ANNALES
POLONICI MATHEMATICI
XXXVI (1979)

The space of multipliers into l_1

by P. WOJTASZCZYK (Warszawa)

Abstract. The reflexivity of the space of multipliers from H_p into l_1 is proved using general Banach space methods. We improve Duren's [2] description of multiplies from H_1 into l_1.

Caveny [1] has proved that the space of multipliers from H_p, $1 \leqslant p \leqslant \infty$, into l_1 is a dual space. In the present note we give a general result (Theorem 1) which implies that for $p > 1$ those spaces are in fact reflexive and for $p = 1$ the space of multipliers is a second dual of a certain Banach space.

The author would like to thank Professor T. Figiel for pointing out an error in the first version of the present paper.

In this note we use standard Banach space techniques. Also our notations are standard in Banach space theory (cf. [7]).

Let us recall that a sequence (x_n, x_n^*), consisting of pairs of elements of a Banach space X and bounded linear functionals from X^*, the dual of X, is said to be *biorthogonal* if $x_n^*(x_n) = \delta_{n,m}$. A biorthogonal sequence is *bounded* if $\sup\|x_n\| < \infty$ and $\sup\|x_n^*\| < \infty$. A sequence is *fundamental* if linear combinations of x_n's are dense in X. All the biorthogonal sequences considered in this paper are assumed to be fundamental and bounded.

If we have an biorthogonal sequence (x_n, x_n^*), the sequence (λ_n) is said to be a *multiplier of* (x_n) *into* l_1 if, for any $x \in X$, $\sum_{n=1}^{\infty} |\lambda_n x_n^*(x)| < \infty$. It is clear that (λ_n) can be considered as an operator from X into l_1. The space of multipliers of (x_n) into l_1 equipped with the operator norm will be denoted by $\{(x_n), l_1\}$. It is easy to see that it is a Banach space.

The subspace of compact operators in $\{(x_n), l_1\}$ will be denoted by $K\{(x_n), l_1\}$.

Let $K(X, Y)$ denote the space of all compact operators from X into Y. The following proposition is well known (cf. [3]).

PROPOSITION 1. *Let X be a Banach space with separable dual X^* and let $\varphi \in K(c_0, X)^*$. Then there exist two sequences $(z_n^*) \subset X^*$ and $(f_n) \subset l_\infty$*

with $\sum_{n=1}^{\infty} \|z_n\| \|f_n\| \leqslant 2\|\varphi\|$ *such that for* $T \in K(c_0, X)$ *we have* $\varphi(T) = \sum_{n=1}^{\infty} T^{**} f_n(z_n^*)$.

LEMMA 2. *Let X be a Banach space with an unconditional basis (x_n). Then*

(a) *X is reflexive if and only if X does not contain neither a subspace isomorphic to l_1 nor c_0,*

(b) *if X does not contain any subspace isomorphic to l_1, then*

$$X^{**} = \Big\{(a_n)\colon \sup_N \Big\| \sum_{n=1}^{N} a_n x_n \Big\| < \infty\Big\},$$

(c) *if X contains a subspace isomorphic to l_1, then there exist an increasing sequence of indices (n_k) and sequence of vectors $z_k = \sum_{n_k+1}^{n_{k+1}} a_i x_i$ such that (z_k) is equivalent to the unit vector basis in l_1.*

This is a well-known lemma and its proof can be found in [7] or [8] or [6].

THEOREM 1. *Let (x_n, x_n^*) be a bounded and fundamental biorthogonal system such that $\overline{\operatorname{span}}(x_n^*) = X^*$. Then the space $\{(x_n^*), l_1\}$ is a second dual of $K\{(x_n^*), l_1\}$. Moreover, the following conditions are equivalent:*

(a) *$\{(x_n^*), l_1\}$ is separable,*

(b) *$\{(x_n^*), l_1\}$ is reflexive,*

(c) *$\{(x_n^*), l_1\}$ does not contain a subspace isomorphic to c_0,*

(d) *$\{(x_n^*), l_1\}$ does not contain a subspace isomorphic to l_1,*

(e) *$\{(x_n^*), l_1\} = K\{(x_n^*), l_1\}$,*

Proof. Let us define the unit multipliers δ_n by $\delta_n = (\delta_{n,m})$, where $\delta_{n,m}$ is a Kronecker symbol. It is clear that the sequence $(\delta_n)_{n=1}^{\infty}$ forms an unconditional basic sequence in $\{(x_n^*), l_1\}$.

Now we want to prove that $\overline{\operatorname{span}}(\delta_n) = K\{(x_n^*), l_1\}$. Let $\Lambda = (\lambda_k)$ be a compact multiplier and let Λ_n denote the multiplier $(\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots)$. Clearly, $\Lambda_n \in \overline{\operatorname{span}}(\delta_n)$. We have

$$\|\Lambda - \Lambda_n\| = \sup_{x^* \in X^*, \|x^*\|=1} \|\Lambda(x^*) - \Lambda_n(x^*)\| = \sup_{x^* \in X^*, \|x^*\|=1} \|R_n \Lambda(x^*)\| = \|R_n \Lambda\|,$$

where $R_n\colon l_1 \to l_1$ is a norm one projection given by $R_n\big(\sum_{k=1}^{\infty} a_k e_k\big) = \sum_{k=n+1}^{\infty} a_k e_k$. Since Λ is a compact operator, $\|R_n \Lambda\| \to 0$ what proves that $\Lambda \in \overline{\operatorname{span}}(\delta_n)$. Obviously, $\overline{\operatorname{span}}(\delta_n) \subset \{(x_n^*), l_1\}$, so our claim is proved.

Now we want to prove that $K(\{x_n^*\}, l_1)$ does not contain a subspace isomorphic to l_1. Observe that each element $\Lambda = (\lambda_n)$ in $K(\{x_n^*\}, l_1)$ is a conjugate operator of the same multiplier considered as a multiplier from c_0 into (x_n). This shows that $K(\{x_n^*\}, l_1)$ can be considered as a subspace of $K(c_0, X)$. Suppose that $K(\{x_n^*\}, l_1)$ contains a subspace isomorphic to l_1. Then by Lemma 2c and Proposition 1 there exist multipliers $\Lambda_k = \sum_{n_k+1}^{n_{k+1}} a_n \delta_n$ equivalent to the unit vector basis in l_1 and sequences $(z_n^*) \subset X^*$ and $(f_n) \subset l_\infty$ such that $\sum_{n=1}^{\infty} \|z_n^*\| \|f_n\|$ is convergent and $\sum_{n=1}^{\infty} \Lambda_k^{**} f_n(z_n^*) = 1$ for every k.

Since (x_n^*) are dense in X^*, we can assume without loss of generality that $z_n^* = \sum_{i=1}^{r_n} \gamma_i x_i^*$. Let us take N such that $\sum_{n=N}^{\infty} \|z_n^*\| \|f_n\| < 0.5$ and k_0 such that $n_{k_0} > \max\{r_n : n \leqslant N\}$. An easy calculation shows that for $k > k_0$ and $n \leqslant N$ we have $\Lambda_k^{**} f_n(z_n^*) = 0$. But it means that for $k > k_0$ we have

$$\Big|\sum_{n=1}^{\infty} \Lambda_k^{**} f_n(z_n^*)\Big| < 0.5.$$

This contradiction shows that $K(\{x_n^*\}, l_1)$ does not contain any subspace isomorphic to l_1.

Since $\{(x_n^*), l_1\} = \big\{(a_n) : \sup_N \big\|\sum_{n=1}^{N} a_n \delta_n\big\| < \infty\big\}$, we have from Lemma 2(b) that $\{(x_n^*), l_1\} = \big(\overline{\operatorname{span}}(\delta_n)\big)^{**}$.

Now we will prove that conditions (a)–(e) are equivalent. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) because $\{(x_n^*), l_1\}$ is a second dual of a space with an unconditional basis not containing any subspace isomorphic to l_1 (cf. [8]). Since $K\{(x_n^*), l_1\} = \overline{\operatorname{span}}(\delta_n)$, we have (e) $\Rightarrow$ (a) and clearly (b) $\Rightarrow$ (e). This completes the proof of the theorem.

Remark. The assumption that we consider the multipliers from a separable dual space into l_1 is important. For example, natural multipliers from c_0 into l_1 are exactly l_1, so they are separable but not reflexive and each multiplier is a compact operator.

COROLLARY 1. *If (x_n) is a bounded and fundamental biorthogonal system in a reflexive space X, then the space of multipliers into l_1 is reflexive.*

Proof. Apply Theorem 1 to the system (x_n^*, x_n) and use the fact that any operator from a reflexive space into l_1 is compact.

Now we consider the multipliers from a Hardy class H_p $(1 \leqslant p \leqslant 2)$ into l_1. For the definitions and properties of H_p see [2]. Note only that elements of H_p are analytic functions in $|z| < 1$ and that we always consider multipliers of the system z^n. Our goal now is to prove.

THEOREM 2. *The space of multipliers from H_p $(1 < p \leqslant 2)$ into l_1 is a reflexive Banach space. The space of multipliers from H_1 into l_1 is a non-reflexive second conjugate space.*

Proof. Only the second part of the Theorem requires a proof. The first part is a special case of Corollary 1. Observe that $H_1 = \big(C(S)/A_0\big)^*$, where $C(S)$ is a space of continuous functions on $|z| = 1$ and A_0 is a subspace of the disc-algebra consisting of functions vanishing at zero (cf. [2], [5]). Therefore we can apply Theorem 1 to the biorthogonal system (f_n, z^n), where f_n is a coset in $C(S)/A_0$ containing z^{-n}. The only thing to be done is to exhibit a non-compact multiplier from H_1 into l_1. This we do in the following

LEMMA 3. *The Hardy multiplier $H = (1/n+1)_{n=0}^{\infty}$ is a non-compact multiplier from H_1 into l_1.*

Proof of the lemma. The sequence $(1/n+1)_{n=0}^{\infty}$ is a multiplier from H_1 into l_1 by the Hardy inequality (cf. [2], [5]). To see that it is not compact, let us consider the functions

$$F_n = z^n \frac{1}{n+1} \sum_{k=0}^{n} \sum_{s=-k}^{k} z^s = \sum_{k=0}^{2n} \frac{n+1-|k-n|}{n+1} z^k.$$

From the properties of the Féjer kernel (cf. [5]) it follows that $\|F_n\| = 1$. Observe that

$$\|H(F_n)\| = \sum_{k=0}^{2n} \frac{1}{k+1} \, \frac{n+1-|k-n|}{n+1} \geqslant 0.5.$$

Since $H(F_n)$ converges coordinatewise to zero in l_1, we infer that H is not compact.

Let $(e_k)_{k=0}^{\infty}$ be the unit vector basis in l_2. The operator $d_n\colon l_2 \to l_2$ is defined by

$$d_n(e_k) = \begin{cases} e_j & \text{if } k \leqslant n \text{ and } j+k = n, \\ 0 & \text{if } k > n. \end{cases}$$

THEOREM 3. *The sequence (λ_n) is a compact multiplier from H_1 into l_1 if and only if the series $\sum_{n=0}^{\infty} \lambda_n d_n$ is unconditionally convergent in $B(l_2)$.*

The proof of this theorem follows easily from the results of Hedlund [4] and the following elementary fact.

LEMMA 4. *$\sum_{n=0}^{\infty} \lambda_n d_n$ is unconditionally convergent in $B(l_2)$ if and only if $\sum_{n=0}^{\infty} |\lambda_n| d_n$ is convergent.*

References

[1] J. Caveny, *Absolute convergence factors for H_p series*, Canadian J. Math. 21 (1969), p. 187–195.

[2] P. L. Duren, *Theory of H_p spaces*, Academic Press, 1970.

[3] M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. 21 (1975), p. 38–49.
[4] J. M. Hedlund, *Multipliers of H_1 and Hankel matrices*, Proc. Amer. Math. Soc. 22 (1969, p. 20–23.
[5] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, 1962.
[6] R. C. James, *On bases and reflexivity in Banach spaces*, Ann. of Math. 52 (1950), p. 518–527.
[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer–Verlag Lecture Notes, 338, 1973.
[8] I. Singer, *Bases in Banach spaces I*, Springer–Verlag, 1970.

INSTITUT OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES, WARSAW

Reçu par la Rédaction le 15. 12. 1975