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On a system of difference inequalities of parabolic type

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We deal here with a certain system of second order difference inequalities that can be used (cf. [1]) to estimate the convergence of a difference scheme for a system of parabolic partial differential equations of the form

$$\frac{\partial u_i}{\partial t} = f_i\left(t, x_1, \ldots, x_n, u_1, \ldots, u_m, \frac{\partial u_i}{\partial x_1}, \ldots, \frac{\partial u_i}{\partial x_n}, \frac{\partial^2 u_i}{\partial x_1^2}, \ldots, \frac{\partial^2 u_i}{\partial x_n^2}\right), \quad i = 1, \ldots, m.$$

This scheme was investigated by Kowalski in [2]. An analogous theorem for the hyperbolic case was given by Pliś in [3].

Notation. We shall consider the nodal points x^M of R^{n+1}, $0 \leqslant M \leqslant P$, where $M = (m_0, m_1, \ldots, m_n)$, $P = (p_0, p_1, \ldots, p_n)$ are given systems of integers m_i, p_i, $i = 0, 1, \ldots, n$, and $0 \leqslant M \leqslant P$ denotes $0 \leqslant m_i \leqslant p_i$ $(i = 0, 1, \ldots, n)$, $x^M = (x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n})$, $x_0^{m_0} = m_0 k$, $x_i^{m_i} = m_i h$, $i = 1, 2, \ldots, n$, where $h = \tau/N$ and $k = \tau/N_1$ are positive numbers. To each nodal point x^M there correspond sequences of m real numbers u_i^M and v_i^M, $+M = (m_0+1, m_1, \ldots, m_n)$, $+jM = (m_0, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_n)$, $-jM = (m_0, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_n)$, $j = 1, \ldots, n$.

We shall use the forward and symmetric differences

$$u_i^{M\sim} = \frac{1}{k}(u_i^{+M} - u_i^M), \qquad u_i^{Mj} = \frac{1}{2h}(u_i^{+jM} - u_i^{-jM})$$

and the n-dimensional vectors

$$u_i^{MI} = (u_i^{M1}, u_i^{M2}, \ldots, u_i^{Mn}).$$

We shall also use the second differences

$$u_i^{Mjj} = \frac{1}{h}(u_i^{+jM} - 2u_i^M + u_i^{-jM})$$

and the n-dimensional vectors

$$u_i^{MII} = (u_i^{M11}, u_i^{M22}, \ldots, u_i^{Mnn}).$$

THEOREM 1. *Suppose that the scalar functions* $f_i(x, u, q, w)$, $i = 1, \dots, m$, $x = (x_0, x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$, $q = (q_1, \dots, q_n)$, $w = (w_1, \dots, w_n)$ *are of the class* C^1 *in the set* $D = [0, \tau]^{n+1} \times R^{2n+m}$ *and satisfy the conditions*

$$0 \leqslant \frac{\partial f_i}{\partial u_k} \leqslant L \quad (i = 1, \dots, m, k = 1, \dots, m; i \neq k), \tag{1}$$

$$\left|\frac{\partial f_i}{\partial q_j}\right| \leqslant \Gamma \quad (i = 1, \dots, m, j = 1, \dots, n), \tag{2}$$

$$0 < g \leqslant \frac{\partial f_i}{\partial w_j} \leqslant G \quad (i = 1, \dots, m, j = 1, \dots, n). \tag{3}$$

The numbers h *and* k (*mesh sizes*) *are chosen so as to satisfy*

$$\frac{g}{h} - \frac{\Gamma}{2} \geqslant 0, \tag{4}$$

$$1 + k\frac{\partial f_i}{\partial u_i} - \frac{2k}{h^2}\sum_{j=1}^{n}\frac{\partial f_i}{\partial w_j} \geqslant 0 \quad (i = 1, \dots, m). \tag{5}$$

Let u_i^M *and* v_i^M, $0 \leqslant M \leqslant P$, *satisfy the difference inequalities*

$$u_i^{M\sim} \leqslant f_i(x^M, u^M, u_i^{MI}, u_i^{MII}), \qquad v_i^{M\sim} \geqslant f_i(x^M, v^M, v_i^{MI}, v_i^{MII}), \qquad i = 1, \dots, m \tag{6}$$

for $0 \leqslant m_0 \leqslant p_0 - 1$, $1 \leqslant m_i \leqslant p_i$ $(i = 1, \dots, n)$, *and the inequalities*

$$u_i^M \leqslant v_i^M \quad \textit{for } m_0 = 0, \quad \textit{or } m_1 = 0, \dots, \textit{ or } m_n = 0, \quad \textit{or } m_1 = N, \dots, \textit{ or } m_n = N. \tag{7}$$

Then the inequalities

$$u_i^M \leqslant v_i^M \quad \textit{for } 0 \leqslant M \leqslant P \tag{8}$$

hold true.

Proof. We shall prove our theorem by induction. Write $r_i^M = u_i^M - v_i^M$. It is sufficient to prove that

$$r_i^M \leqslant 0 \quad \text{for } 0 \leqslant M \leqslant P, \ i = 1, \dots, m. \tag{9}$$

Inequalities (9) are satisfied for $m_0 = 0$ in virtue of assumption (7). Suppose it is satisfied for $m_0 = j$. Let

$$\max_{\substack{m_0 = j+1 \\ 0 \leqslant M \leqslant P}} r_i^M = r_i^{A(i)}$$

and $B(i)$ be such a multi-index that $+B(i) = A(i)$ (cf. Fig. 1). The equality

$$r_i^{A(i)} = r_i^{B(i)} + kr_i^{B(i)\sim} \tag{10}$$

follows from the definition of $u_i^{M\sim}$ and $v_i^{M\sim}$.

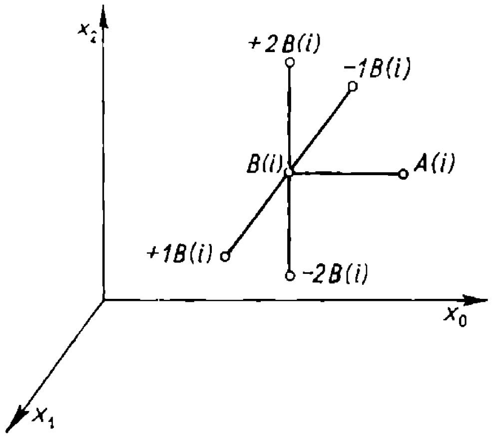

Fig. 1

From (6), by the mean value theorem, it follows that

$$\begin{aligned} r_i^{B(i)\sim} &= u_i^{B(i)\sim} - v_i^{B(i)\sim} \\ &\leqslant f_i(x^{B(i)}, u^{B(i)}, u_i^{B(i)I}, u_i^{B(i)II}) - f_i(x^{B(i)}, v^{B(i)}, v_i^{B(i)I}, v_i^{B(i)II}) \\ &= \sum_{s=1}^{m} \frac{\partial f_i}{\partial u_s} r_s^{B(i)} + \frac{1}{2h} \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j} (r_i^{+jB(i)} - r_i^{-jB(i)}) + \\ &\quad + \frac{1}{h^2} \sum_{j=1}^{n} \frac{\partial f_i}{\partial w_j} (r_i^{+jB(i)} - 2r_i^{B(i)} + r_i^{-jB(i)}) . \end{aligned} \tag{11}$$

Now we return to (10). Using (11), after a suitable regrouping of terms, we can write

$$\begin{aligned} r_i^{A(i)} \leqslant r_i^{B(i)} &\left(1 + k \frac{\partial f_i}{\partial u_i} - \frac{2k}{h^2} \sum_{j=1}^{n} \frac{\partial f_i}{\partial w_j}\right) + \\ &+ \frac{k}{h} \sum_{j=1}^{n} \left(\frac{1}{2} \frac{\partial f_i}{\partial q_j} + \frac{1}{h} \frac{\partial f_i}{\partial w_j}\right) r_i^{+jB(i)} + \\ &+ \frac{k}{h} \sum_{j=1}^{n} \left(\frac{1}{h} \frac{\partial f_i}{\partial w_j} - \frac{1}{2} \frac{\partial f_i}{\partial q_j}\right) r_i^{-jB(i)} + k \sum_{\substack{s=1 \\ s \neq i}}^{m} \frac{\partial f_i}{\partial u_s} r_s^{B(i)} . \end{aligned} \tag{12}$$

We shall prove that the right-hand side of this inequality is non-positive. In fact, we have

$$r_i^M \leqslant 0$$

for $M = +B(i)$, $M = +jB(i)$, $M = -jB(i)$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ because of the induction assumption. We have also:

$$\frac{1}{h}\frac{\partial f_i}{\partial w_j}+\frac{1}{2}\frac{\partial f_i}{\partial q_j} \geqslant \frac{g}{h}-\frac{\Gamma}{2} \geqslant 0\,, \quad \frac{1}{h}\frac{\partial f_i}{\partial w_j}-\frac{1}{2}\frac{\partial f_i}{\partial q_j} \geqslant \frac{g}{h}-\frac{\Gamma}{2} \geqslant 0\,,$$

because of (2), (3) and (4). The first term of right-hand side of inequality (12) is non-positive in virtue of (5). Hence (10) hold true for $m_0 = j+1$, $0 \leqslant M \leqslant P$. This completes the proof.

Remark. For $i = 1$, $u \in R^1$, and Assumption (1) becomes useless.

References

[1] A. Fitzke, *A convergence proof of difference scheme for a parabolic system*, this fasc., pp. 317-321.

[2] Z. Kowalski, *A difference method for a non-linear system of parabolic differential equations without mixed derivatives*, Bull. Acad. Polon. Sci., sér. sci., math., astr. et phys. 15 (1967), pp. 683-689.

[3] A. Pliś, *On difference inequalities corresponding to partial differential inequalities of the first order*, Ann. Polon. Math. 20 (1968), pp. 179-181.

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