A NUMERICAL CONSTANT
ASSOCIATED WITH GENERALIZED CONVOLUTIONS

BY

K. URBANIK (WROCŁAW)

The main topic of this paper is a description of generalized convolutions in terms of some invariance properties of their characteristic functions. We denote by \( C_\beta \) the space of bounded continuous real-valued functions on the positive half-line \( R^+ \) with the topology of uniform convergence on every compact subset of \( R^+ \). Further, by \( \Psi \) we shall denote the set of all probability measures defined on Borel subsets of \( R^+ \). The set \( \Psi \) is endowed with the topology of weak convergence. For \( a \in R^+ \) we define the mapping \( T_a: R^+ \to R^+ \) by \( T_a x = ax \). For a function \( f \in C_\beta \), \( T_a f \) denotes the function \((T_a f)(x) = f(ax)\) and for a measure \( \mu \in \Psi \), \( T_a \mu \) denotes the measure defined by \((T_a \mu)(E) = \mu(a^{-1} E)\) if \( a > 0 \) and \( T_0 \mu = \delta_0 \), where for \( a \in R^+, B \subset R^+ \) we put \( aB = \{ab: b \in B\} \) and \( \delta_c \) is the probability measure concentrated at the point \( c \). We shall also use the notation \( AB = \{ab: a \in A, b \in B\} \) for \( A, B \subset R^+ \). We say that two functions \( f \) and \( g \) from \( C_\beta \) are similar if \( f = T_a g \) for a certain positive number \( a \).

A continuous commutative and associative \( \Psi \)-valued binary operation \( \circ \) defined on \( \Psi \) is called a generalized convolution if the following conditions are fulfilled:

(i) the measure \( \delta_0 \) is a unit element, i.e. \( \mu \circ \delta_0 = \mu (\mu \in \Psi) \),
(ii) \((c\mu + (1-c)\lambda) \circ \nu = c(\mu \circ \nu) + (1-c)(\nu \circ \lambda)\) (\( 0 \leq c \leq 1, \mu, \nu, \lambda \in \Psi \)),
(iii) \((T_a \mu) \circ (T_a \nu) = T_a (\mu \circ \nu)\) (\( a \in R^+, \mu, \nu \in \Psi \)),
(iv) there exists a sequence \( c_1, c_2, \ldots \) of positive numbers such that the sequence \( \delta_0^{c_i} \) converges to a measure different from \( \delta_0 \).

The power \( \delta_0^{c_i} \) is taken here in the sense of the operation \( \circ \).

The set \( \Psi \) with the operation \( \circ \) and the operations of convex linear combinations is called a generalized convolution algebra and denoted by \( (\Psi, \circ) \). For basic properties of generalized convolution algebras we refer to [1] and [3]–[13]. In particular, generalized convolution algebras admitting a non-trivial homomorphism into the algebra of real numbers with the operations of multiplication and convex linear combinations are called
regular. We recall that a homomorphism \( h \) is trivial if either \( h \equiv 0 \) or \( h \equiv 1 \). All generalized convolution algebras under consideration in the sequel will tacitly be assumed to be regular. It has been proved in [10] (Theorem 6) that each regular generalized convolution algebra admits a characteristic function, i.e. a homeomorphic map from \( \mathfrak{P} \) into \( C_h \): \( \mu \to \hat{\mu} \) such that \( (c\mu + (1-c)\nu)^{\wedge} = c\hat{\mu} + (1-c)\hat{\nu} \) \((0 \leq c \leq 1)\), \((\mu \circ \nu)^{\wedge} = \hat{\mu}\hat{\nu} \) and \( (T_\alpha \mu)^{\wedge} = T_\alpha \hat{\mu} \) \((\alpha \in R^+)\) for all \( \mu, \nu \in \mathfrak{P} \). The characteristic function plays the same fundamental role in generalized convolution algebras as the Laplace transform in the ordinary convolution algebra. Moreover, each characteristic function is an integral transform

\[
\hat{\mu}(t) = \int_0^\infty \Omega(tx) \mu(dx)
\]

with a continuous kernel \( \Omega \) satisfying the conditions \(|\Omega(t)| \leq 1 \) \((t \in R^+)\) and

\[(1) \quad \Omega(t) = 1 - t^\kappa L(t), \]

where \( \kappa > 0 \) and the function \( L \) is slowly varying and continuous at the origin. By Theorem 2.1 in [12] all kernels corresponding to characteristic functions of a generalized convolution algebra are similar. Consequently, the constant \( \kappa \) in (1) does not depend upon the choice of a characteristic function and is called the characteristic exponent of the generalized convolution \( \circ \), in symbols \( \kappa(\circ) = \kappa \). Further, by \( *_{s,1} \) \((\kappa > 0)\) we shall denote the Kingman convolution defined by the formula

\[
\int_0^\infty \int_0^\infty f(x)(\mu *_{s,1} \nu)(dx)(dy) = \frac{1}{2} \int_0^\infty \int_0^\infty [f((x^\kappa + y^\kappa)^{1/\kappa}) + f((|x^\kappa - y^\kappa|^{1/\kappa})] \mu(dx) \nu(dy)
\]

for all \( f \in C_h \). In this case the kernel \( \Omega \) is given by the formula

\[(2) \quad \Omega(t) = \cos t^\kappa \]

and

\[(3) \quad \kappa(*_{s,1}) = 2\kappa. \]

For any pair \( \mu, \nu \) from \( \mathfrak{P} \) by \( \mu \nu \) we shall denote the probability distribution of the product \( XY \) of two independent random variables \( X \) and \( Y \) with probability distributions \( \mu \) and \( \nu \) respectively. By Proposition 1.3 in [12] for every characteristic function \( \mu \to \hat{\mu} \) of a generalized convolution the formula

\[(4) \quad (\mu \nu)^{\wedge}(t) = \int_0^\infty \hat{\mu}(tx) \nu(dx) \]

is true. Further, by Theorem 7 in [10] for each characteristic function of a generalized convolution \( \circ \) with \( \kappa(\circ) = \kappa \) there exists a probability measure \( \sigma_{\kappa} \) called the characteristic measure of \( \circ \) such that

\[(5) \quad \sigma_{\kappa}(t) = \exp(-t^\kappa) \quad (t \in R^+). \]
By Theorems 5 and 6 in [10] for any kernel $\Omega$ there exists a positive number $t_0$ such that $\Omega(t) < 1$ whenever $0 < t < t_0$. Consequently, without loss of generality passing to similar kernels if necessary we may assume that the kernel in question has one of the following properties:

\[ (\ast) \quad \Omega(1) = 1, \quad \Omega(t) < 1 \quad \text{whenever } 0 < t < 1, \]

\[ (\ast\ast) \quad \Omega(t) < 1 \quad \text{for all } t > 0. \]

Consider two generalized convolutions $\circ$ and $\circ'$. The convolution $\circ$ is said to be \textit{representable in} $\circ'$, in symbols $\circ < \circ'$, if there exists a continuous non-trivial homomorphism from the algebra $(\Psi, \circ)$ into the algebra $(\Psi, \circ')$ commuting with the semigroup $T_a (a \in R^+)$ ([12], Chapter 3). We recall that a homomorphism $h$ is \textit{non-trivial} if $h(\mu) \neq \delta_0$ for $\mu \in \Psi$.

For any measure $\mu$ from $\Psi$ by $S(\mu)$ we shall denote its support. Given a generalized convolution $\circ$ we put

\[ A(\mu) = \{ t: \hat{\mu}(t) = 1, \ t \in R^+ \} \]

where $\mu \rightarrow \hat{\mu}$ is the characteristic function of $\circ$. Of course, we may assume that its kernel fulfills one of the conditions $(\ast)$, $(\ast\ast)$. It is clear that the set $A(\mu)$ is closed,

\[ (6) \quad 0 \in A(\mu), \]

\[ (7) \quad A(\mu) \cap A(\nu) = A(\mu \cap A(\nu). \]

Moreover, by formula (4),

\[ (8) \quad S(\nu) A(\nu) \subseteq A(\mu). \]

Hence we get the inclusion

\[ A(\mu) \subseteq \bigcap_{x \in S(\nu) \setminus \{0\}} x^{-1} A(\mu) \]

for $\nu \neq \delta_0$. The converse inclusion is also true. Namely, if $\nu \neq \delta_0$ and $t \in x^{-1} A(\mu)$ for every $x \in S(\nu) \setminus \{0\}$, then, by (6), $tx \in A(\mu)$ for every $x \in S(\nu)$ which, by (4), yields $t \in A(\mu)$. Thus

\[ (9) \quad A(\nu) = \bigcap_{x \in S(\nu) \setminus \{0\}} x^{-1} A(\mu) \quad \text{if } \nu \neq \delta_0. \]

Setting $\nu = \delta_1 \circ \delta_1$ and $\mu = \delta_1$ into (8) and applying (7) we get the inclusion

\[ (10) \quad S(\delta_1 \circ \delta_1) A(\delta_1) \subseteq A(\delta_1). \]

Further, it is easy to see that

\[ (11) \quad 1 \in A(\delta_1), \quad A(\delta_1) \cap (0, 1) = \emptyset \quad \text{in the case } (\ast), \]

and

\[ (12) \quad A(\mu) = \{0\} \quad (\mu \in \Psi) \quad \text{in the case } (\ast\ast). \]
Lemma 1. In the case (\(\ast\)) we have the relation \(S(\delta_1 \circ \delta_1) \cap (1, \infty) \neq \emptyset\).

Proof. Suppose the contrary, i.e. \(S(\delta_1 \circ \delta_1) \subset [0, 1]\). Then, by (10) and (11), \(S(\delta_1 \circ \delta_1) \subset \{0\} \cup \{1\}\) or, in other words, \(\delta_1 \circ \delta_1 = c \delta_0 + (1 - c) \delta_1\) where \(0 \leq c \leq 1\). Passing to the characteristic functions we have the equation \(\Omega^2(t) = c + (1 - c) \Omega(t)\) which shows that \(\Omega\) assumes at most two values 1 and \(-c\). Since \(\Omega(0) = 1\), by continuity of \(\Omega\), we get \(\Omega \equiv 1\). But this gives the contradiction. The Lemma is thus proved.

Invariance properties of the set \(A(\delta_1)\) are described by the multiplicative semigroup

\[ N = \{a: T_a A(\delta_1) \subset A(\delta_1), a > 1\}. \]

Setting \(\mu = \delta_1\) into (9) we obtain the formula

\[ A(v) = \bigcap_{x \in S(v) \setminus \{0\}} x^{-1} A(\delta_1) \]

if \(v \neq \delta_0\). As an immediate consequence of the above formula and the equation \(A(\delta_0) = R^+\) we get the inclusion

\[(13) \quad NA(v) \subset A(v) \quad (v \in \mathfrak{B}).\]

In particular,

\[(14) \quad N \subset A(v) \quad \text{if} \quad \bar{\gamma}(1) = 1.\]

Further, by (12),

\[(15) \quad N = (1, \infty) \quad \text{in the case} \quad (\ast\ast)\]

and, by (10),

\[(16) \quad S(\delta_1 \circ \delta_1) \cap (1, \infty) \subset N \quad \text{in the case} \quad (\ast)\]

which, by Lemma 1, shows that always \(N \neq \emptyset\). Moreover, in the case (\(\ast\)) the semigroup \(N\) is closed, i.e. 1 does not belong to the closure of \(N\). In fact, the contrary would imply \(N = (1, \infty)\) and, by (14), \((1, \infty) \subset A(\delta_1)\). In other words, \(\Omega(t) = \Omega(0) = 1\) for \(t > 1\). But then the left-hand side of the equation

\[ \int_0^\infty \Omega(tx) \sigma_x(dx) = \exp(-t^r), \]

where \(\sigma_x\) is the characteristic measure of the convolution in question, would tend to 1 when \(t \to \infty\). This contradiction shows that in the case (\(\ast\)) the semigroup \(N\) is closed.

We associate with every generalized convolution \(\circ\) a numerical constant by setting

\[ \eta(\circ) = \inf N. \]
Of course

(17) \[ \eta(\circ) > 1 \text{ in the case } (\star) \]

and, by (15),

(18) \[ \eta(\circ) = 1 \text{ in the case } (\star\star). \]

We note that for the convolution \( \star_{s,1} \) \( N = \{n^{1/\alpha}: n \geq 2\} \) and

(19) \[ \eta(\star_{s,1}) = 2^{1/\alpha}. \]

**Theorem 1.** If \( \circ < \circ' \), then \( \eta(\circ) \leq \eta(\circ') \).

**Proof.** Suppose that \( h \) is a non-trivial continuous homomorphism from \( (\Psi, \circ) \) into \( (\Psi, \circ') \) commuting with the semigroup \( T_a \ (a \in R^+) \). By Lemma 3.1 in [12] the map \( h \) is of the form \( h(\mu) = \lambda \mu \) for a certain non-degenerate \( \lambda \in \Psi \). Moreover, if \( \mu \to \mu' \) is a characteristic function of \( \circ' \), then \( \mu \to [(h(\mu))^\sim]' \) is a characteristic function of \( \circ \). Therefore denoting by \( A(\mu) \) and \( A'(\mu) \) the sets for the convolutions \( \circ \) and \( \circ' \) respectively we may assume without loss of generality that

\[ A(\mu) = A'(\lambda \mu) \ (\mu \in \Psi). \]

Let \( N' \) be the invariance semigroup for \( \circ' \). The last equation and (13) imply the inclusion \( N' A(\delta_1) \subseteq A(\delta_1) \). Thus \( N' \subseteq N \) and, consequently, \( \eta(\circ) \leq \eta(\circ') \) which completes the proof.

**Lemma 2.** For every generalized convolution \( \circ \) with \( \kappa(\circ) = \kappa \) the inequality

\[ \int_0^\infty x^\kappa(\delta_1 \circ \delta_1)(dx) \leq 2 \]

is true.

**Proof.** The formula

\[ \Omega^2(t) = \int_0^\infty \Omega(tx)(\delta_1 \circ \delta_1)(dx) \]

implies

\[ 1 + \Omega(t) = \int_0^\infty \frac{1 - \Omega(tx)}{1 - \Omega(t)}(\delta_1 \circ \delta_1)(dx) \]

whence, by (1) and the Fatou lemma when \( t \to 0^+ \) our assertion follows.

**Lemma 3.** For every generalized convolution \( \circ \) with the property (\( \star \)) we have the formula

\[ \delta_1 \circ \delta_1 = p\delta_0 + q\delta_1 + (1 - p - q)v \]
where \( v \in \Psi \), \( S(v) \subset [\eta(\omega), \infty) \), \( p, q \geq 0 \) and

\[
4p + 3q \leq 2.
\]

**Proof.** By (10) and (11) we have the inclusion \( S(\delta_1 \circ \delta_1) \cap [0, 1] \subset \{0\} \cup \{1\} \) which together with (16) yields the inclusion

\[
S(\delta_1 \circ \delta_1) \subset \{0\} \cup \{1\} \cup [\eta(\omega), \infty).
\]

Consequently, the measure \( \delta_1 \circ \delta_1 \) can be written in the form

\[
\delta_1 \circ \delta_1 = p\delta_0 + q\delta_1 + (1 - p - q) v
\]

where \( v \in \Psi \), \( S(v) \subset [\eta(\omega), \infty) \) and \( p, q \geq 0 \). It remains to prove inequality (20). For every number \( c \) (\( 0 \leq c \leq 1 \)) let \( I_c \) denote the set of all pairs of non-negative real numbers \((x, y)\) satisfying the conditions

\[
y^2 + 4cy + 4(1+c)x - 4c \leq 0, \quad y \leq 1-c.
\]

Put \( a_c = \min(2(\sqrt{c^2 + c} - c), 1 - c) \) and

\[
\varphi_c(y) = \frac{4c - y^2 - 4cy}{4(1 + c)}.
\]

It is clear that the set \( I_c \) is convex and closed. Moreover, its boundary is the union of the sets

\[
\{(x, y): x = \varphi_c(y), \ 0 \leq y \leq a_c\}, \quad \{(0, y): 0 \leq y \leq a_c\} \quad \text{and} \quad \left\{ (x, 0): 0 \leq x \leq \frac{c}{1 + c} \right\}.
\]

Hence it follows that the maximum \( M_c = \max \{4x + 3y: (x, y) \in I_c\} \) is attained on the curve (22). Since the function \( 4\varphi_c(y) + 3y \) is monotone non-decreasing in the interval \( 0 \leq y \leq a_c \) we finally get the formula \( M_c = 4\varphi_c(a_c) + 3a_c \) which by a simple computation yields \( M_c = 6(\sqrt{c^2 + c} - c) \) if \( 0 \leq c \leq \frac{1}{3} \) and \( M_c = 2 \) if \( \frac{1}{3} \leq c \leq 1 \). Consequently, \( M_c \leq 2 \) if \( 0 \leq c \leq 1 \) and to prove inequality (20) it suffices to show that \((p, q) \in I_c \) for a certain \( c \) (\( 0 \leq c \leq 1 \)). Put

\[
b = \inf \{ \Omega(t): t \in R^+ \}.
\]

Of course, \( b \leq 1 \) and \( \mu(t) \geq -b \ (t \in R^+, \ \mu \in \Psi) \) which, by (5), yields \( b \geq 0 \). Further, by (21),

\[
\Omega^2(t) = p + q\Omega(t) + (1 - p - q) \int_0^\infty \Omega(tx) v(dx).
\]

Since \( 1 \geq q \geq 0 \geq -b \) we can find, by the continuity of \( \Omega \), a sequence \( t_1, t_2, \ldots \) of positive real numbers with the property \( \Omega(t_n) \to q/2 \). Setting
$t = t_n$ into (23) we get, when $n \to \infty$

$$\frac{q^2}{4} \geq p + \frac{q^2}{2} - (1 - p - q) b$$

or, equivalently,

$$q^2 + 4bq + 4(1 + b)p - 4b \leq 0.$$  

Further, taking a sequence $u_1, u_2, \ldots$ with the property $\Omega(u_n) \to -b$ and setting $t = u_n$ into (23) we get, when $n \to \infty$, $b^2 \leq p - bq + 1 - p - q$ or, equivalently, $q \leq 1 - b$. This shows that $(p, q) \in I_b$ which completes the proof of the Lemma.

A relation between constants $\chi(\circ)$ and $\eta(\circ)$ is given by the following quite surprising Theorem.

**Theorem 2.** For every generalized convolution $\circ$ the inequality

(24) \hspace{1cm} \eta(\circ)^{\chi(\circ)} \leq 4

is true. The equation

(25) \hspace{1cm} \eta(\circ)^{\chi(\circ)} = 4

holds if and only if $\circ = \ast_{\alpha, 1}$ where $\alpha = \frac{1}{2} \chi(\circ)$.

**Proof.** By (18) inequality (24) is obvious in the case $(\ast\ast)$. Consider the case $(\ast)$. Then, by Lemma 3,

(26) \hspace{1cm} \delta_1 \circ \delta_1 = p\delta_0 + q\delta_1 + (1 - p - q)v

where $p, q \geq 0$,

(27) \hspace{1cm} 4p + 3q \leq 2,$$

$v \in \Psi$ and $S(v) \subset [\eta(\circ), \infty)$. The last inclusion yields the inequality

(28) \hspace{1cm} \int_0^\infty x^{\chi(\circ)}v(dx) \geq \eta(\circ)^{\chi(\circ)}.$$

Applying Lemma 2 we have

(29) \hspace{1cm} q + (1 - p - q) \int_0^\infty x^{\chi(\circ)}v(dx) \leq 2

which, by (28), implies the inequality

(30) \hspace{1cm} (1 - p - q) \eta(\circ)^{\chi(\circ)} \leq 2 - q.$$

But, by (27), $4(1 - p - q) \geq 2 - q > 1$ which together with (30) gives (24). Further, by (3) and (19), equation (25) holds for $\circ = \ast_{\alpha, 1}$. Suppose now that for a generalized convolution $\circ$ equation (25) is true. Of course, in this case $\circ$ has property $(\ast)$ and, by (30), $4(1 - p - q) \leq 2 - q$ which together with (27)
yields $4p + 3q = 2$. Thus

$$p = \frac{2 - 3q}{4}.$$  \hspace{1cm} (31)

Further, by (28), $\int_0^\infty x^{\kappa(o)} v(dx) \geq 4$. Setting (31) into (29) we obtain the converse inequality $\int_0^\infty x^{\kappa(o)} v(dx) \leq 4$. Thus $\int_0^\infty x^{\kappa(o)} v(dx) = 4$. Since $S(v) \subseteq [\eta(o), \infty)$ the last equation and (25) show that the measure $v$ is concentrated at the point $\eta(o)$. Consequently, by (31), formula (26) can be rewritten in the form

$$\delta_1 \circ \delta_1 = \frac{2 - 3q}{4} \delta_0 + q \delta_1 + \frac{2 - q}{4} \delta_{\kappa(o)}$$

or, equivalently, in terms of the characteristic functions

$$\Omega^2(t) = \frac{2 - 3q}{4} + q\Omega(t) + \frac{2 - q}{4} \Omega(\eta(o)t).$$  \hspace{1cm} (32)

Put

$$F(t) = \frac{2\Omega(t^{1/\kappa}) - q}{2 - q} \quad (t \in R^+)$$  \hspace{1cm} (33)

where $x = \frac{1}{2} x(o)$. By (25) and (32) the function $F$ fulfills the equation

$$F^2(t) = \frac{1}{2} + \frac{1}{2} F(2t).$$

Moreover, the function $F$ is continuous on $R^+$, $F(0) = 1$, $F(t) \leq 1 \quad (t \in R^+)$, $F$ is not identically equal to 1 and, by (1) and the continuity of the function $L$, the limit

$$\lim_{t \to 0^+} \frac{1 - F(t)}{t^2}$$

exists. Applying the Forder theorem ([2], p. 216) we infer that $F(t) = \cos ct$ for a certain positive constant $c$. Consequently, by (33),

$$\Omega(t) = \left(1 - \frac{q}{2}\right)\cos ct^2 + \frac{q}{2}.$$  \hspace{1cm} (34)

Put $\mu = \frac{2}{2 + q} \delta_1 \circ \delta_{2^{1/\kappa}} + \frac{q}{2 + q} \delta_0$. Then

$$\mu(\{0\}) \geq \frac{q}{2 + q}.$$  \hspace{1cm} (35)
and, by (34),
\[
\hat{\mu}(t) = \frac{2-q}{2(2+q)} \Omega(3^{1/\alpha} t) + \frac{1}{2} \Omega(t) + \frac{q}{2+q} \Omega(2^{1/\alpha} t)
\]
or, equivalently,
\[
\mu = \frac{2-q}{2(2+q)} \delta_{3^{1/\alpha}} + \frac{1}{2} \delta_{2} + \frac{q}{2+q} \delta_{2^{1/\alpha}}.
\]
Thus \(0 \notin \mathcal{S}(\mu)\) which, according to (35), yields \(q = 0\). By (34) we infer that \(\Omega(t) = \cos ct^\alpha\) where \(c > 0\). In other words, by (2), \(\Omega\) is the kernel of the characteristic function of \(*_{a,1}\). Since the characteristic function determines the generalized convolution we have the equation \(\circ = *_{a,1}\) which completes the proof of the Theorem.

**Corollary 1.** The convolutions \(*_{a,1}\) \((\alpha > 0)\) are maximal elements under the partial order \(\prec\), i.e. the relation \(*_{a,1} \prec \circ\) yields \(\circ = *_{a,1}\).

**Proof.** Suppose that \(*_{a,1} \prec \circ\). Then, by Theorem 1, \(\eta(\circ) \geq \eta(*_{a,1})\) and, by Theorem 3.1 in [12], \(\chi(\circ) \geq \chi(*_{a,1})\). Consequently, \(\eta(\circ)^{\chi(\circ)} \geq 4\). Applying Theorem 2 we infer that \(\eta(\circ)^{\chi(\circ)} = 4\), \(\eta(\circ) = \eta(*_{a,1})\), \(\chi(\circ) = \chi(*_{a,1}) = 2\alpha\) which finally yields \(\circ = *_{a,1}\).

We conclude this paper with the following simple remark. For any function \(f \in C_b\) by Conv\((f)\) we denote the closed convex set spanned by \(\{T_a f: a \in R^+\}\). As a consequence of Theorem 2 we get the following statement.

**Corollary 2.** Suppose that \(f \in C_b\) and the following conditions are fulfilled:

(a) \(f(t) = 1 - t^{2\alpha} L(t)\) where \(\alpha > 0\) and the function \(L\) is slowly varying at the origin,

(b) there exists a probability measure \(\lambda\) for which
\[
\lim_{t \to \infty} \int_0^\infty f(tx) \lambda(dx) < 1,
\]

(c) the set Conv\((f)\) is closed under pointwise multiplication of functions,

(d) there exists a function \(g \in \text{Conv}(f)\) with the properties \(g(1) = 1\) and \(g(t) < 1\) for \(t \in (1, 2^{1/\alpha})\).

Then \(f(t) = \cos ct^\alpha\) for a certain positive constant \(c\).

**Proof.** By Theorem 2 in [8] conditions (a), (b) and (c) are sufficient for the function \(f\) to be the kernel of a characteristic function of a generalized convolution, say \(\circ\). Moreover, Conv\((f) = \{\hat{\mu}: \mu \in \mathcal{B}\}\). In particular \(g = \hat{\nu}\) for a certain \(\nu \in \mathcal{B}\). By (d) \(1 \in A(\nu)\) and \(A(\nu) \cap (1, \infty) \subset [2^{1/\alpha}, \infty)\) which by (14) yields the inclusion \(N \subset [2^{1/\alpha}, \infty)\). Consequently, \(\eta(\circ) \geq 2^{1/\alpha}\). On the other hand, by (a), \(\chi(\circ) = 2\alpha\). Thus we have the inequality \(\eta(\circ)^{\chi(\circ)} \geq 4\). Applying Theorem 2 we infer that \(\circ = *_{a,1}\) and, consequently, the function \(f\) is similar to the function (2) which completes the proof.
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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
WROCŁAW, POLAND

Reçu par la Rédaction le 15.09.1983