The space of multipliers into $l_1$

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Abstract. The reflexivity of the space of multipliers from $H_p$ into $l_1$ is proved using general Banach space methods. We improve Duren's [2] description of multipliers from $H_1$ into $l_1$.

Caveny [1] has proved that the space of multipliers from $H_p$, $1 \leq p \leq \infty$, into $l_1$ is a dual space. In the present note we give a general result (Theorem 1) which implies that for $p > 1$ those spaces are in fact reflexive and for $p = 1$ the space of multipliers is a second dual of a certain Banach space.

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In this note we use standard Banach space techniques. Also our notations are standard in Banach space theory (cf. [7]).

Let us recall that a sequence $(x_n, x_n^*)$, consisting of pairs of elements of a Banach space $X$ and bounded linear functionals from $X^*$, the dual of $X$, is said to be biorthogonal if $x_n^*(x_m) = \delta_{n,m}$. A biorthogonal sequence is bounded if $\sup \|x_n\| < \infty$ and $\sup \|x_n^*\| < \infty$. A sequence is fundamental if linear combinations of $x_n$'s are dense in $X$. All the biorthogonal sequences considered in this paper are assumed to be fundamental and bounded.

If we have a biorthogonal sequence $(x_n, x_n^*)$, the sequence $(\lambda_n)$ is said to be a multiplier of $(x_n)$ into $l_1$ if, for any $x \in X$, $\sum_{n=1}^{\infty} |\lambda_n x_n^*(x)| < \infty$.

It is clear that $(\lambda_n)$ can be considered as an operator from $X$ into $l_1$. The space of multipliers of $(x_n)$ into $l_1$ equipped with the operator norm will be denoted by $\{(x_n), l_1\}$. It is easy to see that it is a Banach space.

The subspace of compact operators in $\{(x_n), l_1\}$ will be denoted by $K\{(x_n), l_1\}$.

Let $K(X, Y)$ denote the space of all compact operators from $X$ into $Y$. The following proposition is well known (cf. [3]).

Proposition 1. Let $X$ be a Banach space with separable dual $X^*$ and let $\varphi \in K(c_0, X)^*$. Then there exist two sequences $(x_n^*) \subset X^*$ and $(f_n) \subset l_\infty$
with $\sum_{n=1}^{\infty} \|z_n\| \|f_n\| \leq 2 \|\varphi\|$ such that for $T \in K(c_0, X)$ we have $\varphi(T) = \sum_{n=1}^{\infty} T^{**} f_n(z_n^*)$.

**Lemma 2.** Let $X$ be a Banach space with an unconditional basis $(x_n)$. Then

(a) $X$ is reflexive if and only if $X$ does not contain neither a subspace isomorphic to $l_1$ nor $c_0$,

(b) if $X$ does not contain any subspace isomorphic to $l_1$, then

$$X^{**} = \{ (a_n) : \sup_{N} \left\| \sum_{n=1}^{N} a_n x_n \right\| < \infty \},$$

(c) if $X$ contains a subspace isomorphic to $l_1$, then there exist an increasing sequence of indices $(n_k)$ and sequence of vectors $z_k = \sum_{n_k+1}^{n_{k+1}} a_n x_i$ such that $(z_k)$ is equivalent to the unit vector basis in $l_1$.

This is a well-known lemma and its proof can be found in [7] or [8] or [6].

**Theorem 1.** Let $(x_n, x_n^*)$ be a bounded and fundamental biorthogonal system such that $\overline{\text{span}}(x_n^*) = X^*$. Then the space $(x_n^*, l_1)$ is a second dual of $K\{ (x_n^*), l_1 \}$. Moreover, the following conditions are equivalent:

(a) $(x_n^*, l_1)$ is separable,

(b) $(x_n^*, l_1)$ is reflexive,

(c) $(x_n^*, l_1)$ does not contain a subspace isomorphic to $c_0$,

(d) $(x_n^*, l_1)$ does not contain a subspace isomorphic to $l_1$,

(e) $(x_n^*, l_1) = K\{ (x_n^*), l_1 \}$.

**Proof.** Let us define the unit multipliers $\delta_n$ by $\delta_n = (\delta_{n,m})$, where $\delta_{n,m}$ is a Kronecker symbol. It is clear that the sequence $(\delta_n)_{n=1}^{\infty}$ forms an unconditional basic sequence in $(x_n^*, l_1)$.

Now we want to prove that $\overline{\text{span}}(\delta_n) = K\{ (x_n^*), l_1 \}$. Let $A = (\lambda_k)$ be a compact multiplier and let $A_n$ denote the multiplier $(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots)$. Clearly, $A_n \in \overline{\text{span}}(\delta_n)$. We have

$$\|A - A_n\| = \sup_{x^* \in X^* \|x^*\| = 1} \|A(x^*) - A_n(x^*)\|$$

$$= \sup_{x^* \in X^* \|x^*\| = 1} \|R_n A(x^*)\| = \|R_n A\|,$$

where $R_n : l_1 \to l_1$ is a norm one projection given by $R_n \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=n+1}^{\infty} a_k e_k$.

Since $A$ is a compact operator, $\|R_n A\| \to 0$ what proves that $A \in \overline{\text{span}}(\delta_n)$. Obviously, $\overline{\text{span}}(\delta_n) = (x_n^*, l_1)$, so our claim is proved.

Now we want to prove that $K\{ (x_n^*), l_1 \}$ does not contain a subspace isomorphic to $l_1$. Observe that each element $A = (\lambda_n)$ in $K\{ (x_n^*), l_1 \}$ is
a conjugate operator of the same multiplier considered as a multiplier from $c_0$ into $(x_n)$. This shows that $K((a_n^*)^*, l_1)$ can be considered as a subspace of $K(c_0, X)$. Suppose that $K((a_n^*)^*, l_1)$ contains a subspace isomorphic to $l_1$. Then by Lemma 2(c) and Proposition 1 there exist multipliers $A_k = \sum_{n=1}^{\infty} a_n^* \delta_n$ equivalent to the unit vector basis in $l_1$ and sequences $(x_n^*) \subset X^*$ and $(f_n) \subset l_\infty$ such that $\sum_{n=1}^{\infty} ||x_n^*|| f_n ||$ is convergent and $\sum_{n=1}^{\infty} A_k^* f_n (x_n^*) = 1$ for every $k$.

Since $(x_n^*)$ are dense in $X^*$, we can assume without loss of generality that $z_n^* = \sum_{i=1}^{\infty} \gamma_i x_i^*$. Let us take $N$ such that $\sum_{n=N}^{\infty} ||z_n^*|| f_n || < 0.5$ and $k_0$ such that $n_{k_0} > \max \{ r_n; n \leq N \}$. An easy calculation shows that for $k > k_0$ and $n \leq N$ we have $A_k^* f_n (z_n^*) = 0$. But it means that for $k > k_0$ we have

$$\left| \sum_{n=1}^{\infty} A_k^* f_n (z_n^*) \right| < 0.5.$$

This contradiction shows that $K((a_n^*)^*, l_1)$ does not contain any subspace isomorphic to $l_1$.

Since $\{ (a_n^*), l_1 \} = \{ (a_n^*); \sup \sum_{n=1}^{N} a_n^* \delta_n || < \infty \}$, we have from Lemma 2(b) that $\{ (a_n^*)^*, l_1 \} = \text{span}(\delta_n)^**$.

Now we will prove that conditions (a)-(e) are equivalent. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) because $\{ (a_n^*), l_1 \}$ is a second dual of a space with an unconditional basis not containing any subspace isomorphic to $l_1$ (cf. [8]). Since $K((a_n^*), l_1) = \text{span}(\delta_n)$, we have (e) $\Rightarrow$ (a) and clearly (b) $\Rightarrow$ (e). This completes the proof of the theorem.

Remark. The assumption that we consider the multipliers from a separable dual space into $l_1$ is important. For example, natural multipliers from $c_0$ into $l_1$ are exactly $l_1$, so they are separable but not reflexive and each multiplier is a compact operator.

**Corollary 1.** If $(x_n)$ is a bounded and fundamental biorthogonal system in a reflexive space $X$, then the space of multipliers into $l_1$ is reflexive.

**Proof.** Apply Theorem 1 to the system $(x_n^*, x_n)$ and use the fact that any operator from a reflexive space into $l_1$ is compact.

Now we consider the multipliers from a Hardy class $H_p$ $(1 \leq p \leq 2)$ into $l_1$. For the definitions and properties of $H_p$ see [2]. Note only that elements of $H_p$ are analytic functions in $|z| < 1$ and that we always consider multipliers of the system $x_n^*$. Our goal now is to prove.

**Theorem 2.** The space of multipliers from $H_p$ $(1 < p \leq 2)$ into $l_1$ is a reflexive Banach space. The space of multipliers from $H_1$ into $l_1$ is a non-reflexive second conjugate space.
Proof. Only the second part of the Theorem requires a proof. The first part is a special case of Corollary 1. Observe that \( H_1 = (C(S)/A_0)^* \), where \( C(S) \) is a space of continuous functions on \(|z| = 1\) and \( A_0 \) is a subspace of the disc-algebra consisting of functions vanishing at zero (cf. [2], [5]). Therefore we can apply Theorem 1 to the biorthogonal system \((f_n, z^n)\), where \( f_n \) is a coset in \( C(S)/A_0 \) containing \( z^{-n} \). The only thing to be done is to exhibit a non-compact multiplier from \( H_1 \) into \( l_1 \). This we do in the following

**Lemma 3.** The Hardy multiplier \( H = (1/n + 1)_{n=0}^{\infty} \) is a non-compact multiplier from \( H_1 \) into \( l_1 \).

Proof of the lemma. The sequence \((1/n + 1)_{n=0}^{\infty}\) is a multiplier from \( H_1 \) into \( l_1 \) by the Hardy inequality (cf. [2], [5]). To see that it is not compact, let us consider the functions

\[
F_n = z^n \frac{1}{n+1} \sum_{k=0}^{n} \sum_{s=-k}^{k} s^k = \sum_{k=0}^{2n} \frac{n+1-|k-n|}{n+1} s^k.
\]

From the properties of the Féjer kernel (cf. [5]) it follows that \( \|F_n\| = 1 \). Observe that

\[
\|H(F_n)\| = \sum_{k=0}^{2n} \frac{1}{k+1} \frac{n+1-|k-n|}{n+1} \geq 0.5.
\]

Since \( H(F_n) \) converges coordinatewise to zero in \( l_1 \), we infer that \( H \) is not compact.

Let \((e_k)_{k=0}^{\infty}\) be the unit vector basis in \( l_2 \). The operator \( d_n : l_2 \to l_2 \) is defined by

\[
d_n(e_k) = \begin{cases} e_j & \text{if } k \leq n \text{ and } j + k = n, \\ 0 & \text{if } k > n. \end{cases}
\]

**Theorem 3.** The sequence \((\lambda_n)\) is a compact multiplier from \( H_1 \) into \( l_1 \) if and only if the series \( \sum_{n=0}^{\infty} \lambda_n d_n \) is unconditionally convergent in \( B(l_2) \).

The proof of this theorem follows easily from the results of Hedlund [4] and the following elementary fact.

**Lemma 4.** \( \sum_{n=0}^{\infty} \lambda_n d_n \) is unconditionally convergent in \( B(l_2) \) if and only if \( \sum_{n=0}^{\infty} |\lambda_n| d_n \) is convergent.

References


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