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On some recurrence formulae for the H-function

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1. Fox ([3], p. 408), introduced the H-function in the form of Mellin-Barnes type integral, which has been symbolically denoted by Gupta and Jain [4]

$$(1.1)\quad H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}x^s ds,$$

where $\{(f_r,\nu_r)\}$ stands for the set of the parameters $(f_1,\nu_1),\ldots,(f_r,\nu_r)$; x is not equal to zero and empty product is interpreted as unity; p,q,m and n are integers satisfying $1\leqslant m\leqslant q$; $0\leqslant n\leqslant p$; α_j $(j=1,2,\ldots,p)$, β_j $(j=1,2,\ldots,q)$ are positive numbers and a_j $(j=1,2,\ldots,p)$, b_j $(j=1,2,\ldots,q)$ are complex numbers such that no pole of $\Gamma(b_h-\beta_h s)$ $(h=1,2,\ldots,m)$ coincides with any pole of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,2,\ldots,n)$, i.e.,

$$(1.2)\quad \alpha_i(b_h+\nu)\neq\beta_h(a_i-\eta-1)\qquad (\nu,\eta=0,1,\ldots;\ h=1,2,\ldots,m;\ i=1,2,\ldots,n).$$

Moreover, we assume that (see [2a], pp. 239-240)

$$\mu=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j\geqslant 0,$$

and that the relation

$$0<|x^s|<\prod_{j=1}^{p}\alpha_j^{-\alpha_j}\prod_{j=1}^{q}\beta_j^{\beta_j}\qquad\text{for}\qquad \mu=0$$

holds. T is a contour in the complex s-plane such that the points $s=(b_j+\nu)/\beta_j$ $(j=1,\ldots,m;\ \nu=0,1,\ldots)$, resp. $s=(a_j-1-\nu)/\alpha_j$ $(j=1,\ldots,n;\ \nu=0,1,\ldots)$ lie to the right, resp. left of T, while further T runs from $s=\infty-ik$ to $s=\infty+ik$. Here k is a constant with $k>|\mathrm{Im}\, b_j|/\beta_j$ $(j=1,\ldots,m)$. The conditions for the contour T can be fulfilled on account of (1.2).

In section 2 of this paper we have established some formulae for the derivative of the H-function, keeping in view the symmetry of the parameters involved in the function; by the process of differentiation under the sign of integration. In section 3, we have derived some recurrence formulae for the H-function, with the help of results of section 2.

2. In this section we establish the following formulae:

$$(2.1)\qquad x\frac{d}{dx}\left\{H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]\right\} = \frac{\delta}{\alpha_1}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)\\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right]+ + \frac{\delta(a_1-1)}{\alpha_1}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

where $n \geqslant 1$,

$$(2.2)\qquad x\frac{d}{dx}\left\{H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]\right\} = \frac{\delta(a_p-1)}{\alpha_p}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]- -\frac{\delta}{\alpha_p}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_{p-1},\alpha_{p-1}),(a_p-1,\alpha_p)\\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right]$$

provided $n \leqslant p-1$,

$$(2.3)\qquad x\frac{d}{dx}\left\{H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]\right\} = \frac{b_1\delta}{\beta_1}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]- -\frac{\delta}{\beta_1}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ (1+b_1,\beta_1),(b_2,\beta_2),\dots,(b_q,\beta_q)\end{matrix}\right.\right]$$

where $m \geqslant 1$,

$$(2.4)\qquad x\frac{d}{dx}\left\{H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]\right\} = \frac{\delta b_q}{\beta_q}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]+ +\frac{\delta}{\beta_q}H_{p,q}^{m,n}\left[x^{\delta}\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ (b_1,\beta_1),\dots,(b_{q-1},\beta_{q-1}),(1+b_q,\beta_q)\end{matrix}\right.\right]$$

provided $1 \leqslant m \leqslant q-1$.

Proof. To prove (2.1), expressing the H-function on the left-hand side as Mellin-Barnes type integral (1.1), changing the order of differentiation and integration, which is easily justifiable, we have

$$(2.5)\qquad \delta\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, s x^{\delta s} ds\,,$$

now, in view of

$$s\Gamma(1-a_1+\alpha_1 s)=\frac{1}{\alpha_1}(1-a_1+\alpha_1 s+a_1-1)\Gamma(1-a_1+\alpha_1 s)$$

$$=\frac{1}{\alpha_1}[\Gamma(2-a_1+\alpha_1 s)+(a_1-1)\Gamma(1-a_1+\alpha_1 s)]\,,$$

(2.5) reduces to

$$\frac{\delta}{\alpha_1}\cdot\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\times$$

$$\times[\Gamma(2-a_1+\alpha_1 s)+(a_1-1)\Gamma(1-a_1+\alpha_1 s)]x^{\delta s}ds$$

again using (1.1), the definition of H-function, we get the result (2.1). Similarly, other relations can easily be established.

3. Now, subtracting (2.1) from (2.2), (2.3) and (2.4) we respectively get

$$(3.1)\qquad (a_p\alpha_1-a_1\alpha_p+\alpha_p-\alpha_1)H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right]$$

$$=\alpha_p H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right]+$$

$$+\alpha_1 H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}(a_1,\alpha_1),\dots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right],$$

where $1\leqslant n\leqslant p-1$,

$$(3.2)\qquad (b_1\alpha_1-a_1\beta_1+\beta_1)H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right]$$

$$=\beta_1 H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right]+$$

$$+\alpha_1 H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(1+b_1,\beta_1),(b_2,\beta_2),\dots,(b_q,\beta_q)\end{matrix}\right],$$

provided $n\geqslant 1$, $m\geqslant 1$, and

$$(3.3)\qquad (b_q\alpha_1-a_1\beta_q+\beta_q)H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right]$$

$$=\beta_q H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right]-$$

$$-\alpha_1 H_{p,q}^{m,n}\left[x^\delta\middle|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_{q-1},\beta_{q-1}),(b_q+1,\beta_q)\end{matrix}\right],$$

where $n\geqslant 1$, $1\leqslant m\leqslant q-1$.

Subtracting (2.2) from (2.3) and (2.4), we have

$$(3.4)\qquad (a_p\beta_1 - b_1 \alpha_p - \beta_1)\, H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right.\right]$$

$$= \beta_1 H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_{p-1}, \alpha_{p-1}), (a_p - 1, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right.\right] -$$

$$- \alpha_p H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (1 + b_1, \beta_1), (b_2, \beta_2), \ldots, (b_q, \beta_q) \end{matrix} \right.\right],$$

provided $m \geqslant 1$, $n \leqslant p - 1$, and

$$(3.5)\qquad (a_p\beta_q - b_q \alpha_p - \beta_q)\, H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right.\right]$$

$$= \beta_q H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_{p-1}, \alpha_{p-1}), (a_p - 1, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right.\right] +$$

$$- \alpha_p H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (1 + b_q, \beta_q) \end{matrix} \right.\right],$$

where $1 \leqslant m \leqslant q - 1$, $n \leqslant p - 1$.

Subtracting (2.3) from (2.4), we obtain

$$(3.6)\qquad (b_1\beta_q - b_q\beta_1)\, H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right.\right]$$

$$= \beta_q H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (1 + b_1, \beta_1), (b_2, \beta_2), \ldots, (b_q, \beta_q) \end{matrix} \right.\right] +$$

$$- \beta_1 H_{p,q}^{m,n}\left[x^\delta \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (1 + b_q, \beta_q) \end{matrix} \right.\right],$$

provided $1 \leqslant m \leqslant q - 1$.

Since the H-function is symmetrical in the pairs $(a_1, \alpha_1), \ldots, (a_n, \alpha_n)$; likewise in $(a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)$; in $(b_1, \beta_1), \ldots, (b_m, \beta_m)$ and in $(b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)$, so the results established in sections 2 and 3 can be written in various other forms.

4. Particular cases

(i) (2.1) with $\delta = \alpha_1$, (2.2) with $\delta = \alpha_p$, (2.3) with $\delta = \beta_1$, (2.4) with $\delta = \beta_q$ reduce to results due to author [1].

(ii) Taking $\delta = \alpha_j = \beta_h = 1$ $(j = 1, 2, \ldots, p,\ h = 1, 2, \ldots, q)$ in (2.1), (3.1) and (3.2), we respectively get the known results ([2], p. 210(13), [2], p. 210 (12) and [2], p. 209(11)).

(iii) Setting $\alpha_1 = \beta_q$ and $\delta = 1$ in (3.3), we get the relation as in [4], p. 103, 4 (ii).

(iv) Due to the symmetry of the parameters (a_1, α_1), (a_2, α_2), ..., (a_n, α_n), (2.1) can also be written as

$$(4.1)\qquad x\frac{d}{dx}\left\{H^{m,n}_{p,q}\left[x^\delta\middle|\begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right]\right\}$$

$$= \frac{\delta}{\alpha_2} H^{m,n}_{p,q}\left[x^\delta\middle|\begin{matrix}(a_1, \alpha_1), (a_2-1, \alpha_2), (a_3, \alpha_3), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right] + \frac{\delta(a_2-1)}{\alpha_2} H^{m,n}_{p,q}\left[x^\delta\middle|\begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right],$$

where $n \geqslant 2$.

Now subtracting (4.1) from (2.1) and putting $\alpha_2 = \alpha_1$ we obtain a known formula ([4], p. 103, 4(i)).

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