* $[A, A]^{\mathrm{super}}$ *is also* p*-normed (normed, resp.).*

Proof. We divide the proof into three parts.

Part 1. We show that $A^{\mathrm{super}}(T)$ is finite for $T \in A^{\mathrm{super}}(E, F)$. Assume that $A^{\mathrm{super}}(T) = \infty$. Then there exists a sequence (T_n), $T_n \in L(E_n, F_n)$, such that T_n f.r. T and

$$\lim_{n\to\infty} A(T_n) = \infty.$$

By Lemma 1.4, for each n there exist an ultrafilter U_n on an index set I_n and operators $J_{n1}\colon E_n \to (E)_{U_n}$ and $Q_n\colon (F')'_{U_n} \to F''_n$ such that $\|J_{n1}\| \leqslant 1$, $\|Q_n\| \leqslant 1$ and

$$K_{F_n} T_n = Q_n J_n (T)_{U_n} J_{n1},$$

where J_n is the canonical embedding of $(F)_{U_n}$ into $(F')'_{U_n}$. This together with the regularity of $[A, A]$ imply

$$\lim_{n\to\infty} A\big(J_n (T)_{U_n}\big) = \infty .$$

Now put $I = \prod_{n\in N} I_n$ and $U_0 = \prod_{n\in N} U_n$ (cf. [4], Ch. I, § 6.7). Let U be an ultrafilter dominating U_0. In the sequel we define maps

$$R_n\colon (E)_{U_n} \to (E)_U$$

and

$$S_n\colon (F)''_U \to (F')'_{U_n}$$

such that $\|R_n\| \leqslant 1$, $\|S_n\| \leqslant 1$ and

$$J_n (T)_{U_n} = S_n K_{(F)_U} (T)_U R_n .$$

Then it follows from our assumption and the regularity of $[A, A]$ that $A\big((T)_U\big) = \infty$. Since $(T)_U$ f.r. T (Theorem 1.2), this will be a contradiction.

We now construct R_n and S_n. Fix n and let $(x_{i_n})_{U_n} \in (E)_{U_n}$. Here i_n runs over I_n. For $i \in I$, $i = (i_1, i_2, \ldots, i_n, \ldots)$ we put $x_i = x_{i_n}$ and

$$R_n (x_{i_n})_{U_n} = (x_i)_U .$$

It follows from the definition of the product filter that

$$\lim_{U_n} \|x_{i_n}\| = 0$$

implies

$$\lim_{U} \|x_i\| = 0 .$$

Thus, R_n is well-defined and $\|R_n\| \leqslant 1$. The operator $S_{n1}\colon (F')_{U_n} \to (F')_U$ is defined analogously:

$$S_{n1} (y'_{i_n})_{U_n} = (y'_i)_U .$$

If $J\colon (F')_U \to (F)'_U$ is the canonical embedding, then we put

$$S_n = S'_{n1} J'\colon (F'')_U \to (F')'_{U_n} .$$

We have $\|S_n\| \leqslant 1$. Now let $u \in (E)_{U_n}$, $u = (x_{i_n})_{U_n}$ and $v \in (F')_{U_n}$, $v = (y'_{i_n})_{U_n}$.

It follows that

$$\begin{aligned}\langle S_n K_{(F)_U}(T)_U R_n u, v\rangle &= \langle S_n K_{(F)_U}(T)_U R_n (x_{i_n})_{U_n}, (y'_{i_n})_{U_n}\rangle \\ &= \langle K_{(F)_U}(Tx_i)_U, J S_{n1}(y'_{i_n})_{U_n}\rangle = \langle (Tx_i)_U, J(y'_i)_U\rangle \\ &= \lim_U \langle Tx_i, y'_i\rangle = \lim_{U^n} \langle Tx_{i_n}, y'_{i_n}\rangle \\ &= \langle J_n(T)_{U_n}(x_{i_n})_{U_n}, (y'_{i_n})_{U_n}\rangle = \langle J_n(T)_{U_n} u, v\rangle.\end{aligned}$$

Thus,

$$J_n(T)_{U_n} = S_n K_{(F)_U}(T)_U R_n.$$

Part 2. Here we shall verify the ideal properties (excl. completeness). Let $T_1, T_2 \in A^{\text{super}}(E, F)$ and suppose that $T_0 \in L(E_0, F_0)$ is finitely representable in $T_1 + T_2$. By Theorem 1.2, there exist an ultrafilter U, a metric injection J, and a metric surjection Q with

$$K_{F_0} T_0 = Q(T_1 + T_2)_U J = Q(T_1)_U J + Q(T_2)_U J.$$

Again by Theorem 1.2, $Q(T_k)_U J$ f.r. T_k, hence

$$Q(T_k)_U J \in A(E_0, F_0''), \quad A\big(Q(T_k)_U J\big) \leqslant A^{\text{super}}(T_k) \quad (k = 1, 2).$$

It follows that $K_{F_0} T_0 \in A(E_0, F_0'')$ and

$$A(K_{F_0} T_0) \leqslant \varkappa\big(A^{\text{super}}(T_1) + A^{\text{super}}(T_2)\big).$$

Now the regularity of $[A, \mathbf{A}]$ implies

$$T_1 + T_2 \in A^{\text{super}}(E, F)$$

and

$$A^{\text{super}}(T_1 + T_2) \leqslant \varkappa\big(A^{\text{super}}(T_1) + A^{\text{super}}(T_2)\big).$$

If A is a p-norm, then the same proof yields that A^{super} is a p-norm, too.

Next suppose that $T \in A^{\text{super}}(E, F)$, $R \in L(E_1, E)$, $S \in L(F, F_1)$, and let $T_0 \in L(E_0, F_0)$ be finitely representable in STR. Then we deduce from Theorem 1.2

$$K_{F_0} T_0 = Q(STR)_U J = Q(S)_U (T)_U (R)_U J \in A(E_0, F_0'').$$

We obtain, as in the above case,

$$A(K_{F_0} T_0) \leqslant \|S\| A^{\text{super}}(T) \|R\|,$$

and hence

$$STR \in A^{\text{super}}(E_1, F_1), \quad A^{\text{super}}(STR) \leqslant \|S\| A^{\text{super}}(T) \|R\|.$$

Finally, it is trivial to verify that $A^{\text{super}}(E, F)$ contains all one-dimensional operators and that A^{super} coincides on this set with the usual operator norm. Let us mention that the regularity of $[A, \mathbf{A}]^{\text{super}}$ follows easily.

Part 3. It remains to show the completeness of $[A, A]^{\text{super}}$. We use the fact (cf. [16], Th. 6.2.5) that each quasi-normed operator ideal $[A, A]$ possesses an equivalent p-norm A_p $(0 < p \leqslant 1)$, i.e. there is a constant $C > 0$ such that

$$CA(T) \leqslant A_p(T) \leqslant A(T) \qquad (T \in A(E, F)).$$

From this, we get immediately

$$CA^{\text{super}}(T) \leqslant A_p^{\text{super}}(T) \leqslant A^{\text{super}}(T),$$

and A_p^{super} is a p-norm, as well. Now it is sufficient (cf. [16], Th. 6.2.3) to show that

$$T_n \in A^{\text{super}}(E, F) \qquad (n = 1, 2, \ldots), \qquad \sum_n A_p^{\text{super}}(T_n)^p$$

imply

$$T = \sum_n T_n \in A^{\text{super}}(E, F), \qquad A_p^{\text{super}}(T)^p \leqslant \sum_n A_p^{\text{super}}(T_n)^p.$$

We have

$$\Big\| \sum_{n=k}^{m} T_n \Big\|^p \leqslant A_p^{\text{super}}\Big(\sum_{n=k}^{m} T_n\Big)^p \leqslant \sum_{n=k}^{m} A_p^{\text{super}}(T_n)^p.$$

Thus, $\sum_n T_n$ is convergent in the operator norm, and we put $T = \sum_n T_n$. Now suppose that T_0 f.r. T. By Theorem 1.2,

$$K_{F_0} T_0 = Q\Big(\sum_n T_n\Big)_U J.$$

The properties of ultraproducts yield

$$K_{F_0} T_0 = \sum_n Q(T_n)_U J.$$

Furthermore, $(T_n)_U$ f.r. T_n, and hence

$$A_p\big(Q(T_n)_U J\big) \leqslant A_p^{\text{super}}(T_n).$$

Consequently,

$$K_{F_0} T_0 = \sum_n Q(T_n)_U J \in A(E_0, F_0'')$$

and

$$A_p(K_{F_0} T_0) \leqslant \sum_n A_p\big(Q(T_n)_U J\big)^p \leqslant \sum_n A_p^{\text{super}}(T_n)^p.$$

Now the regularity of $[A, A]$ implies the desired result. This concludes the proof of Theorem 2.2.

Since we used permanently the regularity of $[A, A]$, the question arises whether this assumption is in fact essential. The following proposition shows that the answer is affirmative.

PROPOSITION 2.3. *Let $[N, N]$ be the ideal of nuclear operators. Then $[N^{\mathrm{super}}, N^{\mathrm{super}}]$ is not a quasi-normed operator ideal.*

Proof. Assume that $[N^{\mathrm{super}}, N^{\mathrm{super}}]$ is a quasi-normed operator ideal. By the above remark in Step 3 of the proof, we may assume that N^{super} is a p-norm. Figiel and Johnson [7] proved that there exist Banach spaces E, F and a non-nuclear operator $T \in L(E, F)$ whose dual is nuclear, i.e. the ideal $[N, N]$ is not regular. In other words,

$$T \notin N(E, F), \qquad K_F T \in N(E, F'').$$

Let

$$K_F T = \sum_{k=1}^{\infty} x_k' \otimes y_k'', \qquad x_k' \in E', \ y_k'' \in F'',$$

be a representation of $K_F T$ with

$$\sum_{k=1}^{\infty} \|x_k'\| \, \|y_k''\| < \infty.$$

Now choose an increasing sequence of indices $(n_l)_{l=1}^{\infty}$ such that

$$\sum_{l=1}^{\infty} \Big(\sum_{k=n_l+1}^{n_{l+1}} \|x_k'\| \, \|y_k''\| \Big)^p < \infty.$$

Put

$$T_l = \sum_{k=n_l+1}^{n_{l+1}} x_k' \otimes y_k'', \qquad T_l \in N(E, F'').$$

We shall show that $T_l \in N^{\mathrm{super}}(E, F'')$. Suppose that T_0 f.r. T_l. Then we have

$$K_{F_0} T_0 = Q (T_l)_U J.$$

On the other hand,

$$(T_l)_U = \sum_{k=n_l+1}^{n_{l+1}} (x_k')_U \otimes (y_k'')_U$$

thus,

$$K_{F_0} T_0 = \sum_{k=n_l+1}^{n_{l+1}} x_{k0}' \otimes y_{k0}'', \qquad x_{k0}' \in E_0', \ y_{k0}'' \in F_0'',$$

and

$$\sum_{k=n_l+1}^{n_{l+1}} \|x_{k0}'\| \, \|y_{k0}''\| \leqslant \sum_{k=n_l+1}^{n_{l+1}} \|x_k'\| \, \|y_k''\|.$$

By the principle of local reflexivity [12] there exists a $(1+\varepsilon)$-isomorphism R from the space

$$\operatorname{span}(\{y''_{k0}\}_{k=n_l+1}^{n_{l+1}}) \supset \operatorname{Im} T_0$$

onto a subspace of F_0 such that $\operatorname{Im} T_0$ remains invariant. We get

$$T_0 = \sum_{k=n_l+1}^{n_{l+1}} x'_{k0} \otimes R y''_{k0}$$

and, for $\varepsilon \to 0$,

$$N(T_0) \leqslant \sum_{k=n_l+1}^{n_{l+1}} \|x'_k\| \, \|y''_k\|.$$

Therefore,

$$N^{\text{super}}(T_l) \leqslant \sum_{k=n_l+1}^{n_{l+1}} \|x'_k\| \, \|y''_k\|$$

and

$$\sum_{l=1}^{\infty} N^{\text{super}}(T_l)^p < \infty.$$

Consequently, since each quasi-normed ideal is, by definition, complete (cf. [16]),

$$K_F T = \sum_{l=1}^{\infty} T_l \in N^{\text{super}}(E, F'').$$

T is finitely representable in $K_F T$; hence $T \in N(E, F)$ — a contradiction.

At the end of this section we give an example that neither of the definitions of finite representability from [1], [2] lead to quasi-normed super-ideals. Therefore, let us assume that we have replaced our notion of finite representability in Definition 2.1 by one of the notions of [1], [2], say that of [1], the remaining case can be treated analogously. Let us denote by $A^s(E, F)$ the set of super-A-operators with respect to Definition 5 of [1], i.e., $A^s(E, F) = \{T \in L(E, F)\colon T_0 \in L(E_0, F_0)$ finitely representable in T in the sense of [1] implies $T_0 \in A(E_0, F_0)\}$. Put $A^s(T) = \sup\{A(T_0)\colon T_0$ f.r. T in the sense of [1]$\}$. We shall show that $[A^s, A^s]$ does not constitute, in general, a quasi-normed operator ideal, even if $[A, A]$ is regular. For this purpose, let $[A, A] = [G, \| \ \|]$ be the ideal of approximable operators (cf. [16], [17]). This ideal is regular. We have

PROPOSITION 2.4. *$[A^s, A^s]$ is not a quasi-normed operator ideal.*

Proof. Assume that $[A^s, A^s]$ forms a quasi-normed operator ideal. Then it follows immediately from Definition 5 of [1] that, for each $T \in L(E, F)$ and each metric injection $J\colon F \to G$, $JT \in A^s(E, G)$ implies $T \in A^s(E, F)$ and $A^s(T) = A^s(JT)$. This shows that $[A^s, A^s]$ is injective (cf. [16], [17]). Now let $T \in L(E, F)$ be a finite rank operator and suppose $T_0 \in L(E_0, F_0)$ is finitely representable in T in the sense of [1]. It follows that T_0 is finite dimensional (hence approximable) and $\|T_0\| \leqslant \|T\|$. We get

$$T \in A^s(E, F), \quad A^s(T) = \|T\|.$$

Each $T_1 \in A(E, F)$ can be approximated in the operator norm by finite rank operators. By the completeness of the ideal $[A^s, A^s]$ it follows that

$$T_1 \in A^s(E, F), \quad A^s(T_1) = \|T_1\|,$$

thus,

$$[A^s, A^s] = [A, A].$$

But the ideal $[A, A] = [G, \| \; \|]$ is not injective (cf. [16]). This contradiction concludes the proof.

3. Connections with Banach space theory

We start with the definition of a stronger notion of finite representability than the usual one.

DEFINITION 3.1. Let E_0 and E be Banach spaces. E_0 is called *finitely dual-representable in* E (E_0 f.d.-r. E_0 in short) if for each $\varepsilon > 0$ and each pair of subspaces (M_{01}, M_{02}) with $M_{01} \in \mathrm{Dim}(E_0)$ and $M_{02} \in \mathrm{Dim}(E_0')$ there exist a pair (M_1, M_2), $M_1 \in \mathrm{Dim}(E)$, $M_2 \in \mathrm{Dim}(E')$, and $(1+\varepsilon)$-isomorphisms $V\colon M_{01} \to M_1$, $W\colon M_{02} \to M_2$ such that

$$\langle Vx, Wx' \rangle = \langle x, x' \rangle \quad (x \in M_{01},\ x' \in M_{02}).$$

Roughly speaking, this definition signifies that E_0 and E_0' are finitely representable (in the usual sense) in E and E', respectively, and the duality between E_0 and E_0' is preserved. This notion corresponds to the finite representability of operators, as the following proposition shows.

PROPOSITION 3.2. *E_0 is finitely dual representable in E iff I_{E_0} is finitely representable in I_E.*

Proof. The "only if" part is obvious. To prove the converse, suppose that I_{E_0} is finitely representable in I_E. Let $\varepsilon > 0$, $M_{01} \in \mathrm{Dim}(E_0)$ and $M_{02} \in \mathrm{Dim}(E_0')$ be given. Choose $\tilde{M}_{02} \supset M_{02}$ such that the restriction to M_{01} of the quotient map $Q_{N_0}\colon E_0 \to E_0/N_0$ is an injection, where $N_0 = \tilde{M}_{02}^0$ is the polar of $\tilde{M}_{02}$ in E_0. By hypothesis, for each $\delta > 0$ there exist $M_1 \in \mathrm{Dim}(E)$, $N \in \mathrm{Cod}(E)$ and $(1+\delta)$-isomorphisms $V\colon M_{01} \to M_1$, $\tilde{W}\colon E/N \to E_0/N_0$ with

$$\|\tilde{W} Q_N J_{M_1} V - Q_{N_0} J_{M_{01}}\| \leqslant \delta.$$

$Q_{N_0}J_{M_{01}}$ is an injection. Choosing δ small enough, we can achieve that $WQ_NJ_{M_1}V$ is an injection, too. Let $\{e_j\}_{j=1}^n$ be a normed basis of M_{01}. Then $y_j = Q_{N_0}J_{M_{01}}e_j$ is a basis of a subspace of E_0/N_0 and $z_j = Q_NJ_{M_1}Ve_j$ is a basis of Q_NM_1. Furthermore, $\|\tilde{W}z_j - y_j\| \leqslant \delta$. Again, by a suitable choice of δ, we can "correct" $\tilde{W}$ on z_j (say $\tilde{\tilde{W}}$) such that $\tilde{\tilde{W}}z_j = y_j$ and $\tilde{\tilde{W}}$ is a $(1+\varepsilon)$-isomorphism of E/N onto E_0/N_0. Now we have

$$\tilde{\tilde{W}}Q_N Vx = Q_{N_0}x \quad (x \in M_{01})$$

and, for $x' \in \tilde{M}_{02}$,

$$\langle \tilde{\tilde{W}}Q_N Vx, x'\rangle = \langle Q_{N_0}x, x'\rangle$$

hence,

$$\langle Vx, \tilde{\tilde{W}}'x'\rangle = \langle x, x'\rangle.$$

$\tilde{\tilde{W}}'$ is a $(1+\varepsilon)$-isomorphism of $\tilde{M}_{02}$ onto N^0 (the polar of N in E'). Now it is sufficient to put $W = \tilde{\tilde{W}}'|_{M_{02}}$ and $M_2 = \operatorname{Im} W$ to obtain the desired result.

Remark. The principle of local reflexivity yields that E'' is finitely dual-representable in E.

The above Proposition 3.2 combined with Theorem 1.2 gives a criterion of finite dual-representability:

PROPOSITION 3.3. *E_0 is finitely dual-representable in E iff there is an ultrafilter U such that E_0'' is isometrically isomorphic to a norm-one-complemented subspace of $(E)_U$. In particular, $(E)_U$ is finitely dual-representable in E.*

Proof. Suppose that E_0 f.d.-r. E. Then, as already remarked, E_0'' f.d.-r. E and hence $I_{E_0''}$ f.r. I_E. By Theorem 1.2, there exist an ultrafilter U and operators J and Q with $\|J\| \leqslant 1$, $\|Q\| \leqslant 1$ and

$$K_{E_0''}I_{E_0''} = Q(I_E)_U J = QI_{(E)_U}J.$$

Consequently,

$$I_{E_0''} = P_{E_0''}QI_{(E)_U}J,$$

where $P_{E_0''}$ denotes the canonical projection of $E_0^{(IV)}$ onto E_0''. Now the assertion follows immediately.

Conversely, if E_0'' is a subspace of $(E)_U$ which admits a projection of norm one, then Theorem 1.2 implies $I_{E_0''}$ f.r. I_E, hence E_0 f.d.-r. E.

Let $\mathbf{A}$ be a space ideal in the sense of Pietsch (cf. [16], [17]), that means a class of Banach spaces which contains the scalar field $(\boldsymbol{R}, \boldsymbol{C})$ and which is stable under the following operations: isomorphisms, direct sum of two members, complemented subspaces. Let us denote by $\mathbf{A}^{\text{d-super}}$

the corresponding super-class with respect to the *finite dual*-representability, i.e.

$$\mathbf{A}^{\text{d-super}} = \{E\colon E_0 \text{ f.d.-r. } E \text{ implies } E_0 \in \mathbf{A}\}.$$

Recall also that an operator ideal A defines in a natural way a space ideal, denoted by $\mathrm{Space}(A)$: $E \in \mathrm{Space}(A)$ iff $I_E \in A(E, E)$ ([16], [17]). The finite dual-representability can be used to describe $\mathrm{Space}(A)^{\text{super}}$ in terms of $\mathrm{Space}(A)$ only.

THEOREM 3.4. *Let $[A, A]$ be a regular quasi-normed operator ideal. Then*

$$\mathrm{Space}(A^{\text{super}}) = \big(\mathrm{Space}(A)\big)^{\text{d-super}}.$$

Proof. Suppose $E \in \mathrm{Space}(A^{\text{super}})$ and let E_0 f.d.-r. E. Then I_{E_0} f.r. I_E and thus,

$$I_{E_0} \in A(E_0, E_0), \qquad E_0 \in \mathrm{Space}(A).$$

Conversely, let $E \in \big(\mathrm{Space}(A)\big)^{\text{d-super}}$ and let $T_0 \in L(E_0, F_0)$ be finitely representable in I_E. By Theorem 1.2 we have

$$K_{F_0} T_0 = Q (I_E)_U J.$$

Now $(E)_U$ f.d.-r. E (Proposition 3.3); hence

$$(E)_U \in \mathrm{Space}(A), \qquad (I_E)_U \in A\big((E)_U, (E)_U\big).$$

It follows that $T_0 \in A(E_0, F_0)$. This completes the proof.

Remark. The existence of a quasi-norm on A in Theorem 3.4 is, in fact, superfluous. But, since we dealt in Sections 1 and 2 with quasi-normed ideals only, we restricted our attention to this case.

Next we shall discuss the relations between different notions of finite representability of Banach spaces. Let us recall, for completeness, the usual definition of finite representability of Banach spaces:

DEFINITION 3.5. E_0 is *finitely representable in* E (E_0 f.r. E) if, for each $\varepsilon > 0$, each finite-dimensional subspace $M_0 \subset E_0$ is $(1+\varepsilon)$-isomorphic to a subspace $M \subset E$.

We write

$$\mathbf{A}^{\text{super}} = \{E\colon E_0 \text{ f.r. } E \text{ implies } E_0 \in \mathbf{A}\}.$$

In [23] Stern introduced a dual notion (with respect to the preceding one), which he called *local quotient*. Let us mention his definition (in an adapted terminology).

DEFINITION 3.6. E_0 is *finitely quotient-representable in* E (E_0 f.q.-r. E) if, for each $\varepsilon > 0$ and each $N_0 \in \mathrm{Cod}(E_0)$ there exists an $N \in \mathrm{Cod}(E)$ such that E_0/N_0 is $(1+\varepsilon)$-isombrphic to E/N.

Analogously we write

$$\mathbf{A}^{\text{q-super}} = \{E\colon E_0 \text{ f.q.-r. } E \text{ implies } E_0 \in \mathbf{A}\}.$$

It is obvious that E_0 f.d.-r. E implies E_0 f.r. E and E_0 f.q.-r. E. Other direct relations do not hold, as easy examples show. But, if we consider special types of space ideals (properties) then these notions yield the same super-space ideals (super-properties). Recall that a space ideal is said to be *injective* if it contains all subspaces of its members. A space ideal is *surjective* if it contains all quotients of its members (cf. [16, 17]).

THEOREM 3.7. *Let* $\mathbf{A}$ *be a space ideal.*

(i) *If* $\mathbf{A}$ *is injective, then* $\mathbf{A}^{\text{d-super}} = \mathbf{A}^{\text{super}}$.

(ii) *If* $\mathbf{A}$ *is surjective and satisfies, in addition, the following condition:* $E'' \in \mathbf{A}$ *implies* $E \in \mathbf{A}$, *then* $\mathbf{A}^{\text{d-super}} = \mathbf{A}^{\text{q-super}}$.

(iii) *If* $\mathbf{A}$ *is injective and surjective, then*

$$\mathbf{A}^{\text{d-super}} = \mathbf{A}^{\text{q-super}} = \mathbf{A}^{\text{super}}.$$

Proof. (i) Let $E \in \mathbf{A}^{\text{d-super}}$ and let E_0 f.r. E. Then E_0 embeds into $(E)_U$ for some ultrafilter U (cf. [24]). By Theorem 1.2 and Proposition 3.2, $(E)_U$ f.d.-r. E; hence, $(E)_U \in \mathbf{A}$. Now the injectivity of $\mathbf{A}$ yields $E_0 \in \mathbf{A}$. The proof of the converse is obvious.

(ii) Let $E \in \mathbf{A}^{\text{d-super}}$ and let E_0 f.q.-r. E. Then E_0' f.r. E' (cf. [23]). Hence there is an ultrafilter U such that $E_0' \subset (E')_U$. On the other hand, $(E')_U \subset (E)_U'$. Therefore, E_0'' may be identified with a quotient of $(E)_U''$. But $(E)_U''$ f.d.-r. $(E)_U$ and $(E)_U$ f.d.-r. E, thus, $(E)_U'' \in \mathbf{A}$. By surjectivity, $E_0'' \in \mathbf{A}$ and therefore $E_0 \in \mathbf{A}$. The converse is obvious, again.

(iii) Each injective and surjective ideal $\mathbf{A}$ satisfies the conditions of (i) and (ii). This concludes the proof.

Remark. Theorem 3.7 cannot be improved, in a certain sense. Namely, if we start with an arbitrary space ideal $\mathbf{A}$, then $\mathbf{A}^{\text{super}}$ is always injective, $\mathbf{A}^{\text{q-super}}$ is always surjective (and satisfies the additional hypothesis). This follows immediately from Definitions 3.5 and 3.6.

And in fact, the (usual) notion of finite representability was always applied to injective space ideals (i.e. injective properties) only. The notion of finite dual-representability allows us to consider also non-injective space ideals (cf. Section 5).

Theorem 3.7 yields some consequences for the corresponding types of operator ideals.

COROLLARY 3.8. *If* $[A, A]$ *is an injective operator ideal, then*

$$\operatorname{Space}(A^{\text{super}}) = \big(\operatorname{Space}(A)\big)^{\text{super}}.$$

If $[A, A]$ *is surjective and regular, then*

$$\operatorname{Space}(A^{\text{super}}) = \big(\operatorname{Space}(A)\big)^{\text{q-super}}.$$

At the end of this section we shall consider the converse question. Given a space ideal $\mathbf{A}$, one can define the operator ideal $\operatorname{Op}(\mathbf{A})$, which consists of all operators factoring through a member of **A** (see [16]). Recall that a space ideal is said to be *regular* if Op(**A**) is regular ([16]). In particular, every injective space ideal is regular. Furthermore, let us remark that Op(**A**) may not be quasi-normed, in general. In this case Definition 2.1 is reduced to its first part. We have now

THEOREM 3.9. *Let* **A** *be a regular space ideal. Then*

$$\mathrm{Op}(\mathbf{A}^{\text{d-super}}) \subset (\mathrm{Op}(\mathbf{A}))^{\text{super}}.$$

Furthermore, if $\mathbf{A}^{\text{d-super}}$ *is regular, then* $\mathrm{Op}(\mathbf{A}^{\text{d-super}})$ *is a super-ideal.*

Proof. Let $T \in L(E, F)$, $T \in \mathrm{Op}(\mathbf{A}^{\text{d-super}})$. Then there exist $G \in \mathbf{A}^{\text{d-super}}$ and operators $R\colon E \to G$, $S\colon G \to F$ with $T = SR$. If $T_0 \in L(E_0, F_0)$ is finitely representable in T, then

$$K_{F_0} T_0 = Q(R)_U (S)_U J,$$

i.e., $K_{F_0} T_0$ factors through $(G)_U \in \mathbf{A}^{\text{d-super}}$. By the assumption, it follows that $T_0 \in \mathrm{Op}(\mathbf{A})$, and, if $\mathbf{A}^{\text{d-super}}$ is regular, $T_0 \in \mathrm{Op}(\mathbf{A}^{\text{d-super}})$.

Finally, let us remark that equality does not hold, in general, as the example in [1], p. 122, shows. Put

$$\mathbf{A} = \{\text{reflexive spaces}\};$$

then

$$\mathrm{Op}(\mathbf{A}) = \{\text{weakly compact operators}\},$$

$$\mathrm{Op}(\mathbf{A})^{\text{super}} = \{\text{uniformly convexifying operators}\}$$

(cf. Section 5), and, by Theorem 3.7,

$$\mathbf{A}^{\text{d-super}} = \{\text{super-reflexive spaces}\}.$$

However, there are uniformly convexifying operators which do not factor through a super-reflexive space ([1]).

4. Procedures on super-ideals

At first, we need a lemma which reflects the self-dual character of finite representability of operators. It is a consequence of the principle of local reflexivity.

LEMMA 4.1. *If* T_0 *is finitely representable in* T, *then* T_0' *is finitely representable in* T'. *Furthermore,* T'' *is finitely representable in* T.

Proof. Let T_0 f.r. T. By Theorem 1.2,

$$K_{F_0} T_0 = Q(T)_U J;$$

thus,

$$T_0' K_{F_0}' = J'(T)_U' Q',$$

where J' is a metric surjection, Q' a metric injection. It is easily seen that T'_0 f.r. $T'_0 K'_{F_0}$ and $T'_0 K'_{F_0}$ f.r. $(T)'_U$. Lemma 1.6 yields that $(T)'_U$ is finitely representable in $(T')_U$. On the other hand, $(T')_U$ f.r. T'. This proves the first assertion.

Let now $\varepsilon > 0$, $M_0 \in \mathrm{Dim}(E'')$ and $N_0 \in \mathrm{Cod}(F'')$. By local reflexivity, there exist $M_1 \in \mathrm{Dim}(F')$ and a $(1+\varepsilon)$-isomorphism $W_1\colon N_0^0 \to M_1$ with

$$\langle T''x, z\rangle = \langle T''x, W_1 z\rangle \quad (z \in N_0^0,\ x \in M_0).$$

Again by local reflexivity, there is an $M \in \mathrm{Dim}(E)$ and a $(1+\varepsilon)$-isomorphism $V\colon M_0 \to M$ such that

$$\langle x, T'W_1 z\rangle = \langle Vx, T'W_1 z\rangle \quad (z \in N_0^0,\ x \in M_0).$$

Thus,

$$\langle W_1' TVx, z\rangle = \langle T''x, z\rangle$$

and, by setting $N = M_1^0 \cap F$, $W = W_1'\colon F/N \to F''/N_0$, we get

$$WQ_N TJ_M V - Q_{N_0} T'' J_{M_0} = 0.$$

This proves the lemma.

We pass now to the central result of this section. Let us mention before that $[A, A]^{\mathrm{inj\,(sur,\,max,\,reg)}}$ denotes the injective (resp., surjective, maximal, regular) hull of the quasi-normed operator ideal $[A, A]$. Furthermore, $[A, A]^{\mathrm{dual}}$ denotes the dual ideal and $[A, A]^{\mathrm{min}}$ the minimal kernel. All definitions can be found in [16], [17].

THEOREM 4.2. *Let $[A, A]$ be a regular quasi-normed operator ideal. Then*

(i) $([A, A]^{\mathrm{super}})^{\mathrm{dual}} = ([A, A]^{\mathrm{dual}})^{\mathrm{super}}$. *If $[A, A]$ is completely symmetric, then $[A, A]^{\mathrm{super}}$ is completely symmetric, as well.*

(ii) $([A, A]^{\mathrm{super}})^{\mathrm{inj}} = ([A, A]^{\mathrm{inj}})^{\mathrm{super}}$. *If $[A, A]$ is injective, then $[A, A]^{\mathrm{super}}$ is injective, as well.*

(iii) $([A, A]^{\mathrm{super}})^{\mathrm{sur}} \subset ([A, A]^{\mathrm{sur}})^{\mathrm{super}}$. *If $[A, A]$ is surjective, then $[A, A]^{\mathrm{super}}$ is surjective, as well. If $T \in A(E, F)$ implies $T'' \in A(E'', F'')$ and $A(T) = A(T'')$, then equality holds.*

(iv) *If $[A, A]$ is ultrastable, then $[A, A]^{\mathrm{super}} = [A, A]$.*

(v) $([A, A]^{\mathrm{super}})^{\mathrm{max}} = ([A, A]^{\mathrm{max}})^{\mathrm{super}} = [A, A]^{\mathrm{max}}$, *provided $[A, A]$ is p-normed.*

(vi) $\big(([A, A]^{\mathrm{min}})^{\mathrm{reg}}\big)^{\mathrm{super}} = ([A, A]^{\mathrm{min}})^{\mathrm{reg}}$, *provided $[A, A]$ is p-normed.*

Proof. We shall verify only the set-theoretical relations between the ideals, the relations between the quasi-norms can be obtained analogously.

(i) Suppose that $T \in (A^{\mathrm{super}})^{\mathrm{dual}}(E, F)$ and $T_0 \in L(E_0, F_0)$ f.r. T. Then $T' \in A^{\mathrm{super}}(F', E')$. It follows from Lemma 4.1 that $T'_0 \in A(F', E')$; hence $T_0 \in A^{\mathrm{dual}}(E_0, F_0)$ and $T \in (A^{\mathrm{dual}})^{\mathrm{super}}(E, F)$.

Conversely, let $T \in (A^{\text{dual}})^{\text{super}}(E, F)$ and suppose that T_0 is finitely representable in T'. By Lemma 4.1, T_0' f.r. T'', thus T_0' f.r. T, and we get $T_0' \in A^{\text{dual}}(F_0', E_0')$, $T_0'' \in A(E_0'', F_0'')$. Regularity implies $T_0 \in A(E_0, F_0)$, thus $T' \in A^{\text{super}}(F', E')$. The second part of the assertion is obvious.

(ii) Let $T \in (A^{\text{super}})^{\text{inj}}(E, F)$ and let $J\colon F \to l_\infty(\Gamma)$ be a metric injection (e.g. $\Gamma = B_{F'}$). Then we have

$$JT \in A^{\text{super}}\big(E, l_\infty(\Gamma)\big)$$

and for each ultrafilter U,

$$(JT)_U \in A\big((E)_U, (l_\infty(\Gamma))_U\big).$$

This implies

$$(T)_U \in A^{\text{inj}}\big((E)_U, (F)_U\big);$$

hence,

$$T \in (A^{\text{inj}})^{\text{super}}(E, F).$$

Conversely, let $T \in (A^{\text{inj}})^{\text{super}}(E, F)$ and let $J\colon F \to l_\infty(\Gamma)$ be as above. For each ultrafilter U we have

$$(JT)_U \in A^{\text{inj}}\big((E)_U, (l_\infty(\Gamma))_U\big).$$

By Gelfand's representation theorem, $\big(l_\infty(\Gamma)\big)_U$ is a $C(K)$ space. $C(K)''$ is isometric to a space $L_\infty(\Omega, \mu)$, which has the metric extension property (cf. [22], p. 452). Therefore,

$$K_{C(K)}(J)_U(T)_U \in A\big((E)_U, C(K)''\big);$$

thus, by regularity,

$$(J)_U(T)_U \in A\big((E)_U, C(K)\big)$$

and

$$JT \in A^{\text{super}}\big(E, l_\infty(\Gamma)\big).$$

We conclude that $T \in (A^{\text{super}})^{\text{inj}}(E, F)$.

(iii) Suppose that $T \in (A^{\text{super}})^{\text{sur}}(E, F)$. Let $\Gamma = B_E$, and let $Q\colon l_1(\Gamma) \to E$ be the canonical surjection. Then

$$TQ \in A^{\text{super}}\big(l_1(\Gamma), F\big)$$

and

$$(T)_U(Q)_U \in A\Big(\big(l_1(\Gamma)\big)_U, (F)_U\Big).$$

Consequently,

$$(T)_U \in A^{\text{sur}}\big((E)_U, (F)_U\big).$$

This shows the inclusion. Now suppose that $[A, A]$ satisfies the additional assumption. Then we have

$$([A, A]^{\text{dual}})^{\text{dual}} = [A, A].$$

The assertion follows from (i), (ii) and the identity

$$([B, B]^{\text{dual}})^{\text{sur}} = ([B, B]^{\text{inj}})^{\text{dual}}$$

for $[B, B] = [A, A]^{\text{dual}}$ (cf. [16], 8.5.9 or [17], 3.9.2).

(iv) This follows immediately from the definition of ultrastability and Theorem 1.2.

(v) Let $T \in (A^{\max})^{\text{super}}(E, F)$ and let $S \in G(E_1, E)$, $R \in G(F, F_1)$, where G is the ideal of approximable operators. Then

$$(S)_U \in G\big((E_1)_U, (E)_U\big), \qquad (R)_U \in G\big((F)_U, (F_1)_U\big)$$

for each ultrafilter U. This follows from the fact that the ultraproduct of operators of the same finite rank has finite rank, itself. By hypothesis,

$$(T)_U \in A^{\max}\big((E)_U, (F)_U\big),$$

thus,

$$(R)_U (T)_U (S)_U \in A\big((E_1)_U, (F_1)_U\big)$$

and therefore, $RTS \in A^{\text{super}}(E_1, F_1)$. This yields

$$T \in (A^{\text{super}})^{\max}(E, F).$$

We have shown the inclusion

$$(A^{\max})^{\text{super}} \subset (A^{\text{super}})^{\max} \subset A^{\max}.$$

Since every maximal p-normed ideal is ultrastable (cf. [10], [11]), we have

$$(A^{\max})^{\text{super}} = A^{\max}.$$

(vi) Suppose that $T \in (A^{\min})^{\text{reg}}(E, F)$. We have $A^{\min} = (A^{\max})^{\min}$ (cf. [16], 8.7.15). Consequently, there are operators $S \in G(E, E_1)$, $T_1 \in A^{\max}(E_1, F_1)$ and $R \in G(F_1, F'')$ such that

$$K_F T = R T_1 S.$$

We get, for each ultrafilter U,

$$(K_F)_U (T)_U = (R)_U (T_1)_U (S)_U.$$

By (v),

$$(T_1)_U \in A^{\max}\big((E_1)_U, (F_1)_U\big).$$

Moreover,

$$(S)_U \in G\big((E)_U, (E_1)_U\big), \qquad (R)_U \in G\big((F_1)_U, (F'')_U\big).$$

Thus,

$$(K_F T)_U \in (A^{\max})^{\min}\big((E)_U, (F'')_U\big) \subset (A^{\min})^{\text{reg}}\big((E)_U, (F'')_U\big).$$

This implies

$$K_F T \in \big((A^{\min})^{\text{reg}}\big)^{\text{super}}$$

and we get, by the regularity of super-ideals (Theorem 2.2),

$$T \in \big((A^{\min})^{\text{reg}}\big)^{\text{super}}.$$

This completes the proof of Theorem 4.2.

Remark. Simple examples show that the inclusion of (iii) may be strict.

Finally, let us make some few comments on products and quotients of super-ideals (see [16] for the definitions). We omit the proof of the following theorem since it uses exactly the same methods as the proof of 4.2.

THEOREM 4.3. *Let* $[A, A]$ *and* $[B, B]$ *be quasi-normed regular operator ideals. Then*

(i) $[A, A]^{\text{super}} \circ [B, B]^{\text{super}} \subset ([A, A] \circ [B, B])^{\text{super}}$ *and* $[A, A]^{\text{super}} \circ \circ [B, B]^{\text{super}}$ *is a super-ideal.*

(ii) $([A, A]^{-1} \circ [B, B])^{\text{super}} \subset ([A, A]^{\text{super}})^{-1} \circ [B, B]^{\text{super}}$.

In neither cases equality holds, in general. An example for (ii) is easily constructed. Let us mention how to proceed in case (i). Put $A = X$ — the ideal of separable operators, and $B = L_2$ the ideal of 2-factorable operators (cf. [16]). L_2 is a super-ideal, $X^{\text{super}} = K$, where K denotes the ideal of compact operators (cf. Section 5). Now, setting

$$T_n = 1/\sqrt{n}\,\Pi_1^{(n)}, \qquad T = \left(\oplus \sum T_n\right)_{l_2},$$

we obtain

$$T \in L_2 \cap K, \qquad T \notin K \circ L_2.$$

Each operator T_0 which is finitely representable in T also belongs to $L_2 \cap K$. It follows from the compactness and the injectivity of the ideal L_2 that T_0 can be written as the product of a separable operator and a 2-factorable operator. Thus,

$$T \in (X \circ L_2)^{\text{super}}.$$

This shows that equality in (i) does not hold, in general.

5. Examples and applications

The definitions of the operator ideals of this section can be found in [16], [17]. Let us remark that all operator ideals considered up to Theorem 5.8 are normed operator ideals with respect to the usual operator norm. Therefore we omit the second component in the notation of these ideals.

Beauzamy [1] introduced the uniformly convexifying operators. Recall that an operator $T \in L(E, F)$ is said to be *uniformly convexifying* if there exists an equivalent norm $|\cdot|$ on E satisfying the following condition: For each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|x| \leqslant 1, \qquad |y| \leqslant 1, \qquad |(x+y)/2| \geqslant 1-\delta$$

imply

$$\|Tx - Ty\| \leqslant \varepsilon.$$

Let $\mathfrak{U}$ be the class of uniformly convexifying operators. Beauzamy proved that an operator $T \in L(E, F)$ is uniformly convexifying if and only if $T(B_E)$ does not possess the finite tree property. We now give further characterizations of $\mathfrak{U}$. For this purpose, let W be the ideal of weakly compact operators, Y the ideal of Radon-Nikodym operators, i.e. those operators which map every μ-continuous E-valued measure of finite variation into a μ-differentiable F-valued measure ([16]), Q the ideal of decomposable operators and R_∞ the ideal of ∞-radonifying operators.

THEOREM 5.1. $\mathfrak{U} = W^{\text{super}} = Y^{\text{super}} = Q^{\text{super}} = R_\infty^{\text{super}}$.

Proof. Every uniformly convexifying operator is weakly compact [1]. Using Theorem 1.2, it is easily verified that $\mathfrak{U}$ is a super-ideal. Consequently, we have the following inclusions:

$$\mathfrak{U} \subset W^{\text{super}} \subset Y^{\text{super}}.$$

Assume that $T \in Y^{\text{super}}(E, F) \setminus \mathfrak{U}(E, F)$. The method applied here is similar to the proof of the corresponding statement for Banach spaces (see [19], [25]). Therefore we do not give all details. By [1], $T(B_E)$ possesses the finite tree property, i.e. there is an $\varepsilon > 0$ and, for each $n \in N$, a system of elements $y^{(n)}_{\varepsilon_1 \ldots \varepsilon_k} \in T(B_E)$ $(1 \leqslant k \leqslant n,\ \varepsilon_i = \pm 1)$ with

$$y^{(n)}_{\varepsilon_1 \ldots \varepsilon_k} = \frac{y^{(n)}_{\varepsilon_1 \ldots \varepsilon_k +1} + y^{(n)}_{\varepsilon_1 \ldots \varepsilon_k -1}}{2}$$

and

$$\|y^{(n)}_{\varepsilon_1 \ldots \varepsilon_k +1} - y^{(n)}_{\varepsilon_1 \ldots \varepsilon_k -1}\| \geqslant \varepsilon.$$

Choose $x^{(n)}_{\varepsilon_1 \ldots \varepsilon_n} \in B_E$ such that $Tx^{(n)}_{\varepsilon_1 \ldots \varepsilon_n} = y^{(n)}_{\varepsilon_1 \ldots \varepsilon_n}$ and define successively, for $k < n$,

$$x^{(n)}_{\varepsilon_1 \ldots \varepsilon_k} = \frac{x^{(n)}_{\varepsilon_1 \ldots \varepsilon_k +1} + x^{(n)}_{\varepsilon_1 \ldots \varepsilon_k -1}}{2}.$$

We get $Tx^{(n)}_{\varepsilon_1 \ldots \varepsilon_k} = y^{(n)}_{\varepsilon_1 \ldots \varepsilon_k}$. Now let U be a non-trivial ultrafilter on the set of natural numbers N. Put

$$\left.\begin{aligned} x_{\varepsilon_1 \ldots \varepsilon_k} &= (x^{(i)}_{\varepsilon_1 \ldots \varepsilon_k})_U \\ y_{\varepsilon_1 \ldots \varepsilon_k} &= (y^{(i)}_{\varepsilon_1 \ldots \varepsilon_k})_U \end{aligned}\right\} \quad (k = 1, 2, \ldots,\ \varepsilon_j = \pm 1).$$

Using the methods of [25], one can construct from these systems an $(E)_U$, resp., $(F)_U$-valued uniformly bounded martingale $\{x_n(\omega)\}$, resp., $\{y_n(\omega)\}$ on a certain probability space (Ω, Σ, μ) with the following properties:

$$(T)_U x_n(\omega) = y_n(\omega) \quad (\mu\text{-a.e.})$$

and $\{y_n(\omega)\}$ is not convergent in the L_1-norm. Now, $(T)_U$ is finitely representable in T, hence a Radon–Nikodym operator. Such an operator maps uniformly bounded martingales into L_1-convergent martingales. This can easily be shown, using the methods of [6], p. 271. Thus, we have got a contradiction, from which we deduce

$$\mathfrak{U} = W^{\text{super}} = Y^{\text{super}}.$$

The remaining two identities follow from $R_\infty = W$, $Q^{\text{dual}} = Y$ (cf. [16], ch. 24, 25) and Theorem 4.2.

COROLLARY 5.2 ([1]). *$\mathfrak{U}$ is an injective, surjective, completely symmetric ideal.*

Proof. This follows from Theorems 2.2, 4.2 and the corresponding properties of W.

COROLLARY 5.3.

$$\begin{aligned}\text{Space}(\mathfrak{U}) &= \{\text{super-reflexive spaces}\}\\ &= \{\text{reflexive spaces}\}^{\text{d-super(s-super)}}.\end{aligned}$$

Now, let U denote the ideal of unconditional summing operators, i.e. operators that map every weakly unconditional convergent series into an unconditionally convergent series. Let $\mathscr{C}$ be the set of operators of cotype Rademacher. An operator $T \in L(E, F)$ is said to be of *cotype Rademacher* if it satisfies

$$\inf_{\substack{x_1,\dots,x_n \in E\\ \|Tx_k\| \geqslant 1}} \int \Big\| \sum_{k=1}^{n} \varepsilon_k(t) x_k \Big\| \, dt \to \infty \qquad (n \to \infty)$$

where $\varepsilon_k(t)$ denotes the kth Rademacher function on $[0, 1]$. These operators were introduced by Beauzamy [3].

THEOREM 5.4. $U^{\text{super}} = \mathscr{C}$.

Proof. We use the following result of Beauzamy [3]: $T \notin \mathscr{C}(E, F)$ iff there is a $\theta > 0$ such that, for each $\varepsilon > 0$ and each n, there exist $\{x_j^{(n)}\}_{j=1}^{n} \subset E$ with

$$(1-\varepsilon)\sup|a_j| \leqslant \Big\|\sum_{j=1}^{n} a_j x_j^{(n)}\Big\| \leqslant \sup|a_j|,$$

$$(1-\varepsilon)\,\theta\sup|a_j| \leqslant \Big\|\sum_{j=1}^{n} a_j T x_j^{(n)}\Big\| \leqslant \theta\sup|a_j|$$

for all scalar sequences $\{a_j\}_{j=1}^{n}$. Now assume that $T \in U^{\text{super}}(E, F)\setminus\mathscr{C}(E, F)$. It is easily seen that there is an ultrafilter U and an operator $S \in L(c_0, (E)_U)$ such that $(T)_U S$ is an isomorphic embedding. Hence

$$(T)_U \notin U\big((E)_U, (F)_U\big)$$

(cf. [16]). This is a contradiction, and we conclude that $U^{\text{super}} \subset \mathscr{C}$.

To check the converse inclusion, let $T \in \mathscr{C}(E, F)$ and let U be any ultrafilter. Assume

$$(T)_U \notin U\big((E)_U, (F)_U\big).$$

Then there exists an operator $S \in L(c_0, (E)_U)$ such that $(T)_U S$ is an isomorphic embedding (cf. [16]). Let $\{e_k\}$ be the canonical basis of c_0. We put $x_k = Se_k$. Then

$$\int \Big\| \sum_{k=1}^{n} \varepsilon_k(t) x_k \Big\| dt \leqslant C_1, \qquad \|(T)_U x_k\| \geqslant C_2 \qquad (k = 1, \ldots, n)$$

for a suitable choice of $C_1 > 0$ and $C_2 > 0$ and for all n. Consequently, $(T)_U \notin \mathscr{C}\big((E)_U, (F)_U\big)$. From this one can easily deduce that $T \notin \mathscr{C}(E, F)$. This contradiction concludes the proof.

COROLLARY 5.5. *$\mathscr{C}$ is an injective ideal and*

$$\begin{aligned} \mathrm{Space}(\mathscr{C}) &= \{\text{spaces of cotype} < \infty\} \\ &= \{\text{spaces which do not contain } c_0\}^{\text{d-super}}. \end{aligned}$$

Proof. Use Theorems 3.4 and 4.2.

A characterization of operators of type Rademacher can be obtained in a similar way. Let us recall from [3] that an operator T is said to be of *type Rademacher* if it has the following property:

$$\sup_{\substack{x_1, \ldots, x_n \in E \\ \|x_k\| \leqslant 1}} \frac{1}{n} \int \Big\| \sum_{k=1}^{n} \varepsilon_k(t) T x_k \Big\| dt \to 0 \qquad (n \to \infty).$$

These operators satisfy a certain law of large numbers (cf. [3]). Denote the set of all operators of type Rademacher by $\mathscr{T}$. Furthermore, let Z be the ideal of all operators $T \in L(E, F)$ that fulfil the following condition: Each sequence $y_n \in T(B_E)$ has a weak-Cauchy subsequence.

THEOREM 5.6. $\mathscr{T} = Z^{\text{super}}$.

Proof. We use again a result of Beauzamy [3]: $T \notin \mathscr{T}(E, F)$ iff there is a $\theta > 0$ such that, for each $\varepsilon > 0$ and each n, there exist $\{x_j^{(n)}\}_{j=1}^{n} \subset E$ with

$$(1-\varepsilon) \sum_{j=1}^{n} |a_j| \leqslant \Big\| \sum_{j=1}^{n} a_j x_j^{(n)} \Big\| \leqslant \sum_{j=1}^{n} |a_j|,$$

$$(1-\varepsilon)\theta \sum_{j=1}^{n} |a_j| \leqslant \Big\| \sum_{j=1}^{n} a_j T x_j^{(n)} \Big\| \leqslant \theta \sum_{j=1}^{n} |a_j|$$

for all scalar sequences $(a_j)_{j=1}^{n}$. It follows from a well-known theorem of Rosenthal [21] that $T \in Z(E, F)$ iff the product TS is not an injection for any $S \in L(l_1, E)$. Now we can proceed as in the proof of Theorem 5.4.

COROLLARY 5.7. *$\mathscr{T}$ is an injective, surjective, completely symmetric ideal and*

$$\begin{aligned} \mathrm{Space}(\mathscr{T}) &= \{B\text{-convex spaces}\} \\ &= \{\text{spaces which do not contain } l_1\}^{\text{d-super}}. \end{aligned}$$

We now determine the super-ideal of the ideal of separable operators X. K denotes the ideal of compact operators.

THEOREM 5.8. $X^{\text{super}} = K$.

Proof. It follows easily from Theorem 1.2 that K is a super-ideal. Consequently, $K \subset X^{\text{super}}$. Assume that $T \in X^{\text{super}}(E, F)\setminus K(E, F)$. Then there exist an $\varepsilon_0 > 0$ and a sequence $\{y_n\}_{n=1}^{\infty} \subset T(B_E)$ with

$$\|y_n - y_m\| \geqslant \varepsilon_0 \quad (m \neq n).$$

Let $x_n \in B_E$, $Tx_n = y_n$. Next choose a non-trivial ultrafilter on the set of natural numbers N. For $r \in R^+$ (positive real numbers), we define

$$n(i, r) = [ir] \quad (i \in N),$$

where $[ir]$ denotes the greatest integer not exceeding ir. Define a map φ: $R^+ \to (F)_U$ by setting

$$\varphi(r) = (y_{n(i,r)})_U.$$

Now, $r \neq s$ implies $n(i, r) \neq n(i, s)$ for $i > i_0$, and therefore

$$\|y_{n(i,r)} - y_{n(i,s)}\| \geqslant \varepsilon_0 \quad (i > i_0);$$

hence

$$\|\varphi(r) - \varphi(s)\| \geqslant \varepsilon_0.$$

Furthermore,

$$\varphi(r) = (y_{n(i,r)})_U = (Tx_{n(i,r)})_U \in (T)_U B_{(E)_U}.$$

This yields that $(T)_U$ is not separable. We have got a contradiction which proves the theorem.

Let us mention that we do not know any direct characterization of V^{super}, where V is the ideal of completely continuous operators.

Now we give an application of the results of Section 3. Pełczyński and Rosenthal [14] introduced the *uniform approximation property* (u.a.p.). A Banach space possesses the λ-u.a.p. iff the following holds: For each $n \in N$ there is a $k(n)$ with the following property: If $M \subset E$, $\dim M = n$, then there exists an operator $T \in L(E, E)$ of rank $\leqslant k(n)$ with $T|_M =$ identity and $\|T\| \leqslant \lambda$. A Banach space has the u.a.p. if it has the λ-u.a.p. for a certain λ. It turns out that u.a.p. is the dual-super-property of the bounded approximation property (b.a.p.). Recall that E has the λ-b.a.p. if there exists a net T_γ $(\gamma \in \Gamma)$ of finite rank operators, which is uniformly bounded by λ and converges pointwise to the identity of E.

THEOREM 5.9. *A Banach space E has the* u.a.p. *iff each Banach space which is finitely dual-representable in E has the* b.a.p.

Proof. Suppose that E has the u.a.p. If E_0 is finitely dual-representable in E, then there exists an ultrafilter U such that E_0'' is a complemented subspace of $(E)_U$ (Proposition 3.3). By a result of Lindenstrauss and

Tzafriri [13], it follows that $(E)_U$ has the u.a.p., too. Therefore, E_0'' and, consequently, E_0 has the u.a.p. (cf. [13]), hence the b.a.p.

Conversely, suppose E_0 f.d.-r. E implies that E_0 has the b.a.p. Then, in particular, $(E)_U$ has the b.a.p., where U is a non-trivial ultrafilter on N. Put $\lambda = \lambda_0 + \varepsilon_0$, $\varepsilon_0 > 0$, where λ_0 is the b.a.p.-constant of $(E)_U$. We shall show that E has the λ-u.a.p. Assume the contrary. Then there exists an n and a sequence of subspaces $M_i \subset E$ $(i \geqslant n)$, $\dim M_i = n$ with the following property: For each $T \in L(E, E)$,

$$T|_{M_i} = \text{identity}, \qquad \|T\| \leqslant \lambda$$

imply

$$\dim \operatorname{Im}(T) \geqslant i.$$

Put $(M_i)_U = M$. If follows from the hypothesis and from [9] that there exists a finite rank operator $T_0 \in L\big((E)_U, (E)_U\big)$ with

$$T_0|_M = \text{identity} \quad \text{and} \quad \|T_0\| \leqslant \lambda_0 + \varepsilon_0/4.$$

Let $M_1 = \operatorname{Im}(T_0)$ and let $\{x_j\}_{j=1}^m$ be a basis of M_1. Since $M \subset M_1$, we may assume that $\{x_j\}_{j=1}^n$ is a basis of M. Then there are functionals $f_j \in (E)_U'$ $(j = 1, \ldots, m)$ such that T_0 has the representation

$$T_0 = \sum_{j=1}^m f_j \otimes x_j.$$

Since

$$\|T_0\| = \|T_0'\| = \sup_{\substack{y' \in (E)_U' \\ \|y'\| = 1}} \Big\| \sum_{j=1}^m \langle x_j, y' \rangle f_j \Big\|,$$

it follows easily from Lemma 1.3 that there is an operator

$$T = \sum_{j=1}^m g_j \otimes x_j$$

with

$$g_j \in (E')_U, \qquad \|T\| \leqslant \lambda_0 + \varepsilon_0/2, \qquad T|_M = \text{identity}.$$

Let $x_j = (x_{ji})_U$ $(x_{ji} \in M_i)$, $g_j = (g_{ji})_U$,

$$T_i = \sum_{j=1}^m g_{ji} \otimes x_{ji}.$$

Then $T_i \in L(E, E)$ and, for each $\delta > 0$, there is a $D_0 \in U$ such that

$$\|T_i\| \leqslant \lambda_0 + \tfrac{3}{4}\varepsilon_0, \qquad \|T_i x_{ji} - x_{ji}\| \leqslant \delta \qquad (j = 1, \ldots, n)$$

for $i \in D_0$. By a suitable choice of δ and by a "small correction" of T_i on x_{ji} $(j = 1, \ldots, n)$, say $\tilde{T}_i$, we can achieve

$$\tilde{T}_i x_{ji} = x_{ji}, \qquad \dim \operatorname{Im}(\tilde{T}_i) \leqslant m, \qquad \|\tilde{T}_i\| \leqslant \lambda_0 + \varepsilon_0 = \lambda$$

for $j = 1, \ldots, n$ and $i \in D_0$. Since $D_0 \cap \{i > m\} \neq \emptyset$, we get a contradiction, which concludes the proof.

The following corollary answers a question of Lindenstrauss and Tzafriri [13].

COROLLARY 5.10. *The* u.a.p. *is a self-dual property, i.e. E has the* u.a.p. *iff E' has the* u.a.p.

Proof. It is known that E possesses the u.a.p. iff E'' does (cf. [13]). Thus, we need to show only that the u.a.p. is carried across from E to E'. By Theorem 5.9 and Proposition 3.3, it suffices to prove that $(E')_U$ has the b.a.p. for each ultrafilter U. Since $(E')_U$ is finitely dual-representable in E', we get from Lemma 4.1 $(E')'_U$ f.d.-r. E''. E'' possesses the u.a.p., hence $(E')'_U$ has u.a.p., too. We conclude that $(E')_U$ has the b.a.p. This follows from a result of Grothendieck ([8], p. 180). There it is shown that the metric approximation property is carried across to preduals, but the corresponding statement for the b.a.p. can be proved in the same way.

The following and final theorem shows that the "finitely" defined operator ideals are, in general, super-ideals. It should be mentioned that this class is essentially larger than the class of ultrastable ideals.

THEOREM 5.11. *The following ideals are super-ideals:*

(i) $[I_{(r,p,q)}, I_{(r,p,q)}]$ — *(r, p, q)-integral operators* ($0 < r \leqslant \infty$, $1 \leqslant p, q \leqslant \infty$, $1+1/r \geqslant 1/p+1/q$);

(ii) $[P_{(r,p,q)}, P_{(r,p,q)}]$ — *absolutely (r, p, q)-summing operators* ($0 < r$, $p, q \leqslant \infty$, $1/r \leqslant 1/p+1/q$);

(iii) $[L_{(p,q)}, L_{(p,q)}]$ — *(p, q)-factorable operators* ($1 \leqslant p, q \leqslant \infty$, $1/p+1/q \geqslant 1$);

(iv) $[M_{(s,p)}, M_{(s,p)}]$ — *(s, p)-mixing operators* ($0 < p \leqslant s \leqslant \infty$);

(v) $[T_{(s,p)}, T_{(s,p)}]$ — *operators of (s, p)-type* ($0 < s \leqslant 2$, $0 < p < s^+$);

(vi) $[C_{(s,p)}, C_{(s,p)}]$ — *operators of (s, p)-cotype* ($0 < s \leqslant 2$, $0 < p < s^+$);

(vii) $[N_{(r,p,q)}, N_{(r,p,q)}]^{\mathrm{reg}}$ — *the regular hull of (r, p, q)-nuclear operators* ($0 < r \leqslant \infty$, $1 \leqslant p$, $q \leqslant \infty$, $1+1/r \geqslant 1/p+1q$);

(viii) $[N_{(r,2,q)}, N_{(r,2,q)}]$ — ($0 < r \leqslant \infty$, $1/2+1/r \geqslant 1/q$);

(ix) $[N_p^Q, N_p^Q]$ — *quasi-p-nuclear operators* ($1 \leqslant p \leqslant \infty$);

(x) R_p — *p-radonifying operators* ($1 < p < \infty$);

(xi) $S_a^{(s)}$ — *operators of s-class, where s is a regular ultrastable additive s-function and a is a sequence ideal*;

(xii) $[S_p^{(s)}, S_p^{(s)}]$ — *S_p-operators of s-class, where $1 < p < \infty$, and $s = a$ (approximation numbers), $s = c$ (Kolmogorov numbers), $s = d$ (Gelfand numbers) or $s = h$ (Hilbert numbers).*

Proof. (i)–(vi), (xii) follow from Theorem 4.2 (v) and the fact that all these ideals are maximal and p-normed. (vii) is a consequence of Theorem 4.2 (vi). From this result we get also (viii) since $'N_{(r,2,q)}$ is regular (cf. [16]).

(ix) We have $[N_p^Q, N_p^Q] = [N_p, N_p]^{\text{inj}} = [N_{(p,p,1)}, N_{(p,p,1)}]^{\text{inj}}$ (cf. [15], [16]). Thus, the result is obtained, using (vii) and Theorem 4.2.(ii).

(x) In this case $R_p = P_p$ ([16]); hence (ii) yields the desired result.

(xi) This follows from the fact that

$$s\big((T)_U\big) \leqslant \lim_U s(T) = s(T)$$

for each ultrafilter U. This completes the proof of the theorem.

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