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On the differentiability of solutions of a functional equation with respect to a parameter

by S. CZERWIK (Katowice)

Introduction. In the present paper we are concerned with the linear functional equation of the first order

$$\varphi[f(x)] = g(x,t)\varphi(x)+F(x,t), \tag{1}$$

where $\varphi(x)$ is an unknown function and $f(x)$, $g(x,t)$, $F(x,t)$ are known real functions of real variables and t is a real parameter.

In Section 1 we show that under some assumptions concerning the given functions $f(x)$, $g(x,t)$ and $F(x,t)$, the solution $\varphi(x,t)$ of equation (1) which is continuous with respect to x is also continuous with respect to the couple of variables (x,t).

In Section 2 we shall prove that this solution has a continuous derivative $\partial\varphi/\partial t$ and in Section 3 that $\varphi(x,t)$ is of class C^p, $1 \leqslant p \leqslant \infty$ with respect to the parameter t.

For the natural parameter the continuous dependence on given functions of solutions of equation (1) has been investigated in [3] and [2] and for the more general equation

$$\varphi(x) = H_n\big(x, \varphi[f_n(x)]\big)$$

in [4] (under different assumptions).

1. Solutions of class C^0 with respect to the parameter. The functions $f(x)$, $g(x,t)$, $F(x,t)$ will be subjected to the following conditions:

(i) The function $f(x)$ is continuous and strictly increasing in an interval $\langle a,b)$, $a < f(x) < x$ in (a,b), $f(a) = a$.

(ii) The function $g(x,t)$ is continuous in $\langle a,b)\times T$, where T is an interval (finite or not), $g(x,t) \neq 0$ in $\langle a,b)\times T$.

(iii) The function $F(x,t)$ is continuous in $\langle a,b)\times T$.

Let us introduce the notation:

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)],$$

$$G_n(x, t) = \prod_{v=0}^{n-1} g[f^v(x), t], \tag{2}$$

$$G(x, t) = \lim_{n\to\infty} G_n(x, t), \tag{3}$$

Definition. We say that a function $H(x, t)$ is *locally bounded* in $\langle a, b)\times T$, if it is bounded in every compact set of the form $\langle a, d\rangle \times \times \langle \alpha, \beta\rangle$, where $\langle a, d\rangle \subset \langle a, b)$ and $\langle \alpha, \beta\rangle \subset T$.

Suppose that:

(iv) There exist constants θ and η, $0 < \theta < 1$, $0 < \eta < b-a$, and a locally bounded function $H_1(x, t)$ such that the inequalities

$$|g(x, t)-1| \leqslant H_1(x, t), \tag{4}$$

$$H_1[f(x), t] \leqslant \theta H_1(x, t), \tag{5}$$

hold in $\langle a, a+\eta)\times T$.

(v) There exist a locally bounded function $H_2(x, t)$ and a constant $0 < \eta_0 < b-a$ such that the inequalities

$$|F(x, t)| \leqslant H_2(x, t), \tag{6}$$

$$H_2[f(x), t] \leqslant \theta H_2(x, t) \tag{7}$$

hold in $\langle a, a+\eta_0)\times T$ (θ is the constant occurring in (5)).

Theorem 1. *Suppose that hypotheses* (i)-(v) *are fulfilled. Then, for every function $c(t)$ continuous in T, there exists exactly one function $\varphi(x, t)$, continuous in $\langle a, b)\times T$, satisfying equation* (1) *and fulfilling the condition $\varphi(a, t) = c(t)$. It is given by the formula*

$$\varphi(x, t) = \varphi_0(x, t) + \frac{c(t)}{G(x, t)}, \tag{8}$$

where

$$\varphi_0(x, t) = -\sum_{n=0}^{\infty} \frac{F[f^n(x), t]}{G_{n+1}(x, t)}. \tag{9}$$

The proof of the above theorem does not differ essentially from that given in [1] (Theorems 6 and 5) and is therefore omitted. Let us note that solution (8) is also the unique solution of equation (1) which, for every $t\in T$, is continuous with respect to x in $\langle a, b)$ (cf. [1]).

Theorem 2. *Suppose that hypotheses* (i)-(iii) *are fulfilled. If, moreover,*

$$|g(a, t)| > 1 \quad \textit{for } t\in T, \tag{10}$$

then equation (1) *has exactly one solution* $\varphi(x, t)$ *continuous in* $\langle a, b) \times T$, *given by the formula*

$$\varphi(x, t) = -\sum_{n=0}^{\infty} \frac{F[f^n(x), t]}{G_{n+1}(x, t)}. \tag{11}$$

Proof. It is enough to prove that series (11) uniformly converges in every interval $\langle a, c\rangle \times \langle \alpha, \beta\rangle$, where $a < c < b$ and $\langle \alpha, \beta\rangle \subset T$, which may be done quite similarly as in [3], p. 54, and thus we do not enter into details here.

Remark 1. Theorems 1 and 2 are also true for the equation

$$\varphi[f(x)] = g(x, t_1, \ldots, t_n)\varphi(x) + F(x, t_1, \ldots, t_n). \tag{12}$$

2. Solutions of class C^1 with respect to the parameter. The following result is known (cf. [4]):

LEMMA. *Let R be a convex region with respect to $y_1, \ldots, y_n$ in the space of the variables $(x_1, \ldots, x_m, y_1, \ldots, y_n)$ and let the function $f(x_1, \ldots, x_m, y_1, \ldots, y_n)$ be of class C^p, $p > 0$, with respect to $y_1, \ldots, y_n$ in R. Then there exist n functions $\Phi_i(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_n)$, $i = 1, 2, \ldots, n$, of class C^{p-1} with respect to $y_1, \ldots, y_n, z_1, \ldots, z_n$ and such that*

$$f(x_1, \ldots, x_m, z_1, \ldots, z_n) - f(x_1, \ldots, x_m, y_1, \ldots, y_n) = \sum_{i=1}^{n} \Phi_i(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_n)\cdot(z_i - y_i).$$

They are of the form

$$\Phi_i(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_n) = \int_0^1 f_i[x_1, \ldots, x_m, y_1 + s(z_1 - y_1), \ldots, y_n + s(z_n - y_n)]\, ds,$$

where

$$f_i = \frac{\partial f(x_1, \ldots, x_m, y_1, \ldots, y_n)}{\partial y_i}, \quad i = 1, 2, \ldots, n.$$

In the sequel we assume:

(vi) There exist the derivative $\partial g(x, t)/\partial t$ continuous in $\langle a, b) \times T$ and a locally bounded function $B_1(x, t)$ continuous with respect to the variable t and a constant $0 < \eta_1 < b - a$ such that the inequalities

$$\left|\frac{\partial g}{\partial t}\right| \leqslant B_1(x, t), \qquad B_1[f(x), t] \leqslant \theta B_1(x, t) \tag{13}$$

hold in $\langle a, a + \eta_1) \times T$.

(vii) There exist the derivative $\partial F/\partial t$ continuous in $\langle a, b)\times T$ and a locally bounded function $D_1(x, t)$ continuous with respect to the variable t and a constant $0<\varrho_1<b-a$ such that the inequalities

$$\left|\frac{\partial F}{\partial t}\right| \leqslant D_1(x, t), \qquad D_1[f(x), t] \leqslant \theta D_1(x, t) \tag{14}$$

hold in $\langle a, a+\varrho_1)\times T$.

Now we shall prove

THEOREM 3. *Suppose that hypotheses* (i)-(vii) *are fulfilled and H_1 is continuous with respect to t. If $c(t)$ is function of class C^1 in T, then the continuous solution $\varphi(x, t)$ of equation* (1), *fulfilling the condition $\varphi(a, t) = c(t)$ has the derivative $\partial\varphi/\partial t$ continuous in $\langle a, b)\times T$.*

Proof. On account of Theorem 1, there exists exactly one function $\varphi(x, t)$ satisfying equation (1), continuous in $\langle a, b)\times T$ and fulfilling the condition $\varphi(a, t) = c(t)$. Let $t\in T$ be fixed and let Δt vary over a compact interval $\langle \tau_1, \tau_2\rangle$ such that $t\in\langle t+\tau_1, t+\tau_2\rangle\subset T$. The function $\varphi(x, t+\Delta t)$ satisfies the equation

$$\varphi[f(x), t+\Delta t] = g(x, t+\Delta t)\varphi(x, t+\Delta t)+F(x, t+\Delta t). \tag{15}$$

By (15) and (1) we obtain

$$\varphi[f(x), t+\Delta t]-\varphi[f(x), t]$$
$$= g(x, t+\Delta t)\varphi(x, t+\Delta t)-g(x, t)\varphi(x, t)+F(x, t+\Delta t)-F(x, t).$$

We write

$$f(x, y_1, y_2) = g(x, y_1)\cdot y_2, \qquad \bar{f}(x, y_1, y_2) = F(x, y_1),$$

$$y_1 = t, \quad y_2 = \varphi(x, t), \quad z_1 = t+\Delta t, \quad z_2 = \varphi(x, t+\Delta t).$$

Then

$$\varphi[f(x), t+\Delta t]-\varphi[f(x), t]$$
$$= f(x, z_1, z_2)-f(x, y_1, y_2)+\bar{f}(x, z_1, z_2)-\bar{f}(x, y_1, y_2),$$

and according to the lemma, we obtain

$$\varphi[f(x), t+\Delta t]-\varphi[f(x), t] \tag{16}$$
$$= \Phi_2\cdot[\varphi(x, t+\Delta t)-\varphi(x, t)]+\Phi_1\cdot\Delta t+\Psi_1\cdot\Delta t,$$

where

$$\Phi_1 = \Phi_1(x, t, \Delta t) = \int_0^1 f_1[x, y_1+s(z_1-y_1), y_2+s(z_2-y_2)]\,ds$$

$$= \int_0^1 \{\varphi(x,t)+s[\varphi(x,t+\Delta t)-\varphi(x,t)]\}g_t(x, t+s\Delta t)\,ds,$$

$$\Phi_2 = \Phi_2(x, t, \Delta t) = \int_0^1 f_2[x, y_1+s(z_1-y_1), y_2+s(z_2-y_2)]\,ds$$

$$= \int_0^1 g(x, t+s\Delta t)\,ds,$$

$$\Psi_1 = \Psi_1(x, t, \Delta t) = \int_0^1 F_t(x, t+s\Delta t)\,ds.$$

Let us write (t being fixed)

$$\varphi[x, t+\Delta t]-\varphi(x, t) = \psi(x, \Delta t),$$

and then we obtain (according to (26))

$$\Psi[f(x), \Delta t] = \Phi_2\cdot\Psi(x, \Delta t)+(\Phi_1+\Psi_1)\cdot\Delta t, \tag{17}$$

or, for $\Delta t \neq 0$,

$$\frac{\Psi[f(x), \Delta t]}{\Delta t} = \Phi_2\cdot\frac{\psi(x, \Delta t)}{\Delta t}+\Phi_1+\Psi_1. \tag{18}$$

Putting

$$\frac{\Psi(x, \Delta t)}{\Delta t} = \gamma(x, \Delta t),$$

we have

$$\gamma[f(x), \Delta t] = \Phi_2\cdot\gamma(x, \Delta t)+\Phi_1+\Psi_1. \tag{19}$$

The function $\varphi(x, t)$ is continuous, whence it follows that the functions Φ_1, Φ_2, Ψ_1 are continuous with respect to x and Δt. Next $\Phi_2(a, t, \Delta t) \equiv 1$, and

$$|\Phi_2-1| \leqslant \int_0^1 |g(x, t+s\cdot\Delta t)-1|\,ds \leqslant \int_0^1 H_1(x, t+s\cdot\Delta t)\,ds \stackrel{\mathrm{df}}{=} \overline{H}_1(x, \Delta t)$$

for $x \in \langle a, a+\eta)$.

Suppose that $a < d < a+\eta_1$. Then

$$|\varphi(x, \tau)| \leqslant L \quad \text{in } \langle a, d\rangle\times\langle t+\tau_1, t+\tau_2\rangle$$

and

$$(20)\qquad |\Phi_1| \leqslant \int_0^1 (L+2L\cdot s)|g_t(x, t+s\cdot\Delta t)|ds$$
$$\leqslant 3L\int_0^1 B_1(x, t+s\cdot\Delta t)\,ds \overset{\mathrm{df}}{=} \bar{B}_1(x, \Delta t).$$

Next

$$(21)\qquad |\Psi_1| = \left|\int_0^1 F_t ds\right| \leqslant \int_0^1 D_1(x, t+s\cdot\Delta t)\,ds = \bar{D}_1(x, \Delta t).$$

It is easily seen that

$$(22)\qquad \begin{aligned} \bar{H}_1[f(x), \Delta t] &\leqslant \theta\bar{H}_1(x, \Delta t),\\ \bar{B}_1[f(x), \Delta t] &\leqslant \theta\bar{B}_1(x, \Delta t),\\ \bar{D}_1[f(x), \Delta t] &\leqslant \theta\bar{D}_1(x, \Delta t)\end{aligned}$$

in a neighbourhood of the point a.

If $\Delta t = 0$, let $\bar{\gamma}(x)$ be the continuous solution of the equation

$$(23)\qquad \bar{\gamma}[f(x)] = g(x, t)\bar{\gamma}(x)+\varphi(x, t)g_t(x, t)+F_t(x, t)$$

fulfilling the condition $\bar{\gamma}(a) = c'(t)$.

According to Theorem 1 this solution exists and is unique. We shall prove that the function $\gamma(x, \Delta t)$, defined for $\Delta t = 0$ to be equal to $\bar{\gamma}(x)$, is continuous also for $\Delta t = 0$.

The functions Φ_2, Φ_1 and Ψ_1, occurring as the coefficients in equation (19), are continuous for $x\in\langle a, b)$ and small Δt with

$$(24)\qquad \begin{gathered}\Phi_2(x, 0) = g(x, t), \qquad \Phi_1(x, 0) = \varphi(x, t)g_t(x, t),\\ \Psi_1(x, 0) = F_t(x, t).\end{gathered}$$

In wiew of (24), equation (23) is the limit case of (19). Theorem 1 applied to equation (19) yields the continuity of $\gamma(x, \Delta t)$ also for $\Delta t = 0$, i.e., we have

$$\lim_{\Delta t\to 0} \gamma(x, \Delta t) = \bar{\gamma}(x),$$

or, what ammounts to the same

$$\lim_{\Delta t\to 0} \frac{\psi(x, \Delta t)}{\Delta t} = \bar{\gamma}(x).$$

Next, from (18), passing to the limit as $\Delta t \to 0$, we obtain

$$(25)\qquad \frac{\partial\varphi[f(x), t]}{\partial t} = g(x, t)\frac{\partial\varphi(x, t)}{\partial t}+\varphi(x, t)\frac{\partial g(x, t)}{\partial t}+\frac{\partial F(x, t)}{\partial t}$$

whence, again on account of Theorem 1, it follows that the function $\partial\varphi(x, t)/\partial t$ is continuous in $\langle a, b)\times T$, which was to be proved.

THEOREM 4. *Let hypotheses of Theorem 2 be fulfilled. If, moreover, there exist the derivatives $\partial g(x,t)/\partial t$ and $\partial F(x,t)/\partial t$ continuous in $\langle a,b)\times T$, then the continuous solution $\varphi(x,t)$ of equation* (1) *possesses the derivative $\partial\varphi/\partial t$ continuous in $\langle a,b)\times T$.*

The proof of the above theorem does not differ from that given for Theorem 3 and is therefore omitted.

Remark 2. Theorems 3 and 4 are also true for equation (12).

3. Solutions of class C^p, $1\leqslant p\leqslant\infty$, with respect to the parameter. We assume:

(viii) There exist the derivatives $\partial^i F(x,t)/\partial t^i$ and $\partial^i g(x,t)/\partial t^i$, $i=1,2,\ldots,p$, continuous in $\langle a,b)\times T$, and locally bounded functions $B_i(x,t)$ and $D_i(x,t)$ continuous with respect to the variable t and constants $0<\eta_i<b-a$, $0<\varrho_i<b-a$ such that the inequalities

$$\left|\frac{\partial^i F(x,t)}{\partial t^i}\right|\leqslant D_i(x,t),\qquad D_i[f(x),t]\leqslant\theta D_i(x,t)\tag{26}$$

hold in $\langle a,a+\eta_i)\times T$, and

$$\left|\frac{\partial^i g(x,t)}{\partial t^i}\right|\leqslant B_i(x,t),\qquad B_i[f(x),t]\leqslant\theta B_i(x,t)\tag{27}$$

in $\langle a,a+\varrho_i)\times T$, $i=1,2,\ldots,p$.

Now we shall prove the following

THEOREM 5. *Suppose that hypotheses* (i)-(v) *and* (viii) *are fulfilled and H_1 is continuous with respect to t. If $c(t)$ is a function of class C^p in T, then the continuous solution $\varphi(x,t)$ of equation* (1), *fulfilling the condition $\varphi(a,t)=c(t)$ has the derivatives $\partial^i\varphi(x,t)/\partial t^i$, $i=1,\ldots,p$, continuous in $\langle a,b)\times T$.*

Proof. The proof will be by induction. For $p=1$ the theorem follows from Theorem 3. Assuming its validity for an r, $1\leqslant r<p$, we have

$$\frac{\partial^r\varphi[f(x),t]}{\partial t^r}=\frac{\partial^r[g(x,t)\varphi(x,t)]}{\partial t^r}+\frac{\partial^r F(x,t)}{\partial t^r},$$

or

$$\frac{\partial^r\varphi[f(x),t]}{\partial t^r}=\frac{\partial^r\varphi(x,t)}{\partial t}g(x,t)+\sum_{i=1}^{r}\binom{r}{i}\frac{\partial^i g(x,t)}{\partial t^i}\cdot\frac{\partial^{r-i}\varphi(x,t)}{\partial t^{r-i}}+\frac{\partial^r F(x,t)}{\partial t^r}.$$

We put:

$$\begin{gathered}\Phi_3(x,t)=\sum_{i=1}^{r}\binom{r}{i}\frac{\partial^i g(x,t)}{\partial t^i}\cdot\frac{\partial^{r-i}\varphi(x,t)}{\partial t^{r-i}},\\ \Psi_1(x,t)=\frac{\partial^r F(x,t)}{\partial t^r},\qquad a(x,t)=\frac{\partial^r\varphi(x,t)}{\partial t^r}.\end{gathered}\tag{28}$$

According to (28), we obtain

$$a[f(x), t] = g(x, t)\,a(x, t) + \Phi_3(x, t) + \Psi_1(x, t). \tag{29}$$

On account of relation (viii) for $i = 1, 2, \ldots, r+1$, we obtain

$$\left|\frac{\partial \psi_1(x, t)}{\partial t}\right| \leqslant D_{r+1}(x, t)$$

and

$$\left|\frac{\partial \Phi_3(x, t)}{\partial t}\right| \leqslant A_1(t)\cdot B_1(x, t) + \ldots + A_{r+1}(t) B_{r+1}(x, t)$$

for $\langle a, a+\varrho\rangle \times T$, $0 < \varrho < b-a$, where

$$A_i(t) = \sup_{\langle a, a+\varrho\rangle} \left|\frac{\partial^{r+1-i}\varphi(x, t)}{\partial t^{r+1-i}}\right| \cdot \binom{r+1}{i}, \quad i = 2, \ldots, r+1,$$

$$A_1(t) = \sup_{\langle a, a+\varrho\rangle} \left|\frac{\partial^{r}\varphi(x, t)}{\partial t^{r}}\right| \cdot r,$$

are continuous in T.

Finally, we have

$$\left|\frac{\partial \Phi_3(x, t)}{\partial t} + \frac{\partial \Psi_1(x, t)}{\partial t}\right| \leqslant D_{r+1}(x, t) + A_1(t)\cdot B_1(x, t) + \ldots$$
$$\ldots + A_{r+1}(t)\cdot B_{r+1}(x, t)$$

for $\langle a, a+\varrho\rangle \times T$ and consequently the theorem follows from Theorem 3.

THEOREM 6. *Let hypotheses of Theorem 2 be fulfilled. If, moreover, there exist the derivatives* $\partial^i F(x, t)/\partial t^i$ *and* $\partial^i g(x, t)/\partial t^i$, $i = 1, 2, \ldots, p$, $1 \leqslant p \leqslant \infty$, *continuous in* $\langle a, b) \times T$, *then the continuous solution* $\varphi(x, t)$ *of equation* (1) *possesses the derivatives* $\partial^i \varphi(x, t)/\partial t^i$, $i = 1, 2, \ldots, p$, *continuous in* $\langle a, b) \times T$.

The proof is analogous to the proof of Theorem 5, and is therefore omitted.

Remark 3. Theorems 5 and 6 are also true for equation (12).

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