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Differential operators of gradient type associated with spherical harmonics

by ALEKSANDER STRASBURGER (Warszawa)

Abstract. The paper develops ideas of Stein and Weiss and Reimann concerning calculus of differential operators of gradient type associated to spherical harmonics. In particular, compositions of these operators are calculated and intrinsically characterized. For M-invariant differential operators acting on functions with values in the space of spherical harmonics the notion of the radial part is introduced in a functorial way resembling the construction of Helgason in the scalar case. As applications new proofs of certain classical identities involving spherical harmonics are obtained, e.g. the Maxwell representation of spherical harmonics and its generalizations and the identity of Hecke–Bochner.

Introduction. This paper is concerned with a certain class of partial differential operators and its applications in euclidean harmonic analysis and in the theory of special functions. It is also closely connected with the theory of representations of the (special orthogonal) group $\mathbf{SO}(d)$ in that it essentially relies on the representation-theoretic interpretation of spherical harmonics. This class of operators, now called operators of gradient type, was singled out in view of its remarkable invariance properties in the classical paper of Stein and Weiss [20] and has recently received considerable attention in the works of Ahlfors [1] and Reimann [17], [18]. In the context of the analysis on symmetric spaces the analogs of these operators also play a significant role (cf. [9] and the references there, and related papers of the present author [21], [22]).

Here we develop some kind of explicit calculus for these operators in pursuing the line of investigations started by Reimann [18]. To describe the content of the paper in more detail let $\mathscr{H}^l$ denote the space of harmonic polynomial functions on $\boldsymbol{R}^d$ homogeneous of degree l, and $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$ the space of smooth functions on $\boldsymbol{R}^d$ with values in $\mathscr{H}^l$. Any function $\Phi \in \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$ can be regarded as a function of two variables $x, z \in \boldsymbol{R}^d$, and we shall write

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$\Phi(x, z)$ for $\Phi(x)(z)$, the value of $\Phi(x)$ at $z \in \boldsymbol{R}^d$. The gradient type operators of Stein and Weiss

$$S_k\colon \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^k) \to \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^{k+1})$$

and their adjoints

$$S_k^*\colon \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^k) \to \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^{k-1})$$

are shown here to be related by the formula

$$\nabla_z \Phi(x, z) = (S_k \Phi)(x, z) + \|z\|^2 (S_k^* \Phi)(x, z),$$

where ∇_z is the directional derivative and the decomposition (pointwise with respect to x) on the right-hand side is the decomposition of a homogeneous polynomial (in z) into its harmonic projection and the orthogonal complement. Moreover, the composition $S_{l-1} \circ \ldots \circ S_0$ is shown to be given by

$$S_{l-1} \circ \ldots \circ S_0 f(x, z) = \sum_{i=1}^{d(l)} Y_i(\partial) f(x) Y_i(z), \qquad f \in \mathscr{E}(\mathscr{R}^d)$$

for any suitable basis $\{Y_i\}$ for the space $\mathscr{H}^l$. Here, of course, $P(\partial)$ is the differential operator corresponding to a polynomial P. As a result one has for any $P \in \mathscr{H}^l$

$$(D^l f(x) | \bar{P}) = P(\partial) f(x),$$

where D^l denotes a suitable scalar multiple of $S_{l-1} \circ \ldots \circ S_0$; this comes as close as possible to a factorization of $P(\partial)$ into first order $\mathbf{SO}(d)$-invariant differential operators.

When this is further combined with an extension of the notion of the radial part of a differential operator, which we introduce in Section 1 summarizing the arguments of the forthcoming paper of the author [24], it leads, somewhat surprisingly, to a new proof of an old and apparently forgotten formula of Hobson (cf. [12], p. 126, (6)) on derivatives of radial functions (cf. Proposition 3.1) and to its various applications which we give in Section 3.

These include a proof of the Hecke–Bochner formula and the theorem stating that the Hermite–Weber functions are eigenfunctions of the Fourier transform, a generalization to polyharmonic polynomials of the Maxwell construction of spherical harmonics and some more. The proofs we give are as simple and direct as possible. The rather intimidating amount of differentiation needed for direct proofs of the mentioned results is now disposed of completely with the use of the formula of Hobson. We get as a by-product a formula on the Hermite polynomials in several variables which does not seem to appear in the standard sources.

For more comments on the applications given here an interested reader can refer to the beginning of Section 3 and for a comparison with another recent presentation of some of these results to the course by Faraut [8].

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1. $\mathbf{SO}(d)$-invariants and radial parts of invariant operators. Consider the space $\boldsymbol{R}^d$ with the inner product $(\cdot|\cdot)$ and the corresponding norm $\|\cdot\|$ and let $S^{d-1} \subset \boldsymbol{R}^d$ be the unit sphere. Let $M = \mathbf{SO}(d)$, the special orthogonal group in d dimensions, act on $\boldsymbol{R}^d$ (on the left) in the usual way, and let M_1 denote the stabilizer of the unit vector e_1, $\{e_i\}$ being the standard basis for $\boldsymbol{R}^d$. The map $M \ni m \to me_1 \in S^{d-1}$ is used to identify the homogeneous space M/M_1 with S^{d-1}.

Let us briefly recall some basic facts concerning representations of class 1 of the group $\mathbf{SO}(d)$—more details can be found e.g. in [2] or [3].

For any $l \in \boldsymbol{Z}_+$, $\boldsymbol{Z}_+$ denoting the nonnegative integers, let $\mathscr{P}^l = \mathscr{P}^l_C(\boldsymbol{R}^d)$ denote the space of complex-valued homogeneous polynomial functions of degree l on $\boldsymbol{R}^d$ and $\mathscr{H}^l = \mathscr{H}^k_C(\boldsymbol{R}^d) \subset \mathscr{P}^l$ the subspace of harmonic polynomial functions. The formula

$$T^l(m)P(x) := P(m^{-1}x), \qquad x \in \boldsymbol{R}^d,$$

defines an irreducible representation T^l of M on $\mathscr{H}^l$. The representations T^l are mutually inequivalent and for each $l \in \boldsymbol{Z}_+$ the space $\mathscr{H}^l$ contains a unique (up to a scalar factor) element which is invariant under the restriction of T^l to M_1. Moreover, these are the only representations of M with that property. The unique M_1-invariant function in $\mathscr{H}^l$ assuming the value 1 at the point e_1 is denoted by Z^l and is known to be given by

$$Z^l(x) = \sum_{k=0}^{[l/2]} d^l_k x_1^{l-2k} \|x\|^{2k}, \tag{1.1}$$

where the coefficients d^l_k are

$$d^l_0 = 2^l \frac{\Gamma(d-2)\Gamma(l+(d-2)/2)}{\Gamma((d-2)/2)\Gamma(l+d-2)},$$

$$d^l_k = (-1)^k \frac{l(l-1)\dots(l-2k+1)}{2^k k!(d+2l-4)\dots(d+2l-2k-2)} d^l_0, \qquad k > 0$$

(cf. e.g. [25], Ch. IX, § 3.1).

An M-invariant product on $\mathscr{P}^l$ is introduced by

$$(1.2a)\qquad (P|Q) := \int_{S^{d-1}} P(b)\overline{Q(b)}d\sigma(b),$$

where by $d\sigma(\cdot)$ we denote the euclidean volume element on S^{d-1} normalized by $\int d\sigma = 1$. We shall also use the M-invariant bilinear form on $\mathscr{P}^l$ defined by

$$(1.2b)\qquad [P, Q] := (P\,|\,\bar{Q}).$$

Now set $x = \|x\|\, me_1$ and define

$$(1.3)\qquad Z_x^l = \|x\|^l T^l(m)Z^l.$$

Then with the notation $d(l) = \dim \mathscr{H}^l$ we have the well-known reproducing property of Z_x^l.

LEMMA 1.1. *If* $P \in \mathscr{H}^l$, *then*

$$(1.4)\qquad d(l)[Z_x^l, P] = P(x), \quad x \in \boldsymbol{R}^d.$$

Consider now the action of M on the space $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$ of smooth $\mathscr{H}^l$-valued functions on $\boldsymbol{R}^d$ defined by

$$m\Phi(x) := T^l(m)\Phi(m^{-1}x), \quad \Phi \in \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l),$$

and let $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)^M$ denote the subspace of M- invariant elements in $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$. Clearly each $\Phi \in \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)^M$ is determined by its restriction to the axis $\boldsymbol{R}e_1$; moreover, for each $t \in \boldsymbol{R}$, $\Phi(te_1) \in (\mathscr{H}^l)^{M_1}$ and hence $\Phi(te_1) = \varphi(t)Z^l$ for some (complex-valued) function φ. By (1.4) we have

$$\varphi(t) := d(l)[\Phi(te_1), Z^l],$$

and by M-invariance

$$(1.5)\qquad \Phi(mte_1) = \varphi(t)T^l(m)Z^l.$$

Thus any M-invariant function from $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$ is uniquely determined by a complex-valued function on $\boldsymbol{R}$, which is essentially its restriction to the axis $\boldsymbol{R}e_1$. However, as we shall see (cf. (1.8) below), this restriction cannot be completely arbitrary.

Let $\mathscr{E}_+(\boldsymbol{R})$ denote the subspace of $\mathscr{E}(\boldsymbol{R})$ consisting of even functions endowed with the subspace topology. It is classical ([26], see also [19] for a detailed description of the topological part of this statement) that $\mathscr{E}_+(\boldsymbol{R})$ is isomorphic via the map $f \to \varphi$, where $\varphi(s) := f(s^{1/2})$, to the space $\mathscr{E}(\bar{\boldsymbol{R}}_+)$ of all smooth functions on the closed half line $\bar{\boldsymbol{R}}_+ := \boldsymbol{R}_+ \cup \{0\}$.

For the proof of the following result and its extension to distributions we refer the reader to the forthcoming article of the author [24].

PROPOSITION 1.2. *Assume* $d > 2$ *and let* $l \in \boldsymbol{Z}_+$. *For any* $f \in \mathscr{E}_+(\boldsymbol{R})$ *define an* $\mathscr{H}^l$-*valued function* $\theta^{l*}f$ *on* $\boldsymbol{R}^d$ *by*

$$(1.6)\qquad \theta^{l*}f(x) := d(l)^{1/2} f(\|x\|)Z_x^l.$$

Then $\theta^{l*} f \in \mathscr{E}(\boldsymbol{R}^d;\ \mathscr{H}^l)^M$ *and the map*

$$\theta^{l*} := \mathscr{E}_+(\boldsymbol{R}) \to \mathscr{E}(\boldsymbol{R}^d;\ \mathscr{H}^l)^M$$

is a topological isomorphism with respect to the subspace topology. ∎

Now let $\{Y_i\}$ be a basis for $\mathscr{H}^l$ orthonormal with respect to $[\cdot, \cdot]$ and such that Y_1 is M_1-invariant and hence $Y_1 = d(l)^{1/2} Z^l$. Proposition 1.2 together with the addition formula

$$d(l) Z^l(x, z) = \sum_{i=1}^{d(l)} Y_i(x) Y_i(z) \tag{1.7}$$

implies that for any $\Phi \in \mathscr{E}(\boldsymbol{R}^d;\ \mathscr{H}^l)^M$ there exists $f \in \mathscr{E}_+(\boldsymbol{R})$ such that the coordinate functions of Φ with respect to $\{Y_i\}$ are

$$\Phi_i(x) = d(l)^{-1/2} f(\|x\|) Y_i(x). \tag{1.8}$$

This f is related to φ introduced above by

$$\varphi(t) = f(t) t^l, \qquad t \in \boldsymbol{R}. \tag{1.9}$$

Remark 1. The description of M-invariant $\mathscr{H}^l$-valued continuous functions on $\boldsymbol{R}^d$ in the form (1.8) is due to Coifman and Weiss, see Theorem 6.6 of [2]. The passage to the smooth case is obtained by noting that (1.9) is a necessary and sufficient condition for the function Φ defined by (1.5) to be smooth.

Remark 2. The assumption $d > 2$ is necessary since we are working with the group $\mathbf{SO}(d)$. However, if for $d = 2$ one takes $M = \mathbf{O}(2)$ instead of $M = \mathbf{SO}(2)$, then the above characterization of M-invariant functions remains true.

If f is any function on $\boldsymbol{R}_+$, then (1.6) may be used to define a unique M-invariant function on $\boldsymbol{R}^d_* = \boldsymbol{R}^d - \{0\}$ which restricts on the half line $\boldsymbol{R}_+$ to the function $t \to t^l f(t) Z^l$. (Here and whenever convenient in the sequel we shall identify $\boldsymbol{R}$ with the axis $\boldsymbol{R}e_1 \subset \boldsymbol{R}^d$.) Abusing slightly notation we shall write

$$\theta^{l*} f(x) = d(l)^{1/2} f(\|x\|) Z^l_x$$

for any function f defined on $\boldsymbol{R}_+$ and note that also this map is an isomorphism of $\mathscr{E}(\boldsymbol{R}_+)$ with $\mathscr{E}(\boldsymbol{R}^d_*;\ \mathscr{H}^l)^M$.

In the case $l = 0$ we shall write θ^* instead of θ^{0*}. Note that for any $f,\ g \in \mathscr{E}_+(\boldsymbol{R})$ we have

$$\theta^{l*}(fg) = \theta^*(f) \theta^{l*}(g),$$

i.e. the structure of an $\mathscr{E}(\boldsymbol{R}^d)^M$-module on $\mathscr{E}(\boldsymbol{R}^d;\ \mathscr{H}^l)^M$ corresponds under the map θ^{l*} to the pointwise multiplication in $\mathscr{E}_+(\boldsymbol{R})$, cf. [16].

Let us now consider an M-invariant differential operator D: $\mathscr{E}(\boldsymbol{R}^d;\ \mathscr{H}^l)$

$\to \mathscr{E}(\boldsymbol{R}^d, \mathscr{H}^k)$ with smooth coefficients; we recall that an operator is called *M-invariant* if it commutes with the action of M. Clearly D maps $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)^M$ into $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^k)^M$ and hence yields a unique mapping, say $\varrho(D)$: $\mathscr{E}_+(\boldsymbol{R}) \to \mathscr{E}_+(\boldsymbol{R})$, such that

$$D\theta^{l*} f = \theta^{k*} \varrho(D) f, \quad f \in \mathscr{E}_+(\boldsymbol{R}). \tag{1.10}$$

In other words, the diagram below commutes.

$$\begin{array}{ccc} \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l) & \xrightarrow{D} & \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^k) \\ {\scriptstyle \theta^{l*}}\uparrow & & {\scriptstyle \theta^{k*}}\uparrow \\ \mathscr{E}_+(\boldsymbol{R}) & \xrightarrow{\varrho(D)} & \mathscr{E}_+(\boldsymbol{R}) \end{array}$$

Considering the diagram in the particular case of a scalar differential operator (by that we mean an operator of type D: $\mathscr{E}(\boldsymbol{R}^d) \to \mathscr{E}(\boldsymbol{R}^d)$) one sees that the definition of $\varrho(D)$ amounts to the familiar construction of the radial part of a differential operator as described e.g. in [10], Ch. I, § 2. In fact, one sees easily that with respect to the action of M on $\boldsymbol{R}^d_*$ the punctured axis $\boldsymbol{R}_*$ satisfies the transversality condition, hence there exists a unique differential operator $\Delta(D)$ on $\boldsymbol{R}_*$ such that

$$(Df)|_{\boldsymbol{R}_*} = \Delta(D)(f|_{\boldsymbol{R}_*})$$

for each locally invariant function on an open subset of $\boldsymbol{R}^d_*$. Recall that a function f is called *locally invariant* if $X^+ f = 0$ for each vector field X^+ on $\boldsymbol{R}^d_*$ induced by the action of M. The operator $\Delta(D)$ is called the *radial part* of D. Now, since D is assumed M-invariant it is easy to see that its radial part is even (in an obvious sense) and hence $D\theta^* f = \theta^* \Delta(D) f$ in agreement with (1.10).

Remark. The commutativity property of the diagram above may also hold for other M-invariant operators than just the differential ones (perhaps with an appropriate modification of the function spaces used). For example if the operator D is replaced by the Fourier transform of $\mathscr{H}^l$-valued functions and the Schwartz spaces $\mathscr{S}(\boldsymbol{R}^d; \mathscr{H}^l)$ rather than $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$ are used, then the diagram may serve as the definition of the Hankel transform (cf. eg. [11], Ch. 5, Th. 7 and Th. 7bis or [2], Th. 6.9). This provides the background for our discussion of the Bochner-type identities in the last section of this paper.

In the remaining part of this section we shall sketch a simple, coordinate-free description of the map $\varrho(D)$ in terms of a differential operator on $\boldsymbol{R}_*$ (usually singular at 0). However, it should be noted that (1.10) defines $\varrho(D)$ on even functions only and hence does not alone suffice to determine the corresponding differential operator uniquely. Therefore we shall further require that the differential operator itself be even, which determines it completely.

So assume D: $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l) \to \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^k)$ is an M-invariant differential operator with smooth coefficients. For any $f \in \mathscr{E}(\boldsymbol{R}_+)$ the function $\theta^{l*}(f) \in \mathscr{E}(\boldsymbol{R}^d_*; \mathscr{H}^l)^M$ and therefore

$$D\theta^{l*}(f)(te_1) = d(k)^{1/2} h(t) t^k Z^k, \quad t \in \boldsymbol{R}_+,$$

for a unique smooth function $h(\cdot)$ on $\boldsymbol{R}_+$. Denoting the map sending f to h by $\varrho_+(D)$ we have

$$\varrho_+(D)f(t) = d(k)^{1/2}t^{-k}[D\theta^{l*}(f)(te_1), Z^k], \quad t \in \boldsymbol{R}_+, \tag{1.11}$$

which shows $\operatorname{supp}\varrho_+(D)f \subseteq \operatorname{supp} f$. Since obviously $h(t)$ is smooth this implies (Peetre's theorem, cf. e.g. [14], 3.3.3) that $\varrho_+(D)$ is a differential operator on $\boldsymbol{R}_+$ with smooth coefficients. In addition the order of $\varrho_+(D)$ does not exceed the order of D. $\varrho_+(D)$ can be extended in an obvious way to an even differential operator on $\boldsymbol{R}_*$, which we shall denote again by $\varrho(D)$ in anticipation of the next result. Since under these conditions $\varrho(D)$ is uniquely determined we obtain the first part of the following:

PROPOSITION 1.3. *Let D: $\mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l) \to \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^k)$ be an M-invariant differential operator with smooth coefficients and let D_0 denote its restriction to $\boldsymbol{R}^d_*$. Then*

(i) *there exists a unique differential operator $\varrho(D)$: $\mathscr{E}(\boldsymbol{R}_*) \to \mathscr{E}(\boldsymbol{R}_*)$ with smooth coefficients on $\boldsymbol{R}_*$ which is even and satisfies*

$$D_0\theta^{l*}(f) = \theta^{k*}\varrho_+(D)f, \quad f \in \mathscr{E}(\boldsymbol{R}_+), \tag{1.12}$$

where $\varrho_+(D) = \varrho(D)|_{\boldsymbol{R}_+}$;

(ii) *for each $f \in \mathscr{E}_+(\boldsymbol{R})$ the function $\varrho(D)(f|_{\boldsymbol{R}_*})$ extends uniquely to a smooth even function on $\boldsymbol{R}$. The resulting map $\mathscr{E}_+(\boldsymbol{R}) \to \mathscr{E}_+(\boldsymbol{R})$ equals the map $\varrho(D)$ determined by* (1.10).

Proof. It remains to prove (ii). Let $f \in \mathscr{E}_+(\boldsymbol{R})$ and let f_+ denote its restriction to $\boldsymbol{R}_+$. Then clearly $\theta^{l*}(f_+) = \theta^{l*}(f)|_{\boldsymbol{R}^d_*}$. Further, let us write h for the function on $\boldsymbol{R}_*$ equal to $\varrho(D)(f|_{\boldsymbol{R}_*})$, $\varrho(D)$ being the differential operator from (i), and h_+ for its restriction to $\boldsymbol{R}_+$, i.e. $h_+(t) = \varrho_+(D)f_+(t)$ for $t > 0$. Then by (1.12) we have

$$\theta^{k*}(h_+) = D_0\theta^{l*}(f_+) = \big(D\theta^{l*}(f)\big)|_{\boldsymbol{R}^d_*} = \theta^{k*}\big(\varrho(D)f\big)|_{\boldsymbol{R}^d_*},$$

where in the rightmost term $\varrho(D)$ denotes the map $\mathscr{E}_+(\boldsymbol{R}) \to \mathscr{E}_+(\boldsymbol{R})$ defined by (1.10). Now applying (1.9) to both sides of this equality we get $t^k h(t) = t^k\big(\varrho(D)f\big)(t)$ for all $t \neq 0$, showing that h is the restriction to $\boldsymbol{R}_*$ of the smooth function $\varrho(D)f$. ∎

The differential operator $\varrho(D)$ on $\boldsymbol{R}_*$ which is described in the above proposition and sometimes also the related map $\varrho(D)$: $\mathscr{E}_+(\boldsymbol{R}) \to \mathscr{E}_+(\boldsymbol{R})$, will be called the *radial part* of D.

Let D_1, D_2 be two differential operators as above and let their composition $D_1 \circ D_2$ be defined. Then (1.12) together with the uniqueness of $\varrho(D)$ satisfying the requirements above implies

COROLLARY 1.4. $\varrho(D_1 \circ D_2) = \varrho(D_1) \circ \varrho(D_2)$.

2. Calculus of harmonic gradients. Let $\mathscr{P} = \mathscr{P}_C(\boldsymbol{R}^d)$ be the algebra of complex-valued polynomial functions of $\boldsymbol{R}^d$ and I its subalgebra consisting of polynomial functions in r^2, where $r^2(x) = \|x\|^2$ is the square of the euclidean norm. The group M acts on the left on $\mathscr{P}$ in the usual way ($P \to P \circ m^{-1}$) and the isotypic decomposition of $\mathscr{P}$ under the action of M is known to be

$$\mathscr{P} = \bigoplus_{l=0}^{\infty} I \cdot \mathscr{H}^l. \tag{2.1}$$

This holds true in the case $d = 2$ as well, provided one replaces $M = \mathbf{SO}(2)$ by the full orthogonal group $\tilde{M} = \mathbf{O}(2)$. Restricting this decomposition to the subspace $\mathscr{P}^l$ of homogeneous polynomial functions of degree l we have

$$\mathscr{P}^l = \bigoplus_{k=0}^{[l/2]} r^{2k} \mathscr{H}^{l-2k} \tag{2.2}$$

and the decomposition is orthogonal with respect to the inner product $(\cdot|\cdot)$ on $\mathscr{P}^l$ defined in (1.2a). Thus the orthogonal projection $H\colon \mathscr{P}^l \to \mathscr{H}^l$, called the *harmonic projection*, is seen to commute with the action of M.

By a standard construction to each polynomial $P \in \mathscr{P}^l$ there corresponds a differential operator $P(\partial)\colon \mathscr{E}(\mathbf{R}^d) \to \mathscr{E}(\mathbf{R}^d)$ with constant coefficients, where $P(\partial) = \sum_{|\alpha|=l} p_\alpha \partial^\alpha$ if $P(x) = \sum_{|\alpha|=l} p_\alpha x^\alpha$ with $p_\alpha \in \mathbf{C}$. We are using here the multi-index notation, where $|\alpha| = \sum_{i=1}^{d} \alpha_i$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}_+^d$, $x^\alpha := x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ for a point $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ and

$$\partial^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdot \ldots \cdot (\partial/\partial x_d)^{\alpha_d}.$$

In this notation the operator ∇_z of the directional derivative in the direction $z \in \mathbf{R}^d$ is written as $(z|\partial)$.

The following definition assigns an M-invariant differential operator of the form

$$D\colon \mathscr{E}(\mathbf{R}^d) \to \mathscr{E}(\mathbf{R}^d; V)$$

to any finite-dimensional M-invariant subspace $V \subset \mathscr{P}$.

DEFINITION. Given an M-invariant subspace $V \subset \mathscr{P}^l$ (not necessarily irreducible under M) let $D^V\colon \mathscr{E}(\mathbf{R}^d) \to \mathscr{E}(\mathbf{R}^d; V)$ be a differential operator defined as follows. For any $f \in \mathscr{E}(\mathbf{R}^d)$ and any $x \in \mathbf{R}^d$ define $D^V f(x) \in V$ as a (uniquely determined) polynomial from V such that

$$(D^V f(x) | \bar{P}) := P(\partial) f(x), \qquad P \in V. \tag{2.3}$$

It is easy to check that $\operatorname{supp} D^V f \subseteq \operatorname{supp} f$ so that $f \to D^V f$ is a differential operator with smooth coefficients and order $\leqslant l$. Moreover, D^V is M-invariant, which is seen by recalling the well-known fact that the map

$$\mathscr{P} \times \mathscr{E}(\mathbf{R}^d) \ni (P, f) \to P(\partial) f \in \mathscr{E}(\mathbf{R}^d)$$

is M-equivariant.

The above definition will be used mainly in the case when $V = \mathscr{H}^l$ and in this case the operator will be denoted simply by D^l. We note that the defining equality for D^l can be stated as

$$(2.4) \qquad [D^l f(x), P] := P(\partial) f(x), \qquad P \in \mathscr{H}^l$$

for any $f \in \mathscr{E}(\boldsymbol{R}^d)$, where we are using the pairing $[\cdot, \cdot]$ given by (1.2b) instead of the inner product (1.2a).

In the notation of the Introduction, writing $\Phi(x, z)$ for $\Phi(x)(z)$, the value of $\Phi(x) \in \mathscr{H}^l$ at $z \in \boldsymbol{R}^d$, where $\Phi \in \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$, we have the following expression for D^l.

LEMMA 2.1. *If $Z_x^l(z) = Z^l(x, z)$ is the reproducing kernel for $\mathscr{H}^l$, then*

$$(2.5) \qquad D^l f(x, z) = d(l) Z^l(\partial, z) f(x), \qquad f \in \mathscr{E}(\boldsymbol{R}^d).$$

In particular, for any $[\cdot, \cdot]$-orthonormal basis $\{Y_i\}$ for $\mathscr{H}^l$

$$(2.6) \qquad D^l f(x, z) = \sum_{i=1}^{d(l)} Y_i(\partial) f(x) Y_i(z), \qquad f \in \mathscr{E}(\boldsymbol{R}^d).$$

The lemma follows immediately from the reproducing property of Z^l (cf. (1.4)) and the addition formula (1.7). ∎

In order to discuss the connection of the operators D^l with the differential operators of gradient type we need to recall the definition of the latter. The general definition of Stein and Weiss [20] will, however, be confined to the present context of class **1** representations of $\mathbf{SO}(d)$ (cf. also [18]).

If $\Phi \in \mathscr{E}(\boldsymbol{R}^d; \mathscr{H}^l)$, let $\nabla\Phi\colon \boldsymbol{R}^d \to L(\boldsymbol{R}^d, \mathscr{H}^l)$ denote its (full) derivative; by complexification we can regard $\nabla\Phi$ as having values in $L(\boldsymbol{C}^d, \mathscr{H}^l)$. This latter space will be identified with $\boldsymbol{C}^d \otimes \mathscr{H}^l$ by means of the bilinear map

$$\boldsymbol{C}^d \times \mathscr{H}^l \ni (u, h) \to T_{u,h} \in L(\boldsymbol{C}^d, \mathscr{H}^l),$$

where we have set $T_{u,h}(v) := (v \mid \bar{u}) h$ for $h \in \boldsymbol{H}^l$ and $v \in \boldsymbol{C}^d$, $(\cdot \mid \cdot)$ being the standard inner product in $\boldsymbol{C}^d$. Thus we obtain the function, denoted again by $\boldsymbol{R}^d \ni x \to \nabla\Phi(x) \in \boldsymbol{C}^d \otimes \mathscr{H}^l$, which is called the *gradient* of Φ. From the construction one sees easily that for any $m \in M$

$$\nabla(m\Phi)(x) = R \otimes T^l(m) \nabla\Phi(m^{-1}x), \qquad x \in \boldsymbol{R}^d,$$

where by $R(\cdot)$ we have denoted the representation of M on $\boldsymbol{C}^d$ obtained by the complexification of the standard matrix representation of M on $\boldsymbol{R}^d$.

The well-known Clebsch–Gordan decomposition of $R \otimes T^l$ shows that the representations of class **1** occurring in the decomposition of $R \otimes T^l$ into irreducibles are T^{l+1} and T^{l-1} if $d \geqslant 4$ and T^{l+1}, T^l, T^{l-1} if $d = 3$, each with multiplicity one. Letting E^{l+1}, E^{l-1} denote the orthogonal projections of $\boldsymbol{C}^d \otimes \mathscr{H}^l$ onto the M-invariant subspaces carrying representations equivalent to T^{l+1}, T^{l-1} resp., one defines operators S_l, S_l^* by

$$S_l = E^{l+1} \circ \nabla, \qquad S_l^* = E^{l-1} \circ \nabla.$$

Since the projections are applied pointwise and commute with M, S_l and S_l^* are first order M-invariant differential operators with constant coefficients. However, in order to be able to regard S_l and S_l^* as unambiguously defined maps

$$S_l\colon \mathscr{E}(\boldsymbol{R}^d;\mathscr{H}^l)\to\mathscr{E}(\boldsymbol{R}^d;\mathscr{H}^{l+1}),\qquad S_l^*\colon \mathscr{E}(\boldsymbol{R}^d;\mathscr{H}^l)\to\mathscr{E}(\boldsymbol{R}^d;\mathscr{H}^{l-1})$$

one has to fix an M-isomorphism of the respective subspace of $\boldsymbol{C}^d\otimes\mathscr{H}^l$ onto $\mathscr{H}^{l+1}$ or $\mathscr{H}^{l-1}$. By Schur's Lemma this results in an arbitrariness of a numerical factor in the definition of those operators. We shall fix those factors in the course of the proof of Proposition 2.2 below and subsequently compare them with the choice made by Reimann [18]. In view of the construction and further properties which will be established below it seems appropriate to call S_l the *harmonic gradient* and S_l^*, which was shown by Reimann to be the formal adjoint to $-S_{l-1}$, the *harmonic divergence*.

The operators can be expressed by explicit formulas obtained previously in [18] by means of rather lengthy tensor calculations. Here we shall simplify both their expression and the derivation by relating them to the properties of spherical harmonics.

First a remark on notation. If $H\colon \mathscr{P}^l\to\mathscr{H}^l$ is the harmonic projection and $\Phi\colon \boldsymbol{R}^d\to\mathscr{P}^l$, then in accordance with the notation introduced earlier we shall write $H_z\Phi(x, z)$ rather than $H\circ\Phi(x, z)$ in order to remind the reader that the projection H concerns only the variable $z\in\boldsymbol{R}^d$ and is evaluated pointwise with respect to $x\in\boldsymbol{R}^d$.

PROPOSITION 2.2. *With a suitable identification of the range of E^{l+1} with $\mathscr{H}^{l+1}$, and of the range of E^{l-1} with $\mathscr{H}^{l-1}$, the operators S_l and S_l^* are given by*

$$S_l\Phi(x, z) = H_z\big((z\,|\,\partial)\Phi(x, z)\big), \tag{2.7}$$

$$S_l^*\Phi(x, z) = \|z\|^{-2}(I-H_z)\big((z\,|\,\partial)\Phi(x, z)\big). \tag{2.8}$$

(*Here I denotes the identity operator on $\mathscr{P}^{l+1}$.*) *In terms of coordinates this reads* (cf. [18])

$$S_l\Phi(x, z) = \sum_{j=1}^{d} z_j\frac{\partial}{\partial x_j}\Phi(x, z)-\frac{1}{d+2l-2}\|z\|^2\sum_{j=1}^{d}\frac{\partial}{\partial z_j}\frac{\partial}{\partial x_j}\Phi(x, z), \tag{2.9}$$

$$S_l^*\Phi(x, z) = \frac{1}{d+2l-2}\sum_{j=1}^{d}\frac{\partial}{\partial z_j}\frac{\partial}{\partial x_j}\Phi(x, z). \tag{2.10}$$

Proof. Consider the multiplication map $\mu\colon \boldsymbol{C}^d\otimes\mathscr{H}^l\to\mathscr{P}^{l+1}$ given by

$$\mu(v\otimes P)(z) = (z\,|\,\bar{v})P(z),\qquad z\in\boldsymbol{R}^d.$$

Since μ commutes with the action of M, by Schur's Lemma the decomposition (2.2) shows μ can be nonvanishing only on M-invariant subspaces carrying representation equivalent to T^{l+1} and T^{l-1}. On the other hand, it is easy to observe that the image of μ intersects nontrivially either one of the subspaces $\mathscr{H}^{l+1}$ and $r^2\mathscr{H}^{l-1}$ of $\mathscr{P}^{l+1}$. Therefore if E^{l+1}, E^{l-1} are the projections onto the invariant subspaces of $C^d\otimes\mathscr{H}^l$ M-equivalent to $\mathscr{H}^{l+1}$, $\mathscr{H}^{l-1}$ resp., then $\mu\circ E^{l+1}$, $\mu\circ E^{l-1}$ map isomorphically and M-equivariantly the range of E^{l+1}, E^{l-1}, resp., onto $\mathscr{H}^{l+1}$ and $r^2\mathscr{H}^{l-1}$ resp. We choose $\mu\circ E^{l+1}$ as the map giving the identification of the range of E^{l+1} with $\mathscr{H}^{l+1}$ and notice that for the harmonic projection H; $\mathscr{P}^{l+1}\to\mathscr{H}^{l+1}$ we get $\mu\circ E^{l+1} = H\circ\mu$. Then

$$S_l\Phi(x, z) = \mu\circ E^{l+1}(\nabla\Phi(x, z)) = H_z\circ\mu\nabla\Phi(x, z) = H_z((z\,|\,\partial)\Phi(x, z)).$$

Since

$$(z\,|\,\partial)\Phi(x, z) = \sum_{j=1}^{d} z_j\frac{\partial}{\partial x_j}\Phi(x, z)$$

is of the form $\sum_{j=1}^d z_j\Phi_j$, where $\Phi_j\in\mathscr{E}(\boldsymbol{R}^d;\,\mathscr{H}^l)$, the stated coordinate expression follows by observing that for any $H\in\mathscr{H}^l$

$$H_z(z_jH) = z_jH-\frac{1}{d+2l-2}\|z\|^2\frac{\partial}{\partial z_j}H, \tag{2.11}$$

cf. e.g. Cor. 3.14 of [3].

In the case of S_l^* we shall make the identification of the range of the projection E^{l-1} in $C^d\otimes\mathscr{H}^l$ with $r^2\mathscr{H}^{l-1}$ by means of the map $\mu\circ E^{l-1}$ and subsequently identify the latter space with $\mathscr{H}^{l-1}$ in the natural way. Thus

$$S_l^*\Phi(x, z) = \|z\|^{-2}\mu\circ E^{l-1}(\nabla\Phi(x, z)) = \|z\|^{-2}\big((z\,|\,\partial)\Phi(x, z)-H_z((z\,|\,\partial)\Phi(x, z))\big).$$

The coordinate expression for S_l^* follows immediately from this formula and the coordinate expression for S_l. ∎

Remark. Our expression for S_l^* differs from the one given by Reimann ([18], Proposition 8) by the factor $(d+2l-2)/l$. This is due to the use of the integral inner product in $\mathscr{P}^l$ given by (1.2a) instead of the differential one used by Reimann. (Actually the construction of [18] rests on the use of the inner product in the space $S^l(\boldsymbol{R}^d)$ of symmetric tensors on $\boldsymbol{R}^d$ leading via the known isomorphism of this latter space with $\mathscr{P}^l(\boldsymbol{R}^d)$ to the differential inner product in this space.) However, this change of the factor of proportionality makes it possible to retain the relation $S_l^* = -(S_{l-1})^*$ for the present choice of the inner product as well.

We shall now state some recurrence relations satisfied by the kernels $Z^l(x, z)$, which will be needed in the sequel. Their proofs are easy and are omitted([1]).

([1]) Some of the proofs omitted here appear in the earlier version of this paper [23].

LEMMA 2.3. *For any* $x,\ z\in \boldsymbol{R}^d$ *and each* $l\in \boldsymbol{Z}_+$

$$H_z\big((z\,|\,x)Z^l(x,\ z)\big) \tag{2.12}$$
$$=(z\,|\,x)Z^l(x,\ z)-\frac{l}{d+2l-2}\|x\|^2\|z\|^2 Z^{l-1}(x,\ z)=\frac{d+l-2}{d+2l-2}Z^{l+1}(x,\ z),$$

$$(z\,|\,\partial)Z^l(x,\ z)=l\cdot\|z\|^2 Z^{l-1}(x,\ z).\ \blacksquare \tag{2.13}$$

The meaning of the fact that the harmonic projection in formula (2.7) is evaluated pointwise with respect to x is clarified by the following general observation. If $P(x,\ z)$ is a polynomial function on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ homogeneous of order k in either one of its variables, then developing $P(x,\ z)=\sum_{|\alpha|=k}p_\alpha(z)x^\alpha$, where $p_\alpha\in\mathscr{P}^k$ we have

$$H_z P(x,\ z)=\sum_{|\alpha|=k}(Hp_\alpha)(z)x^\alpha.$$

This implies in particular that for any $f\in\mathscr{E}(\boldsymbol{R}^d)$ we have

$$H_z\big(P(\partial,\ z)f(x)\big)=(H_z P)(\partial,\ z)f(x)$$

where on the left-hand side the projection is applied to the polynomial $z\to P(\partial,\ z)f(x)$ for any fixed $x\in\boldsymbol{R}^d$. This observation is used in the proof of the next result.

PROPOSITION 2.4. *For any* $l\in\boldsymbol{Z}_+$

$$S_l\circ D^l=\frac{l+1}{2l+d}D^{l+1}. \tag{2.14}$$

Moreover,

$$D^l=c_l S_{l-1}\circ\ldots\circ S_0, \tag{2.15}$$

where

$$c_l=d(d+2)\ldots(d+2l-2)/l!.$$

Proof. By (2.7) and Lemma 2.1

$$S_l\circ D^l f(x,\ z)=H_z\big((z\,|\,\partial)D^l f(x,\ z)\big)=d(l)H_z\big((z\,|\,\partial)Z^l(z,\ \partial)f(x)\big).$$

Using the observations made just above and Lemma 2.3 we see that

$$S_l\circ D^l=\frac{d(l)}{d(l+1)}\cdot\frac{d+l-2}{d+2l-2}D^{l+1}.$$

Since

$$d(l)=(d+2l-2)\frac{(d+l-3)!}{(d-2)!\,l!}$$

formula (2.14) follows. Noting that $S_0=D^1$ we get the second one. ■

Remark. The factor c_l in (2.15) is just the proportionality factor between the integral and the differential inner products in $\mathscr{H}^l$.

PROPOSITION 2.5. *For any* $l \in \boldsymbol{Z}_+$

$$S_l^* \circ D^l = \frac{d+l-3}{d+2l-4} D^{l-1} \circ \Delta. \tag{2.16}$$

Proof. Let $P \in \mathscr{H}^{l-1}$ and $f \in \mathscr{E}(\boldsymbol{R}^d)$. Then using (2.10) we obtain

$$(*)\ (S_l^* \circ D^l f(x) | \bar{P}) = \frac{1}{d+2l-2} \sum_{i=1}^{d} \left(\frac{\partial}{\partial z_i} \frac{\partial}{\partial x_i} D^l f(x) \middle| \bar{P} \right) = \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i} D^l f(x) \middle| z_i \bar{P} \right)$$

since for $Q \in \mathscr{H}^l$, $P \in \mathscr{H}^{l-1}$ we have

$$\left(\frac{\partial}{\partial z_i} Q \middle| \bar{P} \right) = (d+2l-2)(Q \,|\, z_i \bar{P}),$$

which can easily be verified by the use of the divergence theorem. For any $x \in \boldsymbol{R}^d$ we have $(\partial/\partial x_i) D^l f(x) = D^l(\partial/\partial x_i) f(x) \in \mathscr{H}^l$ and therefore the right-hand side of $(*)$ is equal to

$$\sum_{i=1}^{d} \left(D^l \frac{\partial}{\partial x_i} f(x) \middle| H(z_i \bar{P}) \right) = \sum_{i=1}^{d} H(z_i \bar{P})(\partial) \frac{\partial}{\partial x_i} f(x).$$

From (2.11) we get further

$$= \left(P(\partial) \Delta f - \frac{1}{d+2l-4} \sum_{j=1}^{d} \frac{\partial P}{\partial z_j}(\partial) \frac{\partial}{\partial x_j} \Delta f \right)(x),$$

and finally since any polynomial identity in $\mathscr{P}$ is preserved under the map $P \to P(\partial)$, we have by the Euler relation

$$= \frac{d+l-3}{d+2l-4} P(\partial) \Delta f(x) = \frac{d+l-3}{d+2l-4} (D^{l-1} \circ \Delta f(x) | \bar{P}). \ \blacksquare$$

COROLLARY 2.6 (cf. [18], Theorem, p. 745). *If* $f \in \mathscr{E}(\boldsymbol{R}^d)$ *satisfies* $\Delta f = \lambda f$ *with* $\lambda \in \boldsymbol{C}$, *then*

$$S_l^* \circ S_{l-1}(D^{l-1} f) = \lambda_{l-1} \cdot D^{l-1} f,$$

where

$$\lambda_{l-1} = \frac{l(d+l-3)}{(d+2l-2)(d+2l-4)} \cdot \lambda. \ \blacksquare$$

PROPOSITION 2.7. *For any* $l \in \boldsymbol{Z}_+$ *the radial parts of the operators* S_l *and* S_l^* *are given by*

$$\varrho(S_l) = a_l \cdot \frac{1}{t} \frac{d}{dt}, \tag{2.17}$$

$$\varrho(S_l^*) = a_{l-1}\cdot\left(t\frac{d}{dt}+d+2l-2\right), \tag{2.18}$$

where a_l is given by

$$a_l = \frac{l+1}{2l+d}\cdot\left(\frac{d(l+1)}{d(l)}\right)^{1/2}$$

Proof. We keep the notation introduced in the proof of Proposition 2.2. Recalling (1.6) we get for any $f\in\mathscr{E}_+(\boldsymbol{R})$

$$\mu\nabla\theta^{l*}f(x,z) = (z\,|\,\partial)\theta^{l*}f(x,z) = d(l)^{1/2}(z\,|\,\partial)\big(f(\|x\|)Z^l(x,z)\big)$$
$$= d(l)^{1/2}\left((z\,|\,x)\left(\frac{1}{t}\frac{d}{dt}f\right)(\|x\|)Z^l(x,z)+f(\|x\|)(z\,|\,\partial)Z^l(x,z)\right).$$

By Lemma 2.3

$$(z\,|\,x)Z^l(x,z) = \frac{d+l-2}{d+2l-2}Z^{l+1}(x,z)+\frac{l}{d+2l-2}\|z\|^2\|x\|^2Z^{l-1}(x,z),$$

hence

$$S_l\theta^{l*}f(x,z) = \mu\circ E^{l+1}\circ\nabla\theta^{l*}f(x,z) = d(l)^{1/2}\frac{d+l-2}{d+2l-2}\left(\frac{1}{t}\frac{d}{dt}f\right)(\|x\|)Z^{l+1}(x,z).$$

Using again (2.13) we get

$$S_l^*\theta^{l*}f(x,z) = \|z\|^{-2}\mu\circ E^{l-1}\circ\nabla\theta^{l*}f(x,z)$$
$$= d(l)^{1/2}\frac{l}{d+2l-2}\left(\left(t\frac{d}{dt}+d+2l-2\right)f\right)(\|x\|)Z^{l-1}(x,z).$$

The proof is finished by observing that

$$\frac{d+l-2}{d+2l-2}\cdot\frac{d(l)}{d(l+1)} = \frac{l+1}{2l+d}. \ \blacksquare$$

The following corollary is basic for the applications in the next section.

COROLLARY 2.8. *For any $l\in\boldsymbol{Z}_+$*

$$\varrho(D^l) = d(l)^{1/2}\left(\frac{1}{t}\frac{d}{dt}\right)^l. \tag{2.19}$$

In particular, if $H\in\mathscr{H}^l$ and $f\in\mathscr{E}_+(\boldsymbol{R})$, then for $F=\theta^*f\in\mathscr{E}(\boldsymbol{R}^d)^M$ one has

$$H(\partial)F(x) = \left[\left(\frac{1}{t}\frac{d}{dt}\right)^l f\right](\|x\|)H(x). \tag{2.20}$$

Proof. Recalling $F(x)=f(\|x\|)$ and using (2.4) together with the definition of the radial part we obtain

$$H(\partial)F(x) = [D^l F(x), H] = [\theta^{l*}(\varrho(D^l)f)(x), H]$$
$$= d(l)^{1/2}(\varrho(D^l)f)(\|x\|)[T^l(m)Z^l, H] = d(l)^{-1/2}(\varrho(D^l)f)(\|x\|)H(x'),$$

where we have put $x' = (1/\|x\|)x = me_1$ with $m \in M$. Thus we are reduced to proving (2.19), which follows by elementary calculations after combining (2.15) with Corollary 1.4 and Proposition 2.7. ∎

3. Maxwell–Bochner type identities. In this section we deal with applications. Specifically, we shall discuss some of the classical identities related to spherical harmonics. The prototype of these identities is Maxwell's construction of spherical harmonics, which, roughly speaking, gives them as a result of differentiation of the radial fundamental solution of the Laplacian (i.e. the point potential). This construction was generalized by van der Pol [15] and Erdélyi [6], cf. also [7], Vol. 2, Section 11.5.2, to the case of the Helmholtz equation $(\Delta + k^2)u = 0$. The aim of this section is to disclose an intimate relation of those identities to the representation-theoretic meaning of spherical harmonics, the relation which gives, in our opinion at least, a deeper insight in the meaning of those identities.

The first step in this direction was made by Coifman and Weiss (cf. [3], Ex. 4, p. 44 ff.) who put Maxwell's construction in the representation-theoretic context and in particular related it to the harmonic projection of a polynomial. This latter fact was previously established entirely in the framework of classical analysis by Hobson (cf. [12]. p. 126 ff.), who derived it using an explicit formula, also obtained by him, for differentiation of radial functions. We show here how this formula can be obtained by using the decomposition (2.2) together with the calculus of radial parts developed above, in which Corollary 2.8 is the key result. Using this formula of Hobson we subsequently obtain a generalized form of Maxwell's construction encompassing also the polyharmonic polynomials and the identities of van der Pol and Erdélyi. In the light of the renewed interest in the constructions of Maxwell's type and in Kelvin transform (cf. Ch. VIII in [8] and references there), this derivation might be of some interest.

Corollary 2.8 together with an elementary property of the Fourier transform, (3.11) below, implies immediately the Hecke–Bochner identity and, with some extra calculations, a (possibly new) proof of the theorem stating that the Hermite–Weber functions are eigenfunctions of the Fourier transform. Here again the decomposition (2.2) of $\mathscr{P}^l$ plays an essential role.

PROPOSITION 3.1 (Hobson). *If $f \in \mathscr{E}_+(\boldsymbol{R})$, then for any $P \in \mathscr{P}^n(\boldsymbol{R}^d)$*

$$P(\partial)\theta^* f = \sum_{k=0}^{[n/2]} \frac{1}{2^k k!} \theta^*\left(\left(\frac{1}{t}\frac{d}{dt}\right)^{n-k} f\right) \Delta^k P. \tag{3.1}$$

Proof. If P is harmonic then this is just Corollary 2.8. Otherwise we decompose

$$P(x) = \sum_{k=0}^{[n/2]} r^{2k} H_{n-2k}(x), \tag{3.2}$$

where H_{n-2k} is harmonic and homogeneous of degree $n-2k$ and setting $F = \theta^* f$ we get

$$P(\partial)F = \sum_{k=0}^{[n/2]} \Delta^k H_{n-2k}(\partial)F = \sum_{k=0}^{[n/2]} \Delta^k (F_k H_{n-2k}), \tag{3.3}$$

where F_k is the radial function given by

$$F_k = \theta^*\left(\left(\frac{1}{t}\frac{d}{dt}\right)^{n-2k} f\right),$$

again by Corollary 2.8. Now, if G is a smooth radial function and $H_l \in \mathscr{H}^l$ then essentially by Schur's Lemma $\Delta^k(GH_l)$ is again the product of H_l with a radial function. The computation of the radial factor is well known (and trivial) for $k = 1$, and the general case is given (without proof, which is purely computational) as Lemma 3.2 below. It is perhaps known, but the author was not able to find it in the literature. In the special case $k = 1$ and $G = r^{2p}$ one has

$$\Delta(r^{2p} H_l) = \lambda_{l,p} r^{2p-2} H_l, \tag{3.4}$$

where $\lambda_{l,p} = 2p(d+2(l+p-1))$.

LEMMA 3.2. *Let $g \in \mathscr{E}_+(\mathbf{R})$, $H_l \in \mathscr{H}^l$ and set $G = \theta^* g$. Further, let L_l^k denote the differential operator*

$$L_l^k = \sum_{j=0}^{k} c_{l,j}^k t^{2j} \left(\frac{1}{t}\frac{d}{dt}\right)^{k+j}, \tag{3.5}$$

where the coefficients $c_{l,j}^k$ are given by

$$c_{l,j}^k = \begin{cases} \dfrac{1}{2^{k-j}(k-j)!} \displaystyle\prod_{i=1}^{k-j} \lambda_{l,j+i}, & k > j, \\ 1, & k = j. \end{cases}$$

Then

$$\Delta^k(\theta^*(g)H_l) = \theta^*(L_l^k g) H_l. \tag{3.6}$$

Assuming the lemma we shall now finish the proof of the formula of Hobson. Since

$$\theta^*(L_l^k g) = \sum_{j=0}^{k} c_{l,j}^k r^{2j} \theta^*\left(\left(\frac{1}{t}\frac{d}{dt}\right)^{k+j} g\right),$$

substituting (3.6) in (3.3) we get

$$P(\partial)F = \sum_{k=0}^{[n/2]} H_{n-2k} \sum_{j=0}^{k} c_{n-2k,j}^k r^{2j} \theta^*\left(\left(\frac{1}{t}\frac{d}{dt}\right)^{n-k+j} f\right).$$

Changing the summation order and setting $m = k-j$ we obtain further

$$= \sum_{m=0}^{[n/2]} \theta^*\left(\left(\frac{1}{t}\frac{d}{dt}\right)^{n-m} f\right) \sum_{k=m}^{[n/2]} c_{n-2k,k-m}^{k} r^{2(k-m)} H_{n-2k}.$$

However, it is easily checked by applying Δ m times to the decomposition (3.2) and using (3.4) together with the expression for the coefficients $c_{l,j}^{k}$ that

$$\frac{1}{2^m m!}\Delta^m P = \sum_{k=m}^{[n/2]} c_{n-2k,k-m}^{k} r^{2(k-m)} H_{n-2k}, \tag{3.7}$$

finishing the proof. ∎

Remark. It seems that the occurrence of formula (3.1) in Hobson's treatise [12] was overlooked in [3], cf. the note on p. 65 loc. cit.

Recall now that the Hermite polynomial associated with $P \in \mathscr{P}^n$ may be defined by the formula (cf. e.g. [11])

$$H_P = (-\tfrac{1}{2})^n \exp(r^2) P(\partial) \exp(-r^2).$$

Hobson's formula (3.1) immediately implies:

COROLLARY 3.3.

$$H_P = \sum_{k=0}^{[n/2]} (-1)^k \frac{1}{2^{2k} k!} \Delta^k P. \tag{3.8}$$

In particular, if $P \in \mathscr{H}^n$ *then* $H_P = P$. ∎

Formula (3.8) generalizes the well-known explicit representation of Hermite polynomials in one variable ([7], 10.13.9). For the case when P is a monomial x^α a formula equivalent to (3.8) occurs in [13], but the present author does not know any reference where it is given for an arbitrary P.

The Hermite–Weber function associated to P is in turn given by

$$W_P = \exp(r^2/2) P(\partial) \exp(-r^2),$$

and one sees immediately that

$$W_P = (-2)^n H_P \exp(-r^2/2). \tag{3.9}$$

Let the Fourier transform be defined by

$$\mathscr{F} f(y) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{i(x|y)} f(x)\, dx.$$

Then from the previous results we deduce the following:

THEOREM 3.4. *Let* $P \in \mathscr{P}^n$. *Then*

$$\mathscr{F}(W_P) = i^n W_P. \tag{3.10}$$

The special case of (3.10) when P is in addition assumed to be harmonic is called the *Hecke (–Bochner) identity.*

Proof. The Hecke identity follows immediately from the elementary property of the Fourier transform

$$\mathscr{F}(Pf) = (-i)^n P(\partial)\mathscr{F}(f), \tag{3.11}$$

valid for any homogeneous polynomial P of degree n, the well-known equality

$$\mathscr{F}(\exp(-r^2/2)) = \exp(-r^2/2),$$

and Corollary 2.8.

To prove the general case we shall follow the notation introduced in the proof of Proposition 3.1. For $P \in \mathscr{P}^n$ formulas (3.8) and (3.9) imply

$$\mathscr{F}(W_P) = (-2)^n \sum_{k=0}^{[n/2]} (-1)^k \frac{1}{2^{2k} k!} \mathscr{F}(\exp(-r^2/2)\Delta^k P).$$

On the other hand, using the decomposition (3.2) and formula (3.7) we can write

$$\mathscr{F}(W_P) = (-2)^n \sum_{k=0}^{[n/2]} (-\tfrac{1}{2})^k \sum_{l=k}^{[n/2]} c^l_{n-2l,l-k} \mathscr{F}(r^{2(l-k)} H_{n-2l} \exp(-r^2/2)).$$

By (3.11) and the Hecke identity

$$\mathscr{F}(r^{2(l-k)} H_{n-2l} \exp(-r^2/2)) = i^n (-1)^k \Delta^{l-k}(H_{n-2l} \exp(-r^2/2)).$$

Now making use of Lemma 3.2 we see that

$$\Delta^{l-k}(H_{n-2l} \exp(-r^2/2)) = \sum_{j=0}^{l-k} (-1)^{l-k+j} c^{l-k}_{n-2l,j} r^{2j} H_{n-2l} \exp(-r^2/2),$$

and therefore

$$(*) \quad \mathscr{F}(W_P)$$
$$= i^n(-2)^n \exp(-r^2/2) \sum_{l=0}^{[n/2]} H_{n-2l} \sum_{k=0}^{l} (-\tfrac{1}{2})^k c^l_{n-2l,l-k} \sum_{j=0}^{l-k} (-1)^{l+j} c^{l-k}_{n-2l,j} r^{2j}.$$

Rearranging the inner double sum we get

$$\sum_{k=0}^{l} \sum_{j=0}^{l-k} (-1)^{l+j} (-\tfrac{1}{2})^k c^l_{n-2l,l-k} c^{l-k}_{n-2l,j} r^{2j}$$
$$= (-1)^l \sum_{j=0}^{l} (-1)^j r^{2j} \sum_{k=0}^{l-j} (-\tfrac{1}{2})^k c^l_{n-2l,l-k} c^{l-k}_{n-2l,j}.$$

From the expression for $c^l_{p,j}$ given in Lemma 3.2 we get

$$c^l_{n-2l,l-k} c^{l-k}_{n-2l,j} = \frac{1}{2^{l-j} k!(l-j-k)!} \prod_{i=1}^{l-j} \lambda_{n-2l,j+i},$$

and using the elementary identity

$$\sum_{k=0}^{l-j} (-\tfrac{1}{2})^k \frac{1}{k!(l-j-k)!} = \frac{1}{2^{l-j}(l-j)!}$$

we see that the above double sum reduces to

$$\sum_{j=0}^{l} (-1)^{l-j} \frac{1}{2^{2(l-j)}(l-j)!} \prod_{i=1}^{l-j} \lambda_{n-2l,j+i} r^{2j}$$
$$= \sum_{p=0}^{l} (-1)^p \frac{1}{2^{2p}p!} \prod_{i=l-p+1}^{l} \lambda_{n-2l,i} r^{2(l-p)}.$$

Now substituting into (*) we see by comparison with (3.7) that the triple sum there gives

$$\sum_{l=0}^{[n/2]} H_{n-2l} \sum_{p=0}^{l} (-1)^p \frac{1}{2^{2p}p!} \prod_{i=l-p+1}^{l} \lambda_{n-2l,i} r^{2(l-p)}$$
$$= \sum_{p=0}^{[n/2]} (-1)^p \frac{1}{2^{2p}p!} \sum_{l=p}^{[n/2]} \Big(\prod_{i=l-p+1}^{l} \lambda_{n-2l,i} \Big) r^{2(l-p)} H_{n-2l} = H_p,$$

which implies the theorem by comparing with (3.9). ∎

Now we turn to an application of the preceding formalism to the Maxwell representation of spherical harmonics and related results. Consider first for $\lambda \in \boldsymbol{C}$ the function $r_\lambda(x) := r^{\lambda-d}(x)$ on $\boldsymbol{R}^d_*$ and let K_λ be the generalized Kelvin transform defined on $\mathscr{E}(\boldsymbol{R}^d_*)$, say, by

$$K_\lambda f(x) := r_\lambda(x) f(x/\|x\|^2), \qquad x \in \boldsymbol{R}^d_*, \tag{3.12}$$

the case $\lambda = 2$ being the ordinary Kelvin transform.

If $P \in \mathscr{P}^n(\boldsymbol{R}^d)$ then a simple homogeneity count shows that $K_\lambda P(\partial) r_\lambda \in \mathscr{P}^n(\boldsymbol{R}^d)$ and a problem arises to describe the map

$$M^n(\lambda)\colon\ \mathscr{P}^n(\boldsymbol{R}^d) \ni P \to K_\lambda P(\partial) r_\lambda \in \mathscr{P}^n(\boldsymbol{R}^d).$$

In the classical ($\lambda = 2$) case the image is precisely the space of spherical harmonics and this is an essential part of the Maxwell representation theorem (cf. [12], p. 127 ff., or [4], p. 514 ff.).

THEOREM 3.5. *The map*

$$M(\lambda)\colon\ \mathscr{P}(\boldsymbol{R}^d) \ni P \to K_\lambda P(\partial) r_\lambda \in \mathscr{P}(\boldsymbol{R}^d),$$

is an automorphism of vector spaces except when λ or $\lambda - d$ is an even integer. For the exceptional cases the following holds:

(i) *If $\lambda = 2p$, $p \in \boldsymbol{Z}_+$ and d is odd, then $M(\lambda)$ is an automorphism of the space* Ker Δ^p *of p-harmonic polynomials and vanishes on its orthocomplement, the space of polynomials divisible by r^{2p}.*

(ii) *If λ and d are odd integers, and setting $\lambda - d = 2q$, $q \in \boldsymbol{Z}_+$, and letting I_k denote the space of M-invariant polynomials on $\boldsymbol{R}^d$ of order $\leqslant 2k$, then $M(\lambda)$ is an* *automorphism of the space* $\bigoplus_{l\leqslant q} I_{q-l}\cdot\mathscr{H}^l$ *and vanishes on its orthocomplement* (cf. (2.1)).

(iii) *If* $\lambda = 2p$ *and* $d = 2q$ *with* $p, q\in \boldsymbol{Z}_+$, *then* $M(\lambda)$ *is an automorphism of the space* $\operatorname{Ker}\Delta^p\cap\bigoplus_{l\leqslant q-p} I_{q-p-l}\cdot\mathscr{H}^l$ *and vanishes on its orthocomplement.*

Proof. By virtue of the decomposition (2.1) it is enough to determine the restriction of $M(\lambda)$ to each subspace of the form $r^{2k}\mathscr{H}^l$. We are going to show that

$$M(\lambda)|_{r^{2k}\mathscr{H}^l} = c(\lambda, k, l)I, \tag{3.13}$$

where I is the identity operator and

$$c(\lambda, k, l) = (\lambda-2)\dots(\lambda-2k)(\lambda-d)\dots(\lambda-d-2k-2l+2).$$

In fact, for $H\in\mathscr{H}^l$ we have $(r^{2k}H)(\partial)r_\lambda = H(\partial)\Delta^k r_\lambda$, and from

$$\Delta^k r_\lambda = (\lambda-2)(\lambda-4)\dots(\lambda-k)(\lambda-d)(\lambda-d-2)\dots(\lambda-d-2k+2)r_{\lambda-2k},$$

and

$$\left(\frac{1}{t}\frac{d}{dt}\right)^l t^{\lambda-d-2k} = (\lambda-d-2k)(\lambda-d-2k-2)\dots(\lambda-2k-2l+2)t^{\lambda-d-2k-2l},$$

we get by Corollary 2.8

$$(r^{2k}H)(\partial)r_\lambda = c(\lambda, k, l)r_{\lambda-2k-2l}H.$$

The claim now follows immediately by applying K_λ to both sides of the equality.

One might observe that the irreducibility of the space $r^{2k}\mathscr{H}^l$ alone implies that the restriction of $M(\lambda)$ to that subspace is a multiple of the identity; this, however, is not enough to establish the result, since one has to determine when this constant is not zero.

The rest of the theorem follows by examining the cases when $c(\lambda, k, l) = 0$. For the connection with the polyharmonic polynomials one recalls the orthogonal decompositions

$$\mathscr{P}^n(\boldsymbol{R}^d) = \ker\Delta^p\cap\mathscr{P}^n(\boldsymbol{R}^d)\oplus r^{2p}\mathscr{P}^{n-2p}(\boldsymbol{R}^d),$$

$$\ker\Delta^p\cap\mathscr{P}^n(\boldsymbol{R}^d) = \bigoplus_{k=0}^{p-1} r^{2k}\mathscr{H}^{n-2k}.$$

The first one is a special case of a result on the decomposition of $\mathscr{P}^n$ given in [5], p. 168, and the latter follows by combining the former with (2.2). We leave the detailed verification of the statements in (i)–(iii) to the interested reader. ∎

For the final example we want to compute $P(\partial)G$, where G is a radial eigenfunction of the Laplacian on $\boldsymbol{R}^d_*$, i.e. $G = \theta^* g$ with $g\in\mathscr{E}(\boldsymbol{R}_+)$ and

$$\Delta G + \lambda^2 G = 0 \tag{3.14}$$

for some $\lambda \in \boldsymbol{R}_*$. By the change of variables $x \to \lambda x$ we may (and shall) assume that $\lambda = 1$ in (3.14). Then one knows that setting $\alpha = (d-2)/2$ the function $h(t) = t^\alpha g(t)$ turns out to be a solution of the Bessel equation of order α,

$$\left[\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}+\left(1-\frac{\alpha^2}{t^2}\right)\right]h(t) = 0. \tag{3.15}$$

Thus h is a combination of any pair of independent standard solutions of the Bessel equation, by which we mean here the Bessel functions of the first, second or third kind (cf. [7], 7.2.1). For our purpose it is sufficient to consider the case when h equals one of these. Note that only when h is a multiple of the Bessel function J_α the function $g(t) = t^{-\alpha}h(t)$ extends to a smooth even function on $\boldsymbol{R}$ and the corresponding eigenfunction $G = \theta^* g$ is (defined and) smooth on the whole $\boldsymbol{R}^d$, otherwise on $\boldsymbol{R}^d_*$ only.

Letting Z_ν denote any of those standard Bessel functions and using the recurrence relation

$$\left(\frac{1}{t}\frac{d}{dt}\right)^l t^{-\nu}Z_\nu(t) = (-1)^l t^{-\nu-l}Z_{\nu+l}(t) \tag{3.16}$$

together with Proposition 3.1 we immediately obtain

COROLLARY 3.6. *Let* $\alpha = (d-2)/2$ *and let* Z_α *denote any of the standard Bessel functions of order* α. *For any* $p \in \boldsymbol{Z}_+$ *let* $Z_{\alpha+p}$ *be the Bessel function determined from* (3.16) *and* $G_p = \theta^*(t^{-\alpha-p}\cdot Z_{\alpha+p})$. (Note $G_0 = G$ is the solution of (3.14) corresponding to Z_α.) *Then for any polynomial* $P \in \mathscr{P}^n(\boldsymbol{R}^d)$

$$P(\partial)G(x) = (-1)^n \sum_{k=0}^{[n/2]} (-1)^k \frac{1}{2^k k!} G_{n-k}(x)\cdot \Delta^k P(x), \tag{3.17}$$

and in particular for harmonic P

$$P(\partial)G(x) = (-1)^n G_n(x)\cdot P(x). \tag{3.18}$$

The case considered in [15] corresponds to $P = Z^n$ (and $d = 3$). From (1.1) one gets using (3.14) with $\lambda = 1$,

$$Z^n(-i\partial)G(x) = \sum_{k=0}^{[n/2]} d^n_k\left(-i\frac{\partial}{\partial x_1}\right)^{n-2k} G(x).$$

On the other hand, recalling that

$$\sum_{k=0}^{[n/2]} d^n_k t^{n-2k} = C^\alpha_n(1)^{-1}C^\alpha_n(t),$$

where $C^\alpha_n(\cdot)$ denotes the Gegenbauer polynomial of degree n and index α ([25], Ch. IX, § 3.1) one immediately obtains from (3.18) the formula found by van der Pol in [15]

$$C^\alpha_n\left(-i\frac{\partial}{\partial x_1}\right)G(x) = (-1)^n(r^n G_n)(x)C^\alpha_n\left(\frac{x_1}{r}\right). \tag{3.19}$$

Similarly writing $P \in \mathscr{H}^n$ in the form

$$P(x) = r^{n-m} C_{n-m}^{\alpha-m}\left(\frac{x_1}{r}\right) P_1(x_2, \ldots, x_d),$$

where P_1 is a harmonic polynomial in the $d-1$ variables $x_2, \ldots, x_d$, homogeneous of degree m ([25], Ch. IX, § 3.5) and reasoning as above one can obtain the formula found (for $d = 3$) by Erdélyi ([6], cf. also [7], 11.5.32)

$$C_{n-m}^{\alpha-m}\left(-i\frac{\partial}{\partial x_1}\right) P_1\left(-i\frac{\partial}{\partial x_2}, \ldots, -i\frac{\partial}{\partial x_d}\right) G(x)$$

$$= (-1)^n (r^n G_n)(x) C_{n-m}^{\alpha-m}\left(\frac{x_1}{r}\right) P_1\left(\frac{x_2}{r}, \ldots, \frac{x_d}{r}\right).$$

Remark. As pointed out to the author by Prof. T. Koornwinder one can derive (3.18), and hence also the formulas of van der Pol and Erdélyi for the case when G is given by the Bessel function J_α, directly from another Bochner type formula:

$$\text{(3.20)} \qquad i^n \cdot \Gamma(\alpha+1) \cdot 2^\alpha \cdot P(x) G_n(x) = \int_{S^{d-1}} e^{i(x|y)} P(y) d\sigma(y),$$

valid for any $P \in \mathscr{H}^n$ (cf. e.g. [8], Ch. II), simply by use of (3.11). On the other hand, exactly the same arguments as those used above in the proof of the Hecke formula together with the recurrence (3.16) allow us to obtain (3.20) for an arbitrary harmonic P from the particular case $P = 1$, which is just the Poisson integral representation formula for the Bessel function J_α. In fact, the Bochner identity and even its generalization can also be derived this way (cf. a forthcoming paper of the author).

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DEPARTMENT OF MATHEMATICAL METHODS OF PHYSICS
FACULTY OF PHYSICS, UNIVERSITY OF WARSAW
Hoża 74, 00-682 Warszawa, Poland

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