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Certain formulas associated with generalized Rice polynomials, II*

by H. M. SRIVASTAVA (Victoria, B. C., Canada)

1. Introduction. For the generalized Rice polynomials

(1.1)
$$H_n^{(a,\beta)}[\zeta, p, v] = \binom{a+n}{n} {}_3F_2\left[\begin{matrix} -n, a+\beta+n+1, \zeta; \\ \\ a+1, p; \end{matrix} \; v\right], \quad n = 0, 1, 2, \ldots,$$

which, when $a = \beta = 0$, reduces to the original form ([9], p. 108)

$$H_n[\zeta, p, v] = {}_3F_2\left[\begin{matrix} -n, \; n+1, \; \zeta; \\ \\ 1, \; p; \end{matrix} \; v\right], \tag{1.2}$$

Deshpande and Bhise ([4], p. 170) have proved the generating function

$$(1-t)^{-\lambda} F_2\left[\lambda, \mu, \nu; \; \varrho, \sigma; \; \frac{t}{t-1}, \frac{(1-y)t}{2(t-1)}\right] = \sum_{n=0}^{\infty} \frac{[\lambda]_n(-t)^n}{[\varrho]_n} H_n^{(\mu-\varrho-n, -\mu-n)}[\nu, \sigma, (1-y)/2], \tag{1.3}$$

where

$$[\lambda]_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & \text{if } n = 1, 2, 3, \ldots, \end{cases} \tag{1.4}$$

and F_2 denotes the Appell function of the second kind defined by ([1], p. 14)

$$F_2[a, \beta, \beta'; \; \gamma, \gamma'; \; x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a]_{m+n}[\beta]_m[\beta']_n}{[\gamma]_m[\gamma']_n} \frac{x^m}{m!} \frac{y^n}{n!} \qquad (|x| + |y| < 1). \tag{1.5}$$

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In view of the known transformation ([1], p. 25; see also [5], p. 240)

$$(1.6)\quad F_2[\alpha,\beta,\beta';\gamma,\gamma';x,y] = (1-x)^{-\alpha}F_2\left[\alpha,\gamma-\beta,\beta';\gamma,\gamma';\frac{x}{x-1},\frac{y}{1-x}\right],$$

the first member of formula (1.3) is essentially the same as

$$(1.7)\quad F_2[\lambda,\varrho-\mu,\nu;\varrho,\sigma;t,-\tfrac{1}{2}(1-y)t].$$

Thus if in formula (1.3) we set $\mu=-\beta$, $\varrho=-\alpha-\beta$, and replace t, $(1-y)/2$ by $-t$ and x respectively, we shall at once arrive at its equivalent form

$$(1.8)\quad \sum_{n=0}^{\infty}\frac{[\lambda]_n}{[-\alpha-\beta]_n}H_n^{(\alpha-n,\beta-n)}[\nu,\sigma,x]t^n = F_2[\lambda,-\alpha,\nu;-\alpha-\beta,\sigma;-t,xt].$$

The generating function (1.8) was proved, a couple of years ago, by Manocha ([6], p. 432, (7)). It may be of interest to remark that in a subsequent paper [11] we extended several formulas involving the generalized Rice polynomials (1.1) to hold for various classes of generalized hypergeometric polynomials. In particular, we proved the formulas (see [11], p. 112)

(1.9)

$$\sum_{n=0}^{\infty}\binom{\lambda+n}{n}\binom{n-\alpha}{n}\binom{n-\alpha-\beta}{n}^{-1}\frac{[(a)]_n}{[(b)]_n}\,{}_{C+2}F_{D+1}\left[\begin{matrix}-n,\ \alpha+\beta-n,\ (c);\\ \alpha-n,\ (d);\end{matrix}\ x\right]t^n = F\left[\begin{matrix}\lambda+1,\ (a):\ 1-\alpha;\ (c);\\ (b):\ 1-\alpha-\beta;\ (d);\end{matrix}\ t,\ -xt\right]$$

and

$$(1.10)\quad \sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_{A+2}F_B\left[\begin{matrix}-m,\ -n,\ (a);\\ (b);\end{matrix}\ x\right]{}_{C+2}F_D\left[\begin{matrix}-k,\ \lambda+n,\ (c);\\ (d);\end{matrix}\ y\right]t^n = (1-t)^{-\lambda}F\left[\begin{matrix}\lambda:\ -m,\ (a);\ -k,\ (c);\\ (b);\ (d);\end{matrix}\ \frac{xt}{t-1},\frac{y}{1-t}\right],$$

where (a) is taken to abbreviate the sequence of A parameters

$$a_1,\ldots,a_j,\ldots,a_A,$$

$[(a)]_n$ has the interpretation

$$\prod_{j=1}^{A} [a_j]_n,$$

with $[a_j]_n$ defined by (1.4), and similarly for (b), $[(b)]_n$, etc. Also ${}_AF_B[z]$ denotes the generalized hypergeometric function, and the notation for the double hypergeometric functions, occurring on the right-hand sides of (1.9) and (1.10), is due to Burchnall and Chaundy ([2], p, 112) in preference, for the sake of brevity and elegance, to the one introduced earlier by Kampé de Fériet ([1], p. 150).

Formula (1.9) provides us with a generalization of the generating function (1.8). Indeed, in view of the relationship (1.1), it will correspond to formula (1.8) in the special case when $A = B$, $C = D = 1$, $a_j = b_j$, $j = 1, \ldots, A$ (or B), $c_1 = \nu$, and $d_1 = \sigma$.

Our earlier proofs of formulas (1.9) and (1.10) made use of Laplace's transform and its inverse in conjunction with the principle of multidimensional mathematical induction. In the present note we first construct a simple and direct proof of (1.9) and then develop its generalizations. We also show that formula (1.10), which is proved in the last section by using series iteration techniques, may be looked upon as a special case of certain results, involving the Kampé de Fériet function, which we derived elsewhere (see [12] and [13]).

2. Alternate proof of formula (1.9). For convenience, let us denote the first member of equation (1.9) by Ω. Making use of the elementary relationships (cf., e.g., [8], p. 32)

$$(2.1)\qquad [\lambda]_{n-k} = \frac{(-1)^k[\lambda]_n}{[1-\lambda-n]_k},\qquad [-n]_k = \frac{(-1)^k n!}{(n-k)!},\qquad [\mu]_{-n} = \frac{(-1)^n}{[1-\mu]_n}$$

$$(0 \leqslant k \leqslant n,\ \mu \neq \text{an integer}),$$

which are immediate consequences of definition (1.4), we see that

$$\Omega = \sum_{n=0}^{\infty} \frac{[\lambda+1]_n[(a)]_n}{[(b)]_n} \sum_{k=0}^{n} \frac{[1-\alpha]_{n-k}[(c)]_k}{[1-\alpha-\beta]_{n-k}[(d)]_k} \frac{(-x)^k}{k!} \frac{t^n}{(n-k)!}$$

$$= \sum_{k=0}^{\infty} \frac{[(c)]_k}{[(d)]_k} \frac{(-x)^k}{k!} \sum_{n=k}^{\infty} \frac{[\lambda+1]_n[(a)]_n[1-\alpha]_{n-k}}{[1-\alpha-\beta]_{n-k}[(b)]_n} \frac{t^n}{(n-k)!}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{[\lambda+1]_{n+k}[(a)]_{n+k}[1-\alpha]_n[(c)]_k}{[(b)]_{n+k}[1-\alpha-\beta]_n[(d)]_k} \frac{t^n}{n!} \frac{(-xt)^k}{k!},$$

which is the same as the second member of equation (1.9).

This completes our direct proof of formula (1.9).

3. Further generalizations and particular cases. The foregoing method of proof of formula (1.9) suggests the existence of a generalization in which the generalized hypergeometric ${}_{C+2}F_{D+1}$ polynomials are replaced by another similar system with any arbitrary number of numerator and denominator parameters like $\alpha+\beta-n$ and $\alpha-n$. Indeed, if we consider the double hypergeometric function

$$F\left[\begin{matrix}(a):\ (g);\ (c);\\ \qquad\qquad t,\ -xt\\ (b):\ (h);\ (d);\end{matrix}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a)]_{m+n}[(g)]_m[(c)]_n}{[(b)]_{m+n}[(h)]_m[(d)]_n}\frac{t^{m+n}}{m!}\frac{(-x)^n}{n!}$$

$$=\sum_{N=0}^{\infty}\frac{[(a)]_N[(g)]_N}{[(b)]_N[(h)]_N}\frac{t^N}{N!}\sum_{n=0}^{N}\frac{[-N]_n[1-(h)-N]_n[(c)]_n}{[1-(g)-N]_n[(d)]_n}\frac{[(-1)^{G-H}x]^n}{n!}$$

$$=\sum_{N=0}^{\infty}\frac{[(a)]_N[(g)]_N}{[(b)]_N[(h)]_N}\,{}_{C+H+1}F_{D+G}\left[\begin{matrix}-N,\ 1-(h)-N,\ (c);\\ (-1)^{G-H}x\\ 1-(g)-N,\ (d);\end{matrix}\right]\frac{t^N}{N!},$$

we shall arrive at the following generalization of formula (1.9):

$$\text{(3.1)}\quad \sum_{n=0}^{\infty}\frac{[(a)]_n[(g)]_n}{[(b)]_n[(h)]_n}\,{}_{C+H+1}F_{D+G}\left[\begin{matrix}-n,\ 1-(h)-n,\ (c);\\ x\\ 1-(g)-n,\ (d);\end{matrix}\right]\frac{t^n}{n!}$$

$$=F\left[\begin{matrix}(a):\ (g);\ (c);\\ \qquad\qquad t,\ (-1)^{G-H+1}xt\\ (b):\ (h);\ (d);\end{matrix}\right].$$

It is easy to observe that formula (3.1) incorporates, as its particular cases, a large number of generating functions, known as well as new. For instance, for the generalized Rice polynomials defined by (1.1) we have

$$\text{(3.2)}\quad \sum_{n=0}^{\infty}\frac{[(a)]_n}{[(b)]_n}H_n^{(\alpha-n,\beta)}[\nu,\sigma,x]t^n$$

$$=F\left[\begin{matrix}(a):\ -\alpha;\ \alpha+\beta+1,\ \nu;\\ \qquad\qquad -t,\ -xt\\ (b):\ -;\ \sigma;\end{matrix}\right],$$

$$\sum_{n=0}^{\infty} \frac{[(a)]_n}{[\alpha+1]_n[(b)]_n} H_n^{(\alpha,\beta-n)}[\nu, \sigma, x]t^n \tag{3.3}$$
$$= F\left[\begin{matrix}(a): & -; & \alpha+\beta+1, & \nu; & \\ & & & & t, \quad -xt \\ (b): & -; & \alpha+1, & \sigma; & \end{matrix}\right]$$

and

$$\sum_{n=0}^{\infty} \frac{[(a)]_n}{[-\alpha-\beta]_n[(b)]_n} H_n^{(\alpha-n,\beta-n)}[\nu, \sigma, x]t^n \tag{3.4}$$
$$= F\left[\begin{matrix}(a): & -\alpha; & \nu; & \\ & & & -t, \quad xt \\ (b): & -\alpha-\beta; & \sigma; & \end{matrix}\right].$$

The last formula (3.4), which follows also as a special case of our earlier result (1.9), is a generalization of the known generating function (1.8). Also since

$$H_n^{(\alpha,\beta)}[\nu, \nu, x] = \binom{\alpha+n}{n} {}_2F_1\left[\begin{matrix}-n, \alpha+\beta+n+1; & \\ & x \\ \alpha+1; & \end{matrix}\right] = P_n^{(\alpha,\beta)}(1-2x), \tag{3.5}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial defined by (see, for instance, [15], p. 68)

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k}\binom{\beta+n}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \tag{3.6}$$

therefore formulas (3.2), (3.3) and (3.4) can be further specialized, by letting $\nu = \sigma$ and replacing x by $(1-x)/2$, to obtain generating functions for the special Jacobi polynomials $P_n^{(\alpha-n,\beta)}(x)$, $P_n^{(\alpha,\beta-n)}(x)$ and $P_n^{(\alpha-n,\beta-n)}(x)$ respectively. Note, however, that the generating functions thus obtained for $P_n^{(\alpha-n,\beta)}(x)$ and $P_n^{(\alpha,\beta-n)}(x)$ will be capable of being transformed into each other, since it is well known that ([15], p. 59)

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \tag{3.7}$$

By reversing the order of summation in (3.6) it is readily seen that (see also [8], p. 255, (8))

$$P_n^{(\alpha,\beta)}(x) = \binom{\beta+n}{n}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, \quad -\alpha-n; & \\ & \dfrac{x+1}{x-1} \\ \beta+1; & \end{matrix}\right]. \tag{3.8}$$

The relationship (3.8) would enable us to write the special case of formula (3.1), when $C = D-1 = 0$, $G = H-1 = 0$, $d_1 = \beta+1$, $h_1 = \alpha+1$, in the form

$$\text{(3.9)} \quad \sum_{n=0}^{\infty} \frac{[(a)]_n}{[\alpha+1]_n[\beta+1]_n[(b)]_n} P_n^{(\alpha,\beta)}(x)t^n = F\left[\begin{matrix}(a): & -; & -; & \\ & & & \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t \\ (b): & \alpha+1; & \beta+1; & \end{matrix}\right],$$

which is a generalization of the late Professor Brafman's generating function (1), p. 271 in [8], viz.

$$\text{(3.10)} \quad \sum_{n=0}^{\infty} \frac{[\gamma]_n[\delta]_n}{[\alpha+1]_n[\beta+1]_n} P_n^{(\alpha,\beta)}(x)t^n = F_4[\gamma, \delta, ;\ \alpha+1, \beta+1;\ \tfrac{1}{2}(x-1)t, \tfrac{1}{2}(x+1)t],$$

where F_4 denotes the fourth Appell function defined by ([1], p. 14)

$$\text{(3.11)} \quad F_4[\alpha, \beta;\ \gamma, \gamma';\ x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[\alpha]_{m+n}[\beta]_{m+n}}{[\gamma]_m[\gamma']_n} \frac{x^m}{m!}\frac{y^n}{n!} \qquad (|x|^{1/2}+|y|^{1/2}<1).$$

We remark in passing that since ([15], p. 81)

$$\text{(3.12)} \quad \begin{aligned} P_n^{(\alpha,\alpha)}(x) &= \binom{\alpha+n}{n}\binom{2\alpha+n}{n}^{-1} C_n^{\alpha+1/2}(x), \\ P_n^{(0,0)}(x) &= C_n^{1/2}(x) = P_n(x), \end{aligned}$$

where $C_n^{\alpha}(x)$ denotes the Gegenbauer (or ultraspherical) polynomial, and $P_n(x)$ is the Legendre polynomial, the aforementioned results, involving Jacobi polynomials, can be appropriately specialized to derive generating functions for ultraspherical and Legendre polynomials. Also since a large variety of known polynomial systems, such as the Laguerre polynomials $L_n^{(\alpha)}(x)$ and $L_n^{(\alpha-n)}(x)$, the Bessel polynomials $Y_n^{(\alpha-n)}(x)$, etc., are merely particular cases of the generalized hypergeometric polynomials involved in formula (3.1), it will not be difficult to deduce generating functions for these polynomials as special cases of our result (3.1).

Next we consider those special cases of formula (3.1) in which the double hypergeometric function, occurring on its right-hand side, would reduce to one or the other of the four Appell functions F_1, F_2, F_3 and F_4 (see [1], p. 14 for details). From (3.1) we thus obtain

$$(3.13)\quad F_1[\lambda, \mu, \mu'; \nu; t, xt] = \sum_{n=0}^{\infty} \frac{[\lambda]_n[\mu]_n}{[\nu]_n} \, {}_2F_1\left[\begin{matrix} -n, & \mu'; \\ & & x \\ & 1-\mu-n; \end{matrix}\right] \frac{t^n}{n!},$$

$$(3.14)\quad F_2[\lambda, \mu, \mu'; \nu, \nu'; t, -xt]$$

$$= \sum_{n=0}^{\infty} \frac{[\lambda]_n[\mu]_n}{[\nu]_n} \, {}_3F_2\left[\begin{matrix} -n, \; 1-\nu-n, \; \mu'; \\ & x \\ 1-\mu-n, \; \nu'; \end{matrix}\right] \frac{t^n}{n!},$$

$$(3.15)\quad F_3[\lambda, \lambda', \mu, \mu'; \nu; t, -xt]$$

$$= \sum_{n=0}^{\infty} \frac{[\lambda]_n[\mu]_n}{[\nu]_n} \, {}_3F_2\left[\begin{matrix} -n, \; \lambda', \; \mu'; \\ & x \\ 1-\lambda-n, \; 1-\mu-n; \end{matrix}\right] \frac{t^n}{n!}$$

and

$$(3.16)\quad F_4[\lambda, \mu; \nu, \nu'; t, xt] = \sum_{n=0}^{\infty} \frac{[\lambda]_n[\mu]_n}{[\nu]_n} \, {}_2F_1\left[\begin{matrix} -n, \; 1-\nu-n; \\ & x \\ \nu'; \end{matrix}\right] \frac{t^n}{n!}.$$

Evidently, formula (3.13) may be rewritten as a generating function for the Jacobi polynomials $P_n^{(\alpha-n,\beta)}(x)$ or $P_n^{(\alpha,\beta-n)}(x)$. Formula (3.16) is equivalent to Brafman's generating function (3.10). On the other hand, formula (3.14) is essentially the same as the known special case (1.8) of the generating relation (3.4). And in view of definition (1.1), formula (3.15) gives us the elegant generating function

$$(3.17)\quad \sum_{n=0}^{\infty} \frac{[1-\sigma]_n}{[\varrho]_n} H_n^{(\alpha-n,\beta)}[\nu, \sigma-n, x]t^n$$

$$= F_3[-\alpha, \alpha+\beta+1, 1-\sigma, \nu; \; \varrho; -t, xt]$$

which does not seem to have been noticed earlier.

It may be of interest to note that the generating relation (3.1) admits itself of an obvious further generalization given formally by

$$(3.18)\quad \sum_{n=0}^{\infty} \delta_n \left\{ \sum_{k=0}^{[n/m]} \Delta_{n,k} \frac{[-n]_{mk} x^k}{k!} \right\} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \delta_{n+mk} \Delta_{n+mk,k} \frac{t^n}{n!} \frac{[x(-t)^m]^k}{k!},$$

where m is a positive integer, and the coefficients δ_n and $\Delta_{n,k}$ are arbitrary constants, real or complex. Thus it would seem fairly easy to deduce from (3.18) several classes of series relationships by suitably specializing these arbitrary coefficients. We, therefore, omit the details.

4. The bilinear generating relation (1.10). We now turn to formula (1.10). Let us first recall our bilinear relation (4.8), p. 237 in [12]:

$$\sum_{n=0}^{\infty}\binom{\nu+n}{n}{}_{A+1}F_B\left[\begin{matrix}\varrho-n, & (a); \\ & & x \\ & (b); \end{matrix}\right]{}_{C+1}F_D\left[\begin{matrix}\sigma+n, & (c); \\ & & y \\ & (d); \end{matrix}\right]t^n \tag{4.1}$$

$$= (1-t)^{-\nu-1}\sum_{n=0}^{\infty}\binom{\nu+n}{n}\frac{[(a)]_n[(c)]_n}{[(b)]_n[(d)]_n}\left[-\frac{xyt}{(1-t)^2}\right]^n \times$$

$$\times F\left[\begin{matrix}(a)+n: & \varrho; & \nu+n+1; & \\ & & & x,\ \dfrac{xt}{t-1} \\ (b)+n: & -; & -; & \end{matrix}\right]\times$$

$$\times F\left[\begin{matrix}(c)+n: & \sigma-\nu-1; & \nu+n+1; & \\ & & & y,\ \dfrac{y}{1-t} \\ (d)+n: & -; & -; & \end{matrix}\right],$$

where $A \leqslant B$, $C \leqslant D$, and $|x|$, $|y|$, $|t|$ are appropriately constrained in order that each side of (4.1) possesses a meaning.

For $\varrho = \sigma-\nu-1 = 0$, formula (4.1) would obviously correspond to our earlier result (3.4), p. 309 in [10] which, when $p = q = r = s = 1$, yields the bilinear relation (19b), p. 345 of Meixner ([7]; see also [5], p. 84, (9)). Note that in another paper [13] we have discussed the possibility of further extending formula (4.1) with the product

$$ {}_{A+1}F_B\left[\begin{matrix}\varrho-n, & (a); \\ & & x \\ & (b); \end{matrix}\right]{}_{C+1}F_D\left[\begin{matrix}\sigma+n, & (c); \\ & & y \\ & (d); \end{matrix}\right] \tag{4.2}$$

replaced by the double hypergeometric function

$$F\left[\begin{matrix}(a): & \varrho-n, & (b); & \sigma+n, & (b'); & \\ & & & & & x, y \\ (c): & & (d); & & (d'); & \end{matrix}\right], \tag{4.3}$$

or by the formal double series

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{[\varrho-n]_r[\sigma+n]_s}{r!\,s!}\,\xi_{rs}x^r y^s, \tag{4.4}$$

where the ξ_{rs} are arbitrary complex numbers, $r, s = 0, 1, 2, \ldots$ For similar generalizations of our formula (2.5), p. 113 in [11] the reader may refer, for instance, to the summation formula (1.4), p. 231 in [12].

In the bilinear relation (4.1) we replace A, C by $A+1$ and $C+1$ respectively, and set $a_{A+1} = -m$, $c_{C+1} = -k$, $\varrho = 0$ and $\sigma = \nu+1 = \lambda$. On simplifying the right-hand side of (4.1), we shall arrive at our formula (1.10). Use may be made of our expansion formula (11′), p. 52 in [14].

For a direct proof of formula (1.10), without using the bilinear relation (4.1), we notice that

$$\begin{aligned} I(x,y) &= \sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_{A+1}F_B\left[\begin{matrix}-n,\ (a); \\ & x \\ (b);\end{matrix}\right]{}_{C+1}F_D\left[\begin{matrix}\lambda+n,\ (c); \\ & y \\ (d);\end{matrix}\right]t^n \\ &= \sum_{s=0}^{\infty}\frac{[\lambda]_s[(c)]_s}{[(d)]_s}\frac{y^s}{s!}\sum_{n=0}^{\infty}\frac{[\lambda+s]_n}{n!}\,{}_{A+1}F_B\left[\begin{matrix}-n,\ (a); \\ & x \\ (b);\end{matrix}\right]t^n. \end{aligned}$$

In order to sum the inner series we make use of the late Professor Chaundy's formula ([3], p. 62, (25))

$$\sum_{n=0}^{\infty}\frac{[\nu]_n}{n!}\,{}_{A+1}F_B\left[\begin{matrix}-n,\ (a); \\ & x \\ (b);\end{matrix}\right]t^n = (1-t)^{-\nu}\,{}_{A+1}F_B\left[\begin{matrix}\nu,\ (a); \\ & \dfrac{xt}{t-1} \\ (b);\end{matrix}\right], \tag{4.5}$$

which corresponds to (4.1) with $\varrho = y = 0$ and follows also as a special case of our formula (1.7), p. 111 in [11]. We thus have

$$\begin{aligned} I(x,y) &= (1-t)^{-\lambda}\sum_{s=0}^{\infty}\frac{[\lambda]_s[(c)]_s}{[(d)]_s}\frac{[y/(1-t)]^s}{s!}\,{}_{A+1}F_B\left[\begin{matrix}\lambda+s,\ (a); \\ & \dfrac{xt}{t-1} \\ (b);\end{matrix}\right] \\ &= (1-t)^{-\lambda}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{[\lambda]_{r+s}[(a)]_r[(c)]_s}{[(b)]_r[(d)]_s}\frac{[xt/(t-1)]^r}{r!}\frac{[y/(1-t)]^s}{s!}, \end{aligned}$$

and we arrive at the bilinear generating function

$$(4.6)\qquad \sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,{}_{A+1}F_B\left[\begin{matrix}-n,\ (a);\\ \\ (b);\end{matrix}\ x\right]{}_{C+1}F_D\left[\begin{matrix}\lambda+n,\ (c);\\ \\ (d);\end{matrix}\ y\right]t^n$$

$$=(1-t)^{-\lambda}F\left[\begin{matrix}\lambda:\ (a);\ (c);\\ \\ -:\ (b);\ (d);\end{matrix}\ \frac{xt}{t-1},\ \frac{y}{1-t}\right],$$

which obviously would yield formula (1.10) when we replace A, C by $A+1$ and $C+1$ respectively and set $a_{A+1}=-m$, $c_{C+1}=-k$.

Incidentally, formula (4.6) provides us with an elegant form of our bilinear relation (3.4), p. 309 in [10]. Also it would suggest that in the special case when $\varrho=0$ and $\sigma=\nu+1=\lambda$, our summation formula (2.3), p. 39 in [13] may be written as

$$(4.7)\qquad \sum_{n=0}^{\infty}\frac{[\lambda]_n}{n!}\,F\left[\begin{matrix}(a):\ -n,\ (b);\ \lambda+n,\ (b');\\ \\ (c):\qquad (d);\qquad (d');\end{matrix}\ x,\ y\right]t^n$$

$$=(1-t)^{-\lambda}F\left[\begin{matrix}\lambda,\ (a):\ (b);\ (b');\\ \\ (c):\ (d);\ (d');\end{matrix}\ \frac{xt}{t-1},\ \frac{y}{1-t}\right],$$

which evidently is an interesting generalization of the bilinear generating relations (1.10) and (4.6).

Formula (4.7) can be obtained, for instance, by comparing the second member of our earlier result (3.1), p. 41 in [13] with our expansion formula (11), p. 50 in [14].

References

[1] P. Appell et J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Paris 1926.

[2] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions*, II, Quart. J. Math. Oxford Ser. 12 (1941), p. 112 - 128.

[3] T. W. Chaundy, *An extension of hypergeometric functions*, ibidem 14 (1943), p. 55 - 78.

[4] V. L. Deshpande and V. M. Bhise, *A generating function for the generalised Rice's polynomial*, Mat. Vesnik. 7 (22) (1970), p. 169 - 172.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*, Vol. I, New York – Toronto – London 1953.
[6] H. L. Manocha, *Some formulae for generalized Rice polynomials*, Proc. Cambridge Philos. Soc. 64 (1968), p. 431 - 434.
[7] J. Meixner, *Umformung gewisser Reihen, deren Glieder Produkte hypergeometrischer Funktionen sind*, Deutsche Math. 6 (1942), p. 341 - 349.
[8] E. D. Rainville, *Special functions*, New York 1960.
[9] S. O. Rice, *Some properties of* ${}_3F_2[-n, n+1, \zeta;\ 1, p;\ v]$, Duke Math. J. 6 (1940), p. 109 - 119.
[10] H. M. Srivastava, *An extension of the Hille–Hardy formula*, Math. Comp. 23 (1969), p. 305 - 311; see also Notices Amer. Math. Soc. 15 (1968), p. 634 - 635, Abstract 68T - 449.
[11] – *Certain formulas associated with generalized Rice polynomials*, Ann. Polon. Math. 23 (1970), p. 109 - 115; see also Notices Amer. Math. Soc. 16 (1969), p. 411, Abstract 69T - B27.
[12] – *A formal extension of certain generating functions*, Glasnik Mat. Ser. III. 5 (25) (1970), p. 229 - 239; see also Notices Amer. Math. Soc. 16 (1969), p. 674, Abstract 69T - B115.
[13] – *A formal extension of certain generating functions, II*, Glasnik Mat. Ser. III. 6 (26) (1971), 35 - 44; see also Notices Amer. Math. Soc. 18 (1971), p. 252, Abstract 71T - B15.
[14] – *Certain pairs of inverse series relations*, J. Reine Angew. Math. 245 (1970), p. 47 - 54.
[15] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, R.I., 1939.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA, CANADA

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