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On the boundary domains of the n-th eigenfunctions for the self-adjoined elliptic equation

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Let G be a bounded Jordan measurable domain (simply or multiply connected) in the plane (x_1, x_2). Assume that the boundary $F(G)$ of G is a rectifiable curve; G may be approximated by an increasing sequence of the domain G_n with regular boundaries (i.e. the boundary $F(G_n)$ of G_n is a piecewise regular curve).

Let us consider the problem of eigenvalues and eigenfunctions of the equations

$$L(u)+\mu\varrho(x_1, x_2)u = 0 \tag{1}$$

with boundary condition

$$\frac{du}{d\nu}-h(x_1, x_2)u = 0 \text{ on } F(G)-\Gamma, \qquad u = 0 \text{ on } \Gamma, \tag{2}$$

where

$$L(u) = \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}\left[a_{ik}(x_1, x_2)\frac{\partial u}{\partial x_k}\right]-q(x_1, x_2)u$$

is a self-adjoined differential operator, and μ is a real parameter. Concerning the coefficients we assume that: $q(x_1, x_2) \geqslant 0$, $\varrho(x_1, x_2) > 0$ are defined and continuous in $\bar{G}$, $a_{ik}(x_1, x_2) = a_{ki}(x_1, x_2)$ $(i, k = 1, 2)$ are of class C^1 in $\bar{G}$, and the quadratic form $\sum_{i,k=1}^{2} a_{ik}(x_1, x_2)\xi_i\xi_k$ is positive definite in G; Γ denotes a part of $F(G)$ (Γ being connected or not); in extreme cases Γ may be the whole boundary of G or the empty set. Here $h(x_1, x_2)$ is a non-negative continuous function defined in $\bar{G}$, and $du/d\nu$ is a transversal derivative of u with respect to equation (1), i.e.

$$\frac{du}{d\nu} = \sum_{i,k=1}^{2} a_{ik}(x_1, x_2)\frac{\partial u}{\partial x_i}\cos(n, x_k), \qquad (x_1, x_2) \in F(G), \tag{3}$$

n being the interior normal to $F(G)$.

The boundary condition (2) is a generalized boundary condition (see [1]). Accordingly, we introduce some bilinear functionals and some linear spaces, in which these functionals will be defined. Namely we put

$$\bar{D}(\varphi, \psi) = \iint_G \Big[\sum_{i,k=1}^{2} a_{ik}(x_1, x_2)\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\psi}{\partial x_k} + q(x_1, x_2)\varphi\psi\Big]\,dx_1 dx_2\,, \tag{4}$$

$$H(\varphi, \psi) = \iint_G \varrho(x_1, x_2)\varphi\psi\, dx_1 dx_2\,, \tag{5}$$

$$D(\varphi, \psi) = \bar{D}(\varphi, \psi) + \int_{F(G)-\Gamma} h(x_1, x_2)\varphi\psi\, ds\,. \tag{6}$$

It immediately follows from (4), (5) and (6) that the functionals $\bar{D}$, D and H are symmetric, i.e. $\bar{D}(\varphi, \psi) = \bar{D}(\psi, \varphi)$, $D(\varphi, \psi) = D(\psi, \varphi)$ and $H(\varphi, \psi) = H(\psi, \varphi)$. Let us put

$$\bar{D}(\varphi) = \bar{D}(\varphi, \varphi), \qquad D(\varphi) = D(\varphi, \varphi) \qquad \text{and} \qquad H(\varphi) = H(\varphi, \varphi)\,. \tag{7}$$

Observe that

$$\bar{D}(\varphi) \geqslant 0, \qquad D(\varphi) \geqslant 0 \qquad \text{and} \qquad H(\varphi) \geqslant 0\,. \tag{8}$$

The equality $H(\varphi) = 0$ may occur only if $\varphi \equiv 0$ in G.

Definition 1. We denote by $\mathcal{H}$ the space of all functions φ of class C^0_σ (¹) such that $H(\varphi) < \infty$.

Definition 2. Denote by $\mathfrak{D}$ space of all functions of class C^1_σ in G such that $H(\varphi) < \infty$ and $D(\varphi) < \infty$.

Definition 3. Denote by $\dot{\mathfrak{D}}$ subspace of $\mathfrak{D}$ of functions which vanish at all points of G whose distance from Γ is less than or equal to ε (ε being a fixed positive number).

Definition 4. Denote by $\mathring{\mathfrak{D}}$ the subspace of $\mathfrak{D}$ of functions φ for which there exists a sequence $\varphi_\nu \in \dot{\mathfrak{D}}$ such that $H(\varphi_\nu - \varphi) \to 0$ and $D(\varphi_\nu - \varphi) \to 0$ for $\nu \to \infty$.

In the sequel by the boundary condition $u = 0$ on Γ we shall mean that $u \in \mathring{\mathfrak{D}}$.

Definition 5. Let $\mathcal{F}$ denote the subspace of $\mathfrak{D}$ of all functions φ of class C^2 in G such that $L(\varphi) \in \mathcal{H}$.

We want to give the meaning to the boundary condition $\dfrac{du}{d\nu} - hu = 0$ on $F(G)-\Gamma$. Accordingly consider a domain G_ε, contained with its regular boundary $F(G_\varepsilon)$ in G, such that the distance of $F(G_\varepsilon)$

(¹) For a definition of a function of class C^n_σ ($n \geqslant 0$), see [1].

from $F(G)$ is less than ε. One may prove (cf. [1]) that if $\varphi \in \mathcal{F}$ and $\psi \in \mathring{\mathfrak{D}}$ then

$$D_\varepsilon(\varphi, \psi) + H_\varepsilon\left(\frac{1}{\varrho}L(\varphi), \psi\right) + \int_{F(G_\varepsilon)-\Gamma_\varepsilon} \psi\left(\frac{d\varphi}{d\nu} - h\varphi\right) ds = 0, \tag{9}$$

where Γ_ε denotes the set of points of $F(G_\varepsilon)$ whose distance from Γ is less than or equal to ε. Now let $\varepsilon \to 0$ in (9). Since there exist limits

$$D(\varphi, \psi) = \lim_{\varepsilon\to 0} D_\varepsilon(\varphi, \psi) \quad \text{and} \quad H\left(\frac{1}{\varrho}L(\varphi), \psi\right) = \lim_{\varepsilon\to 0} H_\varepsilon\left(\frac{1}{\varrho}L(\varphi), \psi\right),$$

then there exists also a limit

$$\int_{F(G)-\Gamma} \left(\frac{d\varphi}{d\nu} - h\varphi\right) \psi\, ds = \lim_{\varepsilon\to 0} \int_{F(G_\varepsilon)-\Gamma_\varepsilon} \left(\frac{d\varphi}{d\nu} - h\varphi\right) \psi\, ds.$$

All the limits involved are connected by the formula

$$D(\varphi, \psi) + H\left(\frac{1}{\varrho}L(\varphi), \psi\right) + \int_{F(G)-\Gamma} \left(\frac{d\varphi}{d\nu} - h\varphi\right)\psi\, ds = 0. \tag{10}$$

The boundary condition $\frac{du}{d\nu} - hu = 0$ on $F(G) - \Gamma$ for $u \in \mathcal{F}$ is now defined by the requirement that the equality

$$\int_{F(G)-\Gamma} \left(\frac{du}{d\nu} - hu\right)\psi\, ds = 0 \tag{11}$$

be valid for all $\psi \in \mathring{\mathfrak{D}}$. The boundary condition (2) is defined by the requirements that $u \in \mathcal{F} \cap \mathring{\mathfrak{D}}$ and (11) be valid for all $\psi \in \mathring{\mathfrak{D}}$.

DEFINITION 6. $\mathcal{F}_{h,\Gamma}(G)$ will denote the space of all functions satisfying (2) in the above sense.

We define eigenvalues and eigenfunctions of problem (1), (2) in the following way (variationally): the first *eigenvalue* λ_1 of problem (1), (2) is defined by

$$\lambda_1 = \min_{\varphi \in \mathring{\mathfrak{D}}} \frac{D(\varphi)}{H(\varphi)} \tag{12}$$

and the first *eigenfunction* u_1 is that φ which realizes minimum (12).

Having defined eigenvalues $\lambda_1, \ldots, \lambda_n$ and the corresponding eigenfunctions $u_1, \ldots, u_n$ we put

$$\lambda_{n+1} = \min_{\varphi \in K_n} \frac{D(\varphi)}{H(\varphi)} \tag{13}$$

where K_n is a subclass of $\mathring{\mathfrak{D}}$ of the functions φ that satisfy the orthogonality conditions

$$(14)\qquad H(\varphi, u_i) = 0, \quad i = 1, \ldots, n,$$

and the u_{n+1} is that $\varphi \in K_n$ which gives minimum (13).

Hypothesis Z. *Given* (1) *and* (2) *there exist a sequence of eigenvalues*

$$(15)\qquad 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots,$$

such that

$$(16)\qquad \lim_{n\to\infty} \lambda_n = \infty,$$

and a corresponding sequence of eigenfunctions

$$(17)\qquad u_1(x_1, x_2),\ u_2(x_1, x_2),\ u_3(x_1, x_2),\ \ldots$$

which belong to $\mathcal{F}$.

It was proved in [1] that if Hypothesis Z is satisfied, then each function u_n of sequence (17) satisfies equation (1) or $\mu = \lambda_n$ $(n = 1, 2, 3, \ldots)$, and $u_n \in \mathcal{F}_{h,\Gamma}(G)$.

Remark 1. If the boundary condition (2) is replaced by

$$(18)\qquad \frac{du}{d\nu} = 0 \quad \text{on} \quad F(G),$$

then the functional $D(\varphi, \psi)$ must be replaced by $\bar{D}(\varphi, \psi)$, and in (12) and (13) space $\mathring{\mathfrak{D}}$ by $\mathfrak{D}$.

Let G^* be a subdomain of G, and let the boundary $F(G^*)$ of G^* be a piecewise regular curve. We shall prove the following

Lemma 1. *If* $u = u(x_1, x_2)$ *is a function that belongs to the space* $\mathcal{F}$ *and* u *satisfies equation* (1) *with* $\mu = \lambda_n$, *then*

$$(19)\qquad \bar{D}^*(u) \leqslant 2\lambda_n H^*(u)$$

for n *sufficiently large.* $\bar{D}^*(u)$ *and* $H^*(u)$ *denote the functionals* $\bar{D}(u)$ *and* $H(u)$, *where the domain* G *is replaced by* G^*.

Proof. We shall use formula (10) for the pair u, u. Since u belongs to $\mathcal{F}$, we get

$$(20)\qquad \bar{D}^*(u) + H^*\left(\frac{1}{\varrho} L(u), u\right) + \int_{F(G^*)} hu^2 ds + \int_{F(G^*)} u\left(\frac{du}{d\nu} - hu\right) ds = 0.$$

Since $L(u) = -\lambda_n \varrho u$, formula (20) takes the form

$$(21)\qquad \bar{D}^*(u) - \lambda_n H^*(u) + \int_{F(G^*)} hu^2 ds + \int_{F(G^*)} u\left(\frac{du}{d\nu} - hu\right) ds = 0.$$

Therefore by (16) there exists a number N such that for $n > N$, we get

(22) $$H^*(u)\lambda_n \geqslant -\int_{F(G^*)} u\left(\frac{du}{d\nu} - hu\right) ds\,.$$

Since $\int_{F(G^*)} hu^2 ds \geqslant 0$, the (21) and (22) imply Lemma 1.

LEMMA 2. *If by $k(n)$ we denote the number of domains for which the n-th eigenfunction of problem* (1), (2) *divides with nodal lines the domain G^*, then for n sufficiently large the following inequality must be fultilled*:

(23) $$k(n) \leqslant \lambda_n A|G^*|$$

where A is a constant and $|G^*|$ is the area of G^*.

Proof. Suppose that the nodal lines of u_n divide G^* into subdomains $G_1, \ldots, G_{k(n)}$. Put

$$U_i = \begin{cases} u_n & \text{in} \quad \bar{G}_i, \\ 0 & \text{in} \quad \bar{G}^* - G_i, \end{cases} \qquad i = 1, \ldots, k(n)\,,$$

and $\Phi = a_1 U_1 + \ldots + a_{k(n)} U_{k(n)}$, where $a_1, \ldots, a_{k(n)}$ are real numbers such that $a_1^2 + \ldots + a_{k(n)}^2 > 0$, and $H^*(\Phi, v_j) = 0$ ($j = 1, \ldots, k(n)-1$); $v_1(x_1, x_2), \ldots, v_{k(n)-1}(x_1, x_2)$ are the eigenfunctions of equation (1) with boundary condition (18) for the domain G^*. Let $\nu_1, \ldots, \nu_{k(n)-1}$ be the eigenvalues corresponding to the eigenfunctions $v_1, \ldots, v_{k(n)-1}$.

By the definition of eigenvalues $\nu_1, \ldots, \nu_{k(n)-1}$ we get

(24) $$\bar{D}^*(\Phi) \geqslant \nu_{k(n)} H^*(\Phi)\,.$$

Since for n sufficiently large the function Φ satisfies all the assumptions of Lemma 1 for each domain G_i, we have

(25) $$\bar{D}^*(\Phi) \leqslant 2\lambda_n H^*(\Phi)\,.$$

In virtue of (24) and (25) we have

(26) $$\nu_{k(n)} \leqslant 2\lambda_n\,.$$

By our assumptions, there exists a number $a > 0$ such that

$$\bar{D}^*(\varphi) \geqslant a \iint_{G^*} (\varphi_{x_1}'^2 + \varphi_{x_2}'^2)\, dx_1 dx_2$$

for each function $\varphi \in \mathfrak{D}$; therefore

(27) $$\nu_n \geqslant a\bar{\nu}_n, \qquad n = 1, 2, \ldots,$$

where $\bar{\nu}_n$ denotes the nth eigenvalue of the equation

(28) $$\Delta w + \mu\varrho w = 0$$

with the boundary condition $dw/dn = 0$ on $F(G^*)$.

It is known ([3]) that

$$\lim_{n\to\infty}\frac{n}{\bar{\nu}_n}=\frac{1}{4\pi}\iint\limits_{G^*}\varrho(x_1,x_2)\,dx_1dx_2\,. \tag{29}$$

Therefore by (29) for n sufficiently large we have the inequality

$$k(n)\leqslant\bar{\nu}_{k(n)}2\frac{1}{4\pi}\iint\limits_{G^*}\varrho(x_1,x_2)\,dx_1dx_2\,. \tag{30}$$

In virtue of (26), (27) and (30) we have (23), where

$$A=\frac{a}{\pi}\max_{G^*}\varrho(x_1,x_2).$$

Lemma 1 and 2 imply the following theorems:

THEOREM 1. *If* $\{G_n^*\}$ *denotes a non-increasing sequence of domains contained in* G *such that* $\lim\limits_{n\to\infty}|G_n^*|=0$, *and* $k(n)$ *denotes the number of domains for which the nodal lines of the* n*-th eigenfunction of problem* (1), (2) *divide* G_n^*, *then*

$$\lim_{n\to\infty}\frac{k(n)}{\lambda_n}=0\,. \tag{31}$$

The proof of Theorem 1 follows from inequality (23).

THEOREM 2. *If* $k(n)$ *denotes the number of nodal domains of* n*-th eigenfunction of problem* (1), (2) *whose boundaries have a common part with the curve* $F(G)-\Gamma$ *(so called boundary domains), then*

$$\lim_{n\to\infty}\frac{k(n)}{\lambda_n}=0\,. \tag{32}$$

Proof. Let G_n^* denote the set of all points of G such that the distance from the curve $F(G)-\Gamma$ is not greater than $1/n$ $(n=1,2,\ldots)$. Since the $F(G)$ is a rectifiable curve, we have

$$|G_n^*|\leqslant d\,\frac{1}{n}, \tag{33}$$

where d denotes the length of curve $F(G)-\Gamma$. Inequality (33) denotes that the sequence of domains G_n^* satisfies all the assumptions of Theorem 1. Therefore by (31) we have (32).

Theorem 2 may be utilized to estimate the number of nodal domains of nth eigenfunction of the following problem:

Let us take the differential equation of the form

$$\big(p(x_1,x_2)u'_{x_1}\big)'_{x_1}+\big(p(x_1,x_2)u'_{x_2}\big)'_{x_2}-q(x_1,x_2)u+\mu\varrho(x_1,x_2)u=0 \tag{34}$$

with the boundary condition

$$\frac{du}{dn} - h(x_1, x_2)u = 0 \text{ on } F(G)-\Gamma, \quad u = 0 \text{ on } \Gamma. \tag{35}$$

We assume that $p(x_1, x_2) > 0$ and $\varrho(x_1, x_2) > 0$ in the closure $\bar{G}$ of G, $p(x_1, x_2)$ is a function of the class C^2, $\varrho(x_1, x_2)$ is continuous in $\bar{G}$ and $\log p(x_1, x_2)$ and $\log \frac{\varrho(x_1, x_2)}{p(x_1, x_2)}$ are functions subharmonic in $\bar{G}$. The functions $q(x_1, x_2)$ and $h(x_1, x_2)$ satisfy the assumptions given at the beginning of this paper. With the above assumptions we shall now prove the following

THEOREM 3. *If $N(n)$ denotes the number of nodal domains of n-th eigenfunction of problem* (34), (35) *then*

$$N(n) \leqslant 0{,}7\, n, \tag{36}$$

for n sufficiently large.

Proof. The following inequality ([2])

$$\lambda_1 \iint\limits_G \frac{\varrho(x_1, x_2)}{p(x_1, x_2)} dx_1 dx_2 \geqslant \pi j_0^2 \tag{37}$$

will be utilized to prove Theorem 3, where λ_1 denotes the first eigenvalue of equation (34) with the boundary condition $u = 0$ on $F(G)$ and j_0 being the smallest positive zero of Bessel's function $J_0(x)$.

Owing to [1] the nth eigenvalue λ_n of problem (1), (2) is the first eigenvalue of equation (1) with the boundary condition $u = 0$ on $F(G_{n,i})$ for every interior nodal domain $G_{n,i}$ (i.e. the common part $F(G_{n,i})$ and $F(G)-\Gamma$ is the empty set). Therefore by (37) for equation (34) we obtain

$$\lambda_n \iint\limits_G \frac{\varrho(x_1, x_2)}{p(x_1, x_2)} dx_1 dx_2 \geqslant \pi j_0^2 \quad \text{for} \quad i = 1, \ldots, N(n)-k(n), \tag{38}$$

where $k(n)$ denotes the number of boundary nodal domains of nth eigenfunction of problem (34), (35). Therefore by (38) we have

$$\iint\limits_G \frac{\varrho(x_1, x_2)}{p(x_1, x_2)} dx_1 dx_2 \geqslant \pi j_0^2 \left(\frac{N(n)}{n} \cdot \frac{n}{\lambda_n} - \frac{k(n)}{\lambda_n}\right). \tag{39}$$

Since (see [3], p. 436)

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{1}{4\pi} \iint\limits_G \frac{\varrho(x_1, x_2)}{p(x_1, x_2)} dx_1 dx_2, \tag{40}$$

we get from (39) by (32)

$$\liminf_{n\to\infty} \frac{N(n)}{n} \leqslant \left(\frac{2}{j_0}\right)^2. \tag{41}$$

Since $j_0 = 2{,}408.$, the last inequality implies (36) for all n sufficiently large.

Remark 2. Inequality (36) sharpens the inequality $N(n) \leqslant n$ proved in [1] for problem (1), (2), and generalizes the inequality proved in [2] for equation (34) to the case of the boundary condition $u = 0$ on $F(G)$.

Remark 3. The result of Theorem 2 may be utilized to estimate the number of nodal domains of the nth eigenfunction of problem (1), (2) in all those cases where this estimation is based on the inequality of form (37), for the first eigenvalue of equation (1) with the boundary condition $u = 0$ on $F(G)$.

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