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ROSENTHAL SETS THAT CANNOT BE SUP-NORM PARTITIONED AND AN APPLICATION TO TENSOR PRODUCTS

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In what follows, Γ is a countable discrete abelian group, and $\hat{\Gamma} = G$ We assume some familiarity with Sidon sets, and refer to [5] for basic notation and facts.

Non-Sidon sets $E \subset \boldsymbol{Z}$ such that

$$L_E^\infty(\boldsymbol{T}) = C_E(\boldsymbol{T})$$

(where the subscript E means that $\operatorname{spec}(f) \subset E$) were first constructed in [4]. We refer to these sets as Rosenthal sets.

Definition. A family of finite and mutually disjoint sets $\{F_j\}$ is a *sup-norm partition* for $E \subset \Gamma$ if

$$\bigcup_j F_j = E \quad \text{and} \quad \bigoplus_{l^1} C_{F_j}(G) \approx C_E(G)$$

(E is said to be *sup-norm partitioned*). The strategy in [4] was to construct a non-Sidon sup-norm partitioned set and to note that if E is such a set, then $L_E^\infty = C_E$. In this note we display Rosenthal sets in (any) Γ that cannot be sup-norm partitioned, and use our observations to give a proof of "the dual of $c_0 \hat{\otimes} c_0$ is $l^1 \check{\otimes} l^1$" (see [4]).

First, we recall some familiar machinery (see [1]). Let D be a dense countable subgroup of G, and let $\varphi_D = \varphi\colon \Gamma \to \hat{D}$ be the canonical injection, i.e., $\big(\varphi(\gamma), d\big) = (\gamma, d)$ for all $\gamma \in \Gamma$ and $d \in D$. In [1] we proved (see remark b following Theorem B in [1])

THEOREM. *Let $E \subset \Gamma$ be such that $\overline{\varphi(E)}$ is countable and such that $\varphi(E)$ contains no limit points of $\varphi(E)$. Then E is a Rosenthal set.*

Example. *Fix (any) $D \subset G$, and let $E = \{\lambda_i\}$ and $S = \{\nu_i\}$ in Γ be such that*

(i) *the only limit point of $\varphi(E)$ and $\varphi(S)$ is 0,*

(ii) $(E+S)\cap(E\cup S) = \emptyset$, *and*

(iii) $(\lambda_i+S)\cap(\lambda_j+S) = \emptyset$ *for all* $i \neq j$.

Then $E+S$ is a Rosenthal set that cannot be sup-norm partitioned.

Verification. Certainly, the set of limit points of $\varphi(E+S)$ is $E\cup S\cup\{0\}$ and, therefore, by the Theorem, $L^{\infty}_{E+S}(G) = C_{E+S}(G)$.

We now suppose that $E+S$ is sup-norm partitioned and, without loss of generality, we assume that S is a Sidon set.

Let $S_j = \lambda_j+\{\nu_i\}_{i<j}$, and suppose that $\{F_k\}$ is a sup-norm partition for

$$\bigcup_{j=1}^{\infty} S_j \subset S+E.$$

Fix an arbitrary S_{j_0}, and proceed by induction to find, for $n>0$, S_{j_n} such that

$$S_{j_n}\cap\Big(\bigcup_{i<n}\ \bigcup_{F_k\cap S_{j_i}\neq\emptyset} F_k\Big) = \emptyset \quad \text{and} \quad j_n > j_{n-1}.$$

It is clear that $\{S_{j_n}\}$ is a sup-norm partition for $\bigcup_{n=1}^{\infty} S_{j_n}$. Furthermore, since S is a Sidon set, the Sidon constants of S_{j_n} are uniformly bounded for all n. Therefore, $\bigcup_n S_{j_n}$ is a Sidon set. But this is impossible, for $\bigcup_n S_{j_n}$ contains arbitrarily large "squares"

$$\{\lambda_{j_n}+\nu_i\}_{\substack{n=K,\ldots,2K\\ i=1,\ldots,K}},$$

where $K>0$ is arbitrary (see Theorem 3.5 in [2], for example), and we reach a contradiction.

A concrete example can be produced as follows. Let $\Gamma = \boldsymbol{Z}$, let $p_1, p_2 > 2$ be two primes, and let $E = \{p_1^j p_2^i\}_{\substack{j=1\\ i\leqslant j}}^{\infty}$, and $S = \{p_1^j\}_{j=1}^{\infty}$. Let $D = \boldsymbol{Z}(p_1^{\infty})$, and identify it canonically with a dense subgroup of $\boldsymbol{T}$, i.e.,

$$\boldsymbol{Z}_{p_1^n} \ni a \leftrightarrow \frac{2\pi a}{p_1^n} \in \boldsymbol{T}.$$

It is not difficult to see that, under $\varphi\colon \boldsymbol{Z}\to\hat{D}$ (the group of p_1-adic integers), $E\cup S$ accumulates precisely at 0, and that

$$(E+S)\cap(E\cup S) = \emptyset.$$

We note that E, of course, is not a Sidon set (its asymptotic density is $(\log N)^2/N$).

Application to tensor products. We recall that, for $\psi\in c_0\times c_0$, we let (the projective tensor norm)

$$\|\psi\|_{\hat{\otimes}} = \inf\Big\{\sum \|f_i\|_\infty\|g_i\|_\infty\colon f_i, g_i\in c_0 \text{ and } \psi = \sum f_i g_i\Big\}.$$

We set

$$c_0\hat{\otimes} c_0 = \{\psi\in c_0\times c_0\colon \|\psi\|_{\hat{\otimes}} < \infty\}.$$

For $(a_{ij})_{i,j=1}^{N}$, a finite array of complex scalars, we let (the injective tensor norm)

$$\|(a_{ij})\|_{\check{\otimes}} = \sup\left\{\frac{|\sum a_{ij}t_i s_j|}{\|(t_i)\|_\infty \|(s_j)\|_\infty} : (t_i), (s_j)\ (\neq 0) \in c_0\right\}.$$

$l^1 \check{\otimes} l^1$ is the completion of all finite arrays under the $\check{\otimes}$-norm. For a complete discussion of tensor products we refer to [3], and note that it is not difficult to see that the projective and injective norms are dual to each other.

The following lemma is folklore whose origin is in the work of Varopoulos (see [6], for example), and we include its proof here for the sake of completeness.

LEMMA. *Let $F_1 = \{n_i\}$ and $F_2 = \{k_j\}$ be mutually disjoint and infinite sets in Γ whose union is dissociate. Then*

(1) $$l^1(F_1) \check{\otimes} l^1(F_2) \approx C_{F_1+F_2}(G)$$

and

(2) $$c_0(F_1) \hat{\otimes} c_0(F_2) \approx A(F_1+F_2).$$

(Recall that $A(F_1+F_2) = L^1(G)^\wedge / \{f \in L^1(G) \colon \hat{f} = 0 \text{ on } F_1+F_2\}$).

The Banach algebra isomorphisms are given as follows:

$$l^1 \check{\otimes} l^1 \ni (a_{ij}) \leftrightarrow \sum a_{ij}(n_i, \cdot)(k_j, \cdot) \in C_{F_1+F_2}$$

and

$$c_0 \hat{\otimes} c_0 \ni (b_{ij}) \leftrightarrow \varphi \in A(F_1+F_2), \quad \text{where } \varphi(n_i+k_j) = b_{ij}.$$

Proof. (1) Since finitely supported functions and trigonometric polynomials are dense in $l^1 \check{\otimes} l^1$ and $C_{F_1+F_2}$, respectively, it suffices to check that, for any $(a_{ij})_{i,j=1}^{K}$,

$$\|(a_{ij})\|_{\check{\otimes}} \geqslant \left\| \sum a_{ij}(n_i, \cdot)(k_j, \cdot) \right\|_\infty \geqslant \frac{1}{4} \|(a_{ij})\|_{\check{\otimes}}.$$

Without loss of generality we assume that $a_{ij} \in \boldsymbol{R}$ for all i and j, and let $(t_i)_{i=1}^{K}$ and $(s_j)_{j=1}^{K}$, $|t_i|, |s_i| \leqslant 1$ for all i and j, be such that

$$\|(a_{ij})\|_{\check{\otimes}} = \left|\sum a_{ij}t_i s_j\right|.$$

Let (Riesz product)

$$\mu(\cdot) = \prod_{i=1}^{K}\left[1+t_i\left(\frac{(n_i, \cdot)+\overline{(n_i, \cdot)}}{2}\right)\right]\prod_{j=1}^{K}\left[1+s_j\left(\frac{(k_j, \cdot)+\overline{(k_j, \cdot)}}{2}\right)\right].$$

Then

$$\left|\sum a_{ij}t_i s_j\right| = \left|\left(\sum a_{ij}(n_i, \cdot)(k_j, \cdot)\right) * \mu(0)\right| \leqslant 2 \left\|\sum a_{ij}(n_i, \cdot)(k_j, \cdot)\right\|_\infty.$$

In the other direction, let $g_0 \in G$ be such that

$$\Big|\sum a_{ij}(n_i, g_0)(k_j, g_0)\Big| = \Big\|\sum a_{ij}(n_i, \cdot)(k_j, \cdot)\Big\|_\infty .$$

But

$$\Big|\sum a_{ij}(n_i, g_0)(k_j, g_0)\Big| \leqslant \|(a_{ij})\|_{\check{\otimes}},$$

and (1) follows.

The norms $\hat{\otimes}$ and $\check{\otimes}$ are dual to each other, and so are the norms $A(F_1+F_2)$ and $C_{F_1+F_2}$. It then follows from part (1) that the norms $\hat{\otimes}$ and $A(F_1+F_2)$ are equivalent. Therefore, as finitely supported functions are dense in both $A(F_1+F_2)$ and $c_0 \hat{\otimes} c_0$, we have

$$A(F_1+F_2) \approx c_0 \hat{\otimes} c_0 .$$

We now "thin" out $E \cup S$ ($E = \{p_1^j p_2^i\}$ and $S = \{p_1^j\}$). Let $E' \subset E$ and $S' \subset S$ be infinite sets such that $E' \cup S'$ is dissociate. By the Lemma,

$$A(E'+S') \approx c_0(E') \hat{\otimes} c_0(S') \quad \text{and} \quad C_{E'+S'}(T) \approx l^1(E') \check{\otimes} l^1(S').$$

But the dual of $A(E'+S')$ is $L^\infty_{E'+S'}$ ($L^\infty_{E'+S'} = C_{E'+S'}$) and, therefore,

$$\big(c_0(E') \hat{\otimes} c_0(S')\big)^* \approx l^1(E') \check{\otimes} l^1(S').$$

Since the tensor algebras do not distinguish between the arithmetic structure of the underlying sets, we have proved

PROPOSITION. $(c_0 \hat{\otimes} c_0)^* \approx l^1 \check{\otimes} l^1$ (*a Banach algebra isomorphism*).

Remark. Even though the $\hat{\otimes}$-norm and the $\check{\otimes}$-norm are dual to each other, we have $l^\infty \hat{\otimes} l^\infty \subsetneq (l^1 \check{\otimes} l^1)^*$ (see [6]).

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