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ISOMORPHISM OF THE CO-UNIVERSAL F-SPACES OF KALTON AND TERRY

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A topological linear space E is co-universal for a class $\mathscr{A}$ of topological linear spaces if each $X \in \mathscr{A}$ is isomorphic to a quotient of E.

By a well-known classical result of Schauder (Banach and Mazur [1], p. 111), the space l_1 is co-universal for all separable Banach spaces. More generally, if $0 < p \leqslant 1$, then l_p is co-universal for all separable p-Banach spaces (see the references to Shapiro, Stiles, and Rolewicz in [2], p. 178). Kalton ([2], Theorem 5.4) and Terry ([5], Theorem 1) have independently shown the existence of a co-universal space in the class of separable F-spaces (metrizable complete topological linear spaces). Their constructions are very similar. Our main goal is to show that the co-universal spaces constructed by Kalton and Terry are, in fact, isomorphic. In addition to this, we correct and partly simplify the construction of Terry.

Recall (cf. [3], p. 163) that every F-space X admits an F-norm $\|\cdot\|$ (i.e., a function $\|\cdot\|$: $X \to \boldsymbol{R}_+$ satisfying the following conditions: $\|x\| = 0$ iff $x = 0$, $\|x+y\| \leqslant \|x\| + \|y\|$, $\|tx\| \leqslant \|x\|$ if $|t| \leqslant 1$, and $\|tx\| \to 0$ if $t \to 0$) such that the metric $d(x, y) = \|x-y\|$ determines the topology of X.

Given F-spaces X and Y, we write $X \approx Y$ if X and Y are (topologically) isomorphic.

Denote by Φ the set of all non-decreasing subadditive continuous functions φ: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that $\varphi(t) = 0$ iff $t = 0$. Then, for any sequence (φ_n) in Φ, the sequence space $l_{(\varphi_n)}$ is defined to be the linear space of all scalar sequences (t_n) for which

$$\|(t_n)\| = \sum_n \varphi_n(|t_n|) < \infty,$$

equipped with the F-norm $\|\cdot\|$ given by the above formula, under which it becomes a separable F-space. (This is a particular case of sequence spaces considered by Musielak and Orlicz [4], Definition 2.3.)

Let Φ_1 be the set of functions $\varphi \in \Phi$ which are bounded by 1. Since each $\varphi \in \Phi_1$ has a unique continuous extension to $\bar{\boldsymbol{R}}_+ = [0, \infty]$, we may consider

Φ_1 as a subset of the separable Banach space $C(\bar{\boldsymbol{R}}_+)$. Hence Φ_1 is a separable metric space under the metric d induced from $C(\bar{\boldsymbol{R}}_+)$, viz.,

$$d(\varphi, \psi) = \sup_{t \in \boldsymbol{R}_+} |\varphi(t) - \psi(t)|.$$

In the sequel we shall make use of the obvious fact that (Φ_1, d) has no isolated points.

The result of Terry, after the correction indicated in the Remark below and in a slightly generalized form, can now be stated as follows:

(T) *If (φ_n) is a dense sequence in (Φ_1, d), then $l_{(\varphi_n)}$ is co-universal for all separable F-spaces.*

We shall prove (T) in two steps (following Kalton [2]).

1° If X is a separable F-space, then there is a sequence (α_n) in Φ_1 such that there exists a continuous linear map u of $l_{(\alpha_n)}$ onto X; thus $X \approx l_{(\alpha_n)}/u^{-1}(0)$.

This is a result of Turpin ([6], p. 36); we include a proof for completeness. Choose an F-norm $\|\cdot\|$ determining the topology of X so that $\|x\| \leqslant 1$ for all $x \in X$. Let (x_n) be a sequence dense in $X \setminus \{0\}$. For each $n \in N$ we define $\alpha_n \in \Phi_1$ by

$$\alpha_n(t) = \|t x_n\|.$$

We claim that the map $u\colon l_{(\alpha_n)} \to X$ defined by

$$u((t_n)) = \sum_{n=1}^{\infty} t_n x_n$$

is as required. First observe that

$$\sum_{n=1}^{\infty} \|t_n x_n\| = \sum_{n=1}^{\infty} \alpha_n(|t_n|) = \|(t_n)\|, \qquad (t_n) \in l_{(\alpha_n)},$$

whence u is well defined and is a continuous linear map (since $\|u((t_n))\| \leqslant \|(t_n)\|$). It remains to check that u is onto. Let $x \in X$. Then we can find, by induction, an increasing sequence (i_m) in N such that

$$\|x - x_{i_1} - \ldots - x_{i_m}\| < 2^{-m} \quad \text{for all } m \in N.$$

Then

$$x = \sum_{m=1}^{\infty} x_{i_m} \quad \text{and} \quad \sum_{m=1}^{\infty} \|x_{i_m}\| < \infty,$$

and so $x = u((t_n))$, where $t_n = 1$ if $n = i_m$ for some m, and $t_n = 0$ otherwise.

2° For every sequence (α_n) in Φ_1 there is an increasing sequence (k_n) in N such that

$$\sum_{n=1}^{\infty} d(\alpha_n, \varphi_{k_n}) < \infty;$$

then $l_{(\alpha_n)}$ is isomorphic to the complemented subspace

$$\{(t_n)\in l_{(\varphi_n)}\colon\ t_n = 0 \text{ for } n\notin\{k_1, k_2, \ldots\}\}$$

of $l_{(\varphi_n)}$. This is obvious; combined with 1° it proves (T).

The result of Kalton is the following:

(K) *If Φ_0 is a countable dense subset of Φ in the topology of uniform convergence on compact subsets of $\boldsymbol{R}_+$ and if the functions $\varphi\wedge q = \min(\varphi, q)$ ($\varphi\in\Phi_0$, $q > 0$ rational) are arranged in a sequence (ψ_n), then $l_{(\psi_n)}$ is co-universal for all separable F-spaces.*

Our purpose is to show that if (φ_n) is as in (T) and (ψ_n) is as in (K), then

$$l_{(\varphi_n)} \approx l_{(\psi_n)}.$$

Observe, however, that $l_{(\psi_n)}$ and $l_{(\psi_n\wedge 1)}$ are equal as sets and have comparable F-norms, whence $l_{(\psi_n)} \approx l_{(\psi_n\wedge 1)}$ by the open mapping theorem. Moreover, the sequence $(\psi_n\wedge 1)$ is dense in (Φ_1, d). Therefore, it is enough to prove the following

PROPOSITION. *If (φ_n) and (ψ_n) are dense sequences in Φ_1, then $l_{(\varphi_n)} \approx l_{(\psi_n)}$.*

Proof. We claim that there exists a bijection σ of $\boldsymbol{N}$ onto $\boldsymbol{N}$ such that

$$\sum_{n=1}^{\infty} d(\varphi_n, \psi_{\sigma(n)}) < \infty.$$

This will imply that the spaces $l_{(\varphi_n)}$ and $l_{(\psi_{\sigma(n)})}$ are equal as sets, and therefore are isomorphic by the closed graph theorem. Since the map $(t_n)\mapsto(t_{\sigma(n)})$ is an isomorphism between $l_{(\psi_n)}$ and $l_{(\psi_{\sigma(n)})}$, our proof will be completed.

To prove the existence of σ we use only the fact that the metric space (Φ_1, d) has no isolated points. Consequently, if a sequence is dense in Φ_1, then it remains such after cancellation of any finite number of its terms.

Define inductively two increasing sequences (n_i) and (p_i) of natural numbers such that

$$\{1, \ldots, k\}\subset\{n_1, \ldots, n_{2k-1}\}\cap\{p_1, \ldots, p_{2k}\}$$

and

$$d(\varphi_{n_k}, \psi_{p_k}) < 2^{-k} \quad \text{for all } k\in\boldsymbol{N}.$$

Then both (n_i) and (p_i) exhaust the whole set $\boldsymbol{N}$. It follows that $\sigma\colon \boldsymbol{N}\to\boldsymbol{N}$ defined by $\sigma(n_i) = p_i$ for $i\in\boldsymbol{N}$ is as desired.

Remark. In his argument in [5], p. 61, line 4 from above, Terry admits incorrectly that if $\|\cdot\|$ is a strictly monotone F-norm on $\boldsymbol{R}$ such that $\sup_{x\in\boldsymbol{R}}\|x\| =: s \leqslant 1$, then the function F defined by

$$F(t) = \begin{cases} \dfrac{1}{s}\left\|\dfrac{t}{1-t}\right\| & \text{for } 0\leqslant t < 1, \\ 1 & \text{for } t = 1 \end{cases}$$

belongs to $H(I)$, i.e., F is an increasing homeomorphism of $I = [0, 1]$ onto itself satisfying the following condition: if $a, b, c \in I$ with $a \leqslant b+c$, then $F(a) \leqslant F(b)+F(c)$. This, however, is not true in general. In fact, let $\|\cdot\|$ be a strictly monotone F-norm on $\boldsymbol{R}$ with the additional property that $\|x\| = |x|$ for $|x| \leqslant 1/2$. (The existence of such a norm $\|\cdot\|$ follows from [6], 0.2.8.) Then

$$F\left(\frac{1}{3}\right) = \frac{1}{2s} > \frac{1}{s}\left(\frac{1}{5} + \frac{1}{5}\right) = F\left(\frac{1}{6}\right) + F\left(\frac{1}{6}\right),$$

and so F fails to satisfy the above subadditivity type condition. Terry's proof can be easily corrected, simply by redefining $H(I)$ to be the set of all functions F of the above form.

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