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WEAK PRODUCT DECOMPOSITIONS OF PARTIALLY ORDERED SETS

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Miller [7] studied weak product decompositions of graphs and proved the following theorem:

(*) *Any connected non-oriented graph without multiple edges is a weak product of irreducible graphs.*

In [6] weak products of lattices were dealt with and as an application there was given a partial solution of a Birkhoff's problem (cf. [1], Problem 8) concerning non-oriented graphs of discrete lattices.

Let P be a partially ordered set. P is called *irreducible* if it cannot be represented as a non-trivial direct product. P is *discrete* if every bounded chain of P is finite. If for any pair $a, b \in P$, with $a < b$, there is a finite sequence $a_0 = a, a_1, \ldots, a_n = b$ such that $a_{i-1} < a_i$ and each interval $[a_{i-1}, a_i]$ is irreducible ($i = 1, \ldots, n$), then P is said to be *almost discrete.*

In this note we prove the following assertion:

(**) *Every almost discrete connected partially ordered set is a weak product of irreducible partially ordered sets.*

Further, we show that if G is a partially ordered group such that the corresponding partially ordered set is connected and almost discrete, then G is a restricted direct product of irreducible partially ordered groups.

Since each discrete partially ordered set is almost discrete, we infer from (**) that

(***) *Every discrete connected partially ordered set is a weak product of irreducible partially ordered sets.*

To any discrete partially ordered set P there corresponds, in a natural way, a non-oriented graph $G(P)$: the vertices of $G(P)$ are the elements of P; two elements $a, b \in P$ are joined by an edge in $G(P)$ if and only if either a covers b or a is covered by b in P.

Let us remark that (***) is not implied by the result of Miller, because in general there does not exist a one-to-one correspondence between the weak product decompositions of $G(P)$ and the weak product decompositions of P (cf. 4.6).

We shall use the standard notations for partially ordered sets and partially ordered groups (cf. Birkhoff [1] and Fuchs [2]). A partially ordered set P is said to be *connected* if, for any $a, b \in P$, there are elements $a_0, \ldots, a_n \in P$ such that $a_0 = a$, $a_n = b$ and the elements a_{i-1}, a_i are comparable ($i = 1, \ldots, n$). Let $a, b \in P$, $a \leqslant b$. The interval $[a, b]$ is the set $\{x \in P\colon a \leqslant x \leqslant b\}$. If a is the least element of P and $[a, b] = \{a, b\}$, $a < b$, then b is an atom of P; we call P an *atomic partially ordered set* if P has the least element and if, for each $x \in P$, there is an atom b of P such that $b \leqslant x$. Let $u, v \in P$; if the least upper bound of the set $\{u, v\}$ does exist in P, then we denote it by $u \vee v$. The meaning of $a \wedge b$ is analogous.

1. Direct factors of a partially ordered set. In the whole paper it is assumed that P is a connected partially ordered set. In this section there are collected some notions and auxiliary results on direct decompositions of P containing two factors, which we shall need in Sections 2 and 3.

Let A and B be partially ordered sets. The (exterior) direct product $A \times B$ is the set of all pairs (a, b) with $a \in A$ and $b \in B$, the partial order on $A \times B$ being defined by the rule $(a_1, b_1) \leqslant (a_2, b_2)$ if $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$. A direct product $A \times B$ is trivial if either $\operatorname{card} A = 1$ or $\operatorname{card} B = 1$.

Let $A_1, A_2 \subset P$, $x_0 \in P$, and $A_1 \cap A_2 = \{x_0\}$. Suppose that there are mappings φ_i of the set P onto A_i ($i = 1, 2$) such that

(i) $x \in A_i \Leftrightarrow \varphi_i(x) = x$,

(ii) $x \in A_1$, $y \in A_2 \Rightarrow \varphi_1(y) = x_0 = \varphi_2(x)$,

(iii) the mapping $\varphi\colon x \to (\varphi_1(x), \varphi_2(x))$ is an isomorphism of P onto $A_1 \times A_2$.

Then we write $P = [A_1 \times A_2]$. Partially ordered sets A_1 and A_2 are said to be *direct factors* of P with respect to the element x_0, and P is an *interior direct product* of A_1 and A_2. If, for $x \in P$, $\varphi_2(x) = x_0$, then, by (i) and (ii), $\varphi(x) = (\varphi_1(x), x_0) = \varphi(\varphi_1(x))$, and so, according to (iii), $x = \varphi_1(x)$; thus $x \in A_1$. Analogously, $\varphi_1(y) = x_0$ implies $y \in A_2$.

Let ψ be an isomorphism of P onto $A \times B$ and $x_0 \in P$. If $x \in P$ and $\psi(x) = (a, b)$, we write $\varphi_1^0(x) = a$ and $\varphi_2^0(x) = b$.

Let

$$A_1 = \{x \in P\colon \varphi_2^0(x) = \varphi_2^0(x_0)\},$$

$$A_2 = \{x \in P\colon \varphi_1^0(x) = \varphi_1^0(x_0)\}.$$

Further, let $\varphi_i(x)$ be an element of A_i such that $\varphi_i^0(\varphi_i(x)) = \varphi_i^0(x)$ for $i = 1, 2$. It is easy to verify that conditions (i)-(iii) are fulfilled, and thus $P = [A_1 \times A_2]$. Moreover, A_1 (A_2) is isomorphic to A (B). Therefore, it suffices to consider only internal decompositions.

Assume that there are given two direct decompositions

$$P = [A_1 \times A_2], \tag{1}$$

$$P = [B_1 \times B_2] \tag{2}$$

with respect to the element x_0. Corresponding mappings, for the direct decomposition (2), will be denoted by ψ_1, ψ_2 and ψ. Let $\mathscr{F}(x_0)$ be the system of all direct factors of P with respect to x_0; $\mathscr{F}(x_0)$ is partially ordered by the inclusion. Obviously, $P = [P \times \{x_0\}]$, and so P and $\{x_0\}$ are the greatest and the least elements of $\mathscr{F}(x_0)$, respectively.

Isomorphisms of direct products of partially ordered sets were investigated in [4]. Let us remark that if X_1 and Y_1 are subsets of P with $\operatorname{card}(X_1 \cap Y_1) = 1$ and if P is isomorphic with $X_1 \times Y_1$, then $P = [X_1 \times Y_1]$ need not hold.

Example. Let $X = Y$ be the set of all reals with the natural order, and $P = X \times Y$. Put $X_1 = \{(x, 0) \in P\colon\ -1 < x < 1\}$, $Y_1 = \{(0, y) \in P\colon -1 < y < 1\}$. Clearly, P is isomorphic with $X_1 \times Y_1$, $X_1 \cap Y_1 = \{(0, 0)\}$ and $P \neq [X_1 \times Y_1]$.

For $x, y \in P$ we put $x \equiv y(R_1)$ if $\varphi_2(x) = \varphi_2(y)$, and $x \equiv y(R_2)$ if $\varphi_1(x) = \varphi_1(y)$. Relations R_1' and R_2' are defined analogously (with ψ_1, ψ_2 instead of φ_1, φ_2).

1.1. *R_1 and R_2 are equivalence relations on the set P. For any $x, y \in P$ there are uniquelly determined elements $u, v \in P$ with*

$$x \equiv u(R_1), \quad u \equiv y(R_2), \tag{3}$$

$$x \equiv v(R_2), \quad v \equiv y(R_1). \tag{4}$$

If $x \equiv y(R_1)$ and $x \equiv y(R_2)$, then $x = y$. Moreover, if $y = x_0$, then $v = \varphi_1(x)$ and $u = \varphi_2(x)$.

This is an immediate consequence of the definition of R_1 and R_2.

The following lemma follows immediately from Lemma 2 in [4]:

1.2. *If (3) is valid and $x \equiv y(R_1')$, then $u \equiv x(R_1')$.*

1.3. *Let $x, y \in P$ be such that $x \equiv y(R_1)$, and $x \equiv y(R_1')$. Then there are elements $x_0, \ldots, x_n \in P$ such that, for $x_0 = x$ and $x_n = y$, $x_i \equiv x(R_1)$, $x_i \equiv x(R_1')$ and x_{i-1} is comparable with x_i, $i = 1, \ldots, n$.*

Proof. Since P is connected, there are elements $z_0, \ldots, z_n$ such that $z_0 = x$ and $z_n = y$, and the elements z_{i-1}, z_i are comparable for $i = 1, \ldots, n$. Further, there are elements $w_0, \ldots, w_n$ with $\varphi_1(w_i) = \varphi_1(z_i)$ and $\varphi_2(z_i) = \varphi_2(x)$, $i = 0, 1, \ldots, n$.

According to (iii), the elements w_{i-1}, w_i are comparable ($i = 1, \ldots, n$), $w_0 = x$, and $w_n = y$. Clearly,

$$w_i \equiv x(R_1). \tag{5}$$

There exist $x_0, \ldots, x_n \in P$ such that

(6) $$\psi_1(x_i) = \psi_1(w_i) \quad \text{and} \quad \psi_2(x_i) = \psi_2(x) \qquad (i = 1, \ldots, n).$$

By the comparability of w_{i-1}, w_i, we infer that the elements x_{i-1}, x_i are also comparable. By (6) we have

(7) $$w_i \equiv x_i(R_2') \quad \text{and} \quad x_i \equiv x(R_1').$$

According to 1.2, we infer, from (5) and (7), that $x_i \equiv x(R_1)$. The proof is complete.

1.4. *Let $x, y, u \in P$, $u \leqslant x$, $u \leqslant y$, and assume that* (3) *is valid. Then $u = x \wedge y$.*

Proof. Let $u_1 \in P$, $u_1 \leqslant x$, and $u_1 \leqslant y$. We have $\varphi_1(u_1) \leqslant \varphi_1(y) = \varphi_1(u)$ and $\varphi_2(u_1) \leqslant \varphi_2(x) = \varphi_2(u)$, and so $u_1 \leqslant u$. Hence $u = x \wedge y$.

1.5. *Let elements $x, y, u \in P$ satisfy the same conditions as in* 1.4 *and let $v \in P$ fulfil* (4). *Then $v = x \vee y$.*

Proof. From (3), (4) and $u \leqslant y$ it follows that $\varphi_1(x) = \varphi_1(v)$ and $\varphi_2(x) = \varphi_2(u) \leqslant \varphi_2(y) = \varphi_2(v)$, whence $x \leqslant v$. Analogously, $y \leqslant v$. According to 1.4, by duality, we obtain $x \vee y = v$.

1.6. *Let $x, y, u, v \in P$, $x \leqslant y$, and assume that* (3) *and* (4) *hold. Then $x = u \wedge v$ and $y = u \vee v$.*

Proof. We have $\varphi_1(x) \leqslant \varphi_1(y) = \varphi_1(u)$ and $\varphi_2(x) = \varphi_2(u)$. Thus $x \leqslant u$ and, similarly, $x \leqslant v$. Therefore, by (3), (4), 1.5 and 1.4, we obtain $u \wedge v = x$ and $u \vee v = y$.

1.7. *Under the same assumptions as in* 1.6, *let $v_1 \in P$ be such that $u \wedge v_1 = x$ and $u \vee v_1 = y$. Then $v_1 = v$.*

Proof. Clearly,

$$\varphi_1(v_1) \leqslant \varphi_1(y) = \varphi_1(u),$$
$$\varphi_2(u) = \varphi_2(x) \leqslant \varphi_2(v_1).$$

Therefore, in the partially ordered set $A_1 \times A_2$, we have

$$\big(\varphi_1(u), \varphi_2(u)\big) \wedge \big(\varphi_1(v_1), \varphi_2(v_1)\big) = \big(\varphi_1(v_1), \varphi_2(x)\big).$$

Since $u \wedge v_1 = x$, we obtain $\varphi_1(v_1) = \varphi_1(x) = \varphi_1(v)$. Analogously, $\varphi_2(v_1) = \varphi_2(y) = \varphi_2(v)$. Hence $v_1 = v$.

1.8. *Let $x, y, u \in P$ be such that the elements x, u are comparable and the elements y, u are comparable. Assume that* (3) *holds. Then there exists $v \in P$ such that whenever $x \equiv u(R_1')$ and $u \equiv y(R_2')$, then $x \equiv v(R_2')$ and $v \equiv y(R_1')$. Moreover, if $u \leqslant x$, $u \leqslant y$ ($u \geqslant x$, $u \geqslant y$), then $v = x \vee y$ ($v = x \wedge y$). If $x \leqslant u \leqslant y$ ($y \leqslant u \leqslant x$), then v is the* (*unique*) *relative complement of u in the interval with the end points x and y.*

This follows from 1.4, 1.5 and 1.7.

1.9. *Let $x, y, u \in P$, $\varphi_1(x) = \varphi_1(y)$ and $x \wedge y = u$. Then $\varphi_1(u) = \varphi_1(x)$.*

Proof. There is $u_1 \in P$ with $\varphi_1(u_1) = \varphi_1(x)$ and $\varphi_2(u_1) = \varphi_2(u)$. Clearly, $\varphi_1(u) \leqslant \varphi_1(x)$, thus $u \leqslant u_1$, $u_1 \leqslant x$, and $u_1 \leqslant y$. This shows that $u = u_1$, and, therefore, $\varphi_1(u) = \varphi_1(x)$.

For any two equivalences T_1 and T_2 on P we write $x \equiv y(T_1 \wedge T_2)$ if $x \equiv y(T_1)$ and $x \equiv y(T_2)$.

1.10. *Let $y, z, t \in P$. If $y \equiv z(R_1 \wedge R_1')$ and $y \equiv t(R_2 \wedge R_2')$, then there is $w \in P$ such that $w \equiv z(R_2 \wedge R_2')$ and $w \equiv t(R_1 \wedge R_1')$.*

Proof. If $y = z$ or $y = t$, the assertion is obvious. Let $z \neq y \neq t$. Then, by 1.3, there are elements $z_0, \ldots, z_n, y_0, \ldots, y_m \in P$ with $z_0 = z$, $z_n = y = y_0$, $y_m = t$ and such that z_{i-1}, z_i are comparable for $i = 1, \ldots, n$, y_{j-1}, y_j are comparable for $j = 1, \ldots, m$, and

$$z_i \equiv z(R_1 \wedge R_1') \quad \text{for } i = 0, \ldots, n,$$

$$y_j \equiv y(R_2 \wedge R_2') \quad \text{for } j = 0, \ldots, m.$$

Consider the elements z_{n-1}, y, y_1. Since the elements z_{n-1}, y are comparable and y, y_1 are comparable, there is, according to 1.8, $v_n \in P$ such that the elements v_n, z_{n-1} are comparable, the elements v_n, y_1 are comparable, and

$$z_{n-1} \equiv v_n(R_2 \wedge R_2'),$$

$$v_n \equiv y_1(R_1 \wedge R_1').$$

After $n-1$ analogous steps we get elements $v_0, v_1, \ldots, v_n$ such that $v_0 = z$, the elements v_{i-1}, v_i are comparable for $i = 1, \ldots, n$, and

$$v_0 \equiv v_1(R_2 \wedge R_2'),$$

$$v_{i-1} \equiv v_i(R_1 \wedge R_1') \quad \text{for } i = 2, \ldots, n.$$

Now, by using induction with respect to $n+m$, we infer that there is $w \in P$ such that $v_1 \equiv w(R_2 \wedge R_2')$ and $w \equiv t(R_1 \wedge R_1')$.

Hence $z \equiv w(R_1 \wedge R_2')$ and $w \equiv t(R_1 \wedge R_1')$.

1.11. $\varphi_1(\psi_1(x)) = \psi_1(\varphi_1(x))$ *for any $x \in P$.*

Proof. Let $x \in P$. Put $\varphi_1(x) = y$. There are elements $z, t \in P$ such that

$$y \equiv t(R_1'), \qquad t \equiv x(R_2'),$$

$$y \equiv z(R_2'), \qquad z \equiv x_0(R_1').$$

According to 1.2, we have $y \equiv t(R_2)$ and $y \equiv z(R_1)$. Thus, by 1.10, there is $v \in P$ satisfying $z \equiv v(R_2 \wedge R_1')$ and $t \equiv v(R_1 \wedge R_2')$.

Hence $x_0 \equiv v(R_1')$, $v \equiv x(R_2')$ and, therefore, $v = \psi_1(x)$.

Further, we have $\varphi_1(v) = z$ and $\psi_1(y) = z$. Thus $\varphi_1(v) = \psi_1(z)$, and so $\varphi_1(\psi_1(x)) = \psi_1(\varphi_1(x))$.

1.12. *If $A_1 = B_1$, then $A_2 = B_2$ and $\varphi_1(x) = \psi_1(x)$ for any $x \in P$.*

Proof. For any $x \in P$ we have $\varphi_1(x) \in A_1 = B_1$. Therefore, according to (i), $\psi_1(\varphi_1(x)) = \varphi_1(x)$ and, similarly, $\varphi_1(\psi_1(x)) = \psi_1(x)$. Thus, with respect to 1.11, $\varphi_1(x) = \psi_1(x)$. Since $A_2 = \{x \in P\colon \varphi_1(x) = x_0\}$ and $B_2 = \{x \in P\colon \psi_1(x) = x_0\}$, we infer that $A_2 = B_2$.

We have shown that, if the element x_0 is kept fixed, $\varphi_1(x)$ is uniquely determined by A_1 and x; with respect to this we write $\varphi_1(x) = x(A_1)$. Further, we denote $A_2 = A^*$, $R_1 = R(A)$, $R_2 = R(A^*)$. For any $X \subset P$ and $A \in \mathscr{F}(x_0)$ we write $X(A) = \{x(A)\colon x \in X\}$. If $A, B \in \mathscr{F}(x_0)$, we write $X(A)(B)$ instead of $(X(A))(B)$. Now we may express 1.11 in the following form:

1.13. *$x(A)(B) = x(B)(A)$ for any $x \in P$ and any $A, B \in \mathscr{F}(x_0)$.*

1.14. *$A(B) = A \cap B$ for any $A, B \in \mathscr{F}(x_0)$.*

Proof. If $x \in A \cap B$, then $x = x(B) \in A(B)$. Conversely, let $x \in A(B)$. Clearly, $x \in B$. There is $a \in A$ such that $a(B) = x$. According to 1.13, $a(B) = a(A)(B) = a(B)(A) \in A$, and thus $x \in A \cap B$.

1.15. *Assume that* (1) *and* (2) *are valid. Then $A_1 = [(A_1 \cap B_1) \times \times (A_1 \cap B_2)]$, $x(A_1 \cap B_1) = x(B_1)$ and $x(A_1 \cap B_2) = x(B_2)$ for any $x \in A_1$.*

Proof. Obviously, $(A_1 \cap B_1) \cap (A_1 \cap B_2) = \{x_0\}$. Let $x \in A_1$. By 1.14, $x(B_1) \in A_1 \cap B_1$ and $x(B_2) \in A_1 \cap B_2$. We have to verify whether partial mappings $(\psi_1)_{A_1}, (\psi_2)_{A_1}, \psi_{A_1}$ fulfil conditions (i), (ii) and (iii) if P, A_1, A_2 are replaced by $A_1, A_1 \cap B_1, A_1 \cap B_2$, respectively. Now (i) and (ii) easily follow from the fact that ψ_1 and ψ_2 satisfy these conditions with respect to P, B_1, B_2. Since ψ is an isomorphism, it only remains to verify whether the partial map $\psi_{A_1}\colon A_1 \to (A_1 \cap B_1) \times (A_1 \cap B_2)$ is onto. Let $b_1 \in A_1 \cap B_1$ and $b_2 \in A_1 \cap B_2$. There is $x \in P$ such that $\psi_1(x) = b_1$ and $\psi_2(x) = b_2$. Consequently, $x \equiv b_1(R_2')$, $x \equiv b_2(R_1')$ and $b_1 \equiv x_0 \equiv b_2(R_1)$, whence, according to 1.2, $x \equiv x_0(R_1)$. Hence $x \in A_1$. Clearly, $\psi(x) = (b_1, b_2)$. The proof is complete.

Assume that (1) holds and that $X \subset A_1$, $Y \subset A_2$ and $x_0 \in X \cap Y$. Let Z be the set of all $z \in P$ such that $z(A_1) \in X$ and $z(A_2) \in Y$. It is easy to verify that $Z = [X \times Y]$, $z(X) = z(A_1)$ and $z(Y) = z(A_2)$ for any $z \in Z$. Put $A_1 \cap B_1 = C_1, A_1 \cap B_2 = C_2$ and let D be the set of all $d \in P$ with $d(A_1) \in C_2$. Then we have

1.16. *$D = [C_2 \times A_2]$, $x(C_2) = \varphi_1(x)$ and $x(A_2) = \varphi_2(x)$ for each $x \in D$.*

By an argument similar to that in the proof of 1.15 (i.e., by checking conditions (i), (ii) and (iii)) the following assertion can easily be verified:

1.17. *$P = [C_1 \times D]$ and $x(C_1) = \psi_1(\varphi_1(x))$ for any $x \in P$. The element $x(D) = y \in D$ is defined by $y(C_2) = \psi_2(\varphi_1(x))$ and $y(A_2) = \varphi_2(x)$.*

From 1.16 and 1.17 it follows that $A_1 \cap B_1 \in \mathscr{F}(x_0)$ whenever A_1 and B_1 belong to $\mathscr{F}(x_0)$ and that the direct product is associative in the sense that $P = \big[[C_1 \times C_2] \times A_2\big]$ implies $P = \big[C_1 \times [C_2 \times A_2]\big]$ (and conversely, since, clearly, $[A_1 \times A_2] = [A_2 \times A_1]$); therefore we can write $P = [C_1 \times C_2 \times A_2]$ and, analogously, for any finite number of factors.

Under the denotations as above we have

1.18. *If $B \in \mathscr{F}(x_0)$, then $B(A_1) = [B(C_1) \times B(C_2)]$.*

Proof. From $A_1 = [C_1 \times C_2]$ and $B(A_1) \subset A_1$ we obtain $B(A_1) \subset [B(A_1)(C_1) \times B(A_1)(C_2)]$. Let $x \in [B(A_1)(C_1) \times B(A_1)(C_2)]$. Then $x \in A_1$ and $\psi_1(x) = u \in B(A_1)(C_1) = B \cap A_1 \cap C_1 = B \cap C_1$, $\psi_2(x) = v \in B(A_1)(C_2) = B \cap A_1 \cap C_2 = B \cap C_2$. Hence $x \equiv u(R_2')$, $x \equiv v(R_1')$ and $u \equiv x_0 \equiv v\big(R(B)\big)$.

Therefore, according to 1.2, $x \equiv u\big(R(B)\big)$, so $x \in B$. Thus $x \in B \cap A_1 = B(A_1)$. By summarizing,

$$B(A_1) = [B(A_1)(C_1) \times B(A_1)(C_2)] = [B(C_1) \times B(C_2)].$$

By induction, from 1.18 we get

1.19. *If $P = [C_1 \times C_2 \times \ldots \times C_n]$ and $B \in \mathscr{F}(x_0)$, then*

$$B([C_1 \times \ldots \times C_{n-1}]) = [B(C_1) \times \ldots \times B(C_{n-1})].$$

Let M_1 and M_2 be distinct elements of $\mathscr{F}(x_0)$. Assume that M_1, M_2 are irreducible and that $M_1 \neq \{x_0\} \neq M_2$. Then $X = M_1 \cap M_2$ belongs to $\mathscr{F}(x_0)$ and $X \neq M_i$, $i = 1, 2$. Hence $X = \{x_0\}$.

1.20. *Let $M_1, \ldots, M_k$ be distinct elements of $\mathscr{F}(x_0)$, and $M_i \neq \{x_0\}$ for $i = 1, \ldots, k$. If each M_i is irreducible, then P can be expressed in the form $P = [M_1 \times M_2 \times \ldots \times M_k \times C_k]$, where $C_k = M_1^* \cap \ldots \cap M_k^*$.*

Proof. For $k = 1$ we have $P = [M_1 \times M_1^*]$. Suppose that $P = [M_1 \times \times \ldots \times M_{k-1} \times C_{k-1}]$ and $C_{k-1} = M_1^* \cap \ldots \cap M_{k-1}^*$. Put $[M_1 \times \ldots \times M_{k-1}] = A$. Then from $P = [M_k \times M_k^*]$ we obtain (by using 1.19)

$$M_k(A) = [M_k(M_1) \times \ldots \times M_k(M_{k-1})] = \{x_0\},$$

since $M_k(M_i) = M_k \cap M_i = \{x_0\}$ for $i = 1, \ldots, k-1$. Thus

$$M_k = [M_k(A) \times M_k(C_{k-1})] = M_k(C_{k-1}) = M_k \cap C_{k-1}$$

and

$$C_{k-1} = [C_{k-1}(M_k) \times C_{k-1}(M_k^*)] = [(C_{k-1} \cap M_k) \times (C_{k-1} \cap M_k^*)]$$
$$= [M_k \times C_k].$$

This implies that $P = [M_1 \times \ldots \times M_{k-1} \times M_k \times C_k]$.

2. Irreducible intervals of P. Throughout Section 2 it is supposed that P is a connected almost discrete partially ordered set with card$P > 1$.

We denote by $\mathscr{P}$ the set of all irreducible intervals of P containing more than one element.

2.1. *Let* (1) *be fulfilled and let* $X = [u, v]$ *be an interval of* P. *Then the mapping* φ: $x \to (\varphi_1(x), \varphi_2(x))$ *for* $x \in X$ *is an isomorphism of* X *onto* $X(A_1) \times X(A_2)$.

Proof. For any subset $X \subset P$ the mapping φ is an isomorphism of X into $X(A_1) \times X(A_2)$. Let $(a_1, a_2) \in X(A_1) \times X(A_2)$. There is $y \in P$ with $\varphi(y) = (a_1, a_2)$. Clearly, $\varphi(u)(\varphi(v))$ is the least element (the greatest element) of $X(A_1) \times X(A_2)$; thus $y \in [u, v]$. Hence $\varphi([u, v]) = X(A_1) \times X(A_2)$.

Let X be an interval of P and $A \in \mathscr{F}(x_0)$. If $X(A^*)$ is a one-element set, then X will be said to be *parallel* to A and we write $X \| A$. From 2.1 it follows:

2.2. *If* $A \in \mathscr{F}(x_0)$ *and* $X \in \mathscr{P}$, *then either* $X \| A$ *or* $X \| A^*$. *Moreover,* $X \| A$ *if and only if the mapping* $x \to \varphi_1(x)$ *is an isomorphism of* X *into* A.

2.3. *Let* $p = [x, y] \in \mathscr{P}$, $A \in \mathscr{F}(x_0)$, $B \in \mathscr{F}(x_0)$, $p \| A$, *and* $p \| B$. *Then* $p \| A \cap B$.

Proof. Assume (to the contrary) that p is not parallel to $A \cap B$. Then, according to 2.2, $p \| (A \cap B)^*$; thus $x(A \cap B) = y(A \cap B)$. In view of 1.16, we have $x(A)(B) = y(A)(B)$, therefore $[x(A), y(A)] \| B^*$. Moreover, from $p \| A$ we obtain $x(A) < y(A)$, so $x(A)(B^*) < y(A)(B^*)$, whence, by 1.13, $x(B^*)(A) < y(B^*)(A)$. But $x(B^*) = y(B^*)$, since $p \| B$, and so $x(B^*)(A) = y(B^*)(A)$. A contradiction.

Let $x, y \in P$, $A = \{a_0, a_1, \ldots, a_n\} \subset P$, $a_0 = x$, and $a_n = y$. Assume that a_{i-1}, a_i are comparable and that the interval with the end points a_{i-1}, a_i is irreducible for $i = 1, \ldots, n$. Then A is said to be a line of length n connecting x and y. Let $x \neq y$ and let $d(x, y)$ be the minimal length of lines connecting x and y; if $d(x, y) = n$, A is said to be a minimal line connecting x and y. A line A is simple if $a_{i-1} \neq a_i$ for $i = 1, \ldots, n$. For any line A connecting distinct elements x and y there is a simple line $A^0 \subset A$ connecting x and y. Any minimal line connecting distinct elements is simple.

2.4. *Let* $A_1 = \{a_0, \ldots, a_n\}$ *be a minimal line connecting* x *and* y. *If* $A \in \mathscr{F}(x_0)$ *and* $x(A) = y(A)$, *then* $a_i(A) = x(A)$ *for* $i = 1, \ldots, n$.

Proof. There are elements $b_i \in P$ such that $b_i(A) = x(A)$, and $b_i(A^*) = a_i(A^*)$ for $i = 1, \ldots, n$. Then $b_0 = x$, $b_n = y$ and the elements b_{i-1}, b_i are comparable. Let p_i (q_i) be the interval with the end points a_{i-1}, a_i (b_{i-1}, b_i). In view of 2.2, since p_i is irreducible, either $\operatorname{card} q_i = 1$ or q_i is isomorphic to p_i, and thus q_i is also irreducible. Therefore, $B = \{b_0, \ldots, b_n\}$ is a line connecting x and y. Since A_1 is minimal, we have $d(x, y) = n$ and from this we get $B_0 = B$. Hence $b_{i-1} \neq b_i$ for $i = 1, \ldots, n$, and now it follows from 2.2 that $a_{i-1}(A) = a_i(A)$ for $i = 1, \ldots, n$.

Let p be a fixed element of $\mathscr{P}$. By $\mathscr{F}(x_0, p)$ we denote the set of all $A \in \mathscr{F}(x_0)$ such that p is paralell to A. Assume that $p_1, p_2 \in \mathscr{P}$ and let x, u (y, v) be the end points of the interval p_1 (p_2).

2.5. *Let* $x \equiv u(R(A))$ *for each* $A \in \mathscr{F}(x_0, p)$ *and* $u \equiv y(R(A_0^*))$ *for some* $A_0 \in \mathscr{F}(x_0, p)$. *Then there is* $v \in P$ *such that* $x \equiv v(R(A_0^*))$ *and* $v \equiv y(R(A))$ *for each* $A \in \mathscr{F}(x_0, p)$.

Proof. First consider the case where $u \leqslant x$ and $u \leqslant y$. Since (3) is valid for $A = A_0$, $v = x \vee y \in P$, by 1.8, and v satisfies (4) for $A = A_0$. Let $A \in \mathscr{F}(x_0, p)$, $A \neq A_0$. If $u \not\equiv y(R(A))$, then ($[u, v]$ being an element of $\mathscr{P}$) $u \equiv y(R(A^*))$, and thus the argument applied for A_0 remains valid for A; therefore (4) holds. If $u \equiv y(R(A))$, then, according to the dual shape of 1.9, $v \equiv y(R(A))$. Hence $v \equiv y(R(A))$ for each $A \in \mathscr{F}(x_0, p)$.

In the case where $u \geqslant x$ and $u \geqslant y$, the proof is dual to that given above.

Let $x \leqslant u \leqslant v$. According to 1.8, the element u has a unique relative complement v in the interval $[x, y]$ and v satisfies (4) for $A = A_0$. Let $A \in \mathscr{F}(x_0, p)$. If $u \equiv y(R(A))$ does not hold, then $u \equiv y(R(A^*))$ and (4) holds for A. And if $u \equiv y(R(A))$, then $x \equiv y(R(A))$, and this implies $y \equiv v(R(A))$, since $[v, y] \subset [x, y]$.

The case $x \geqslant u \geqslant v$ is dual.

2.6. *Let* $x, y, u, v \in P$ *be such that* (3) *and* (4) *are valid,* $u < y$ *and* $[u, y] \in \mathscr{P}$. *Then* $x < v$ *and* $[x, v] \in \mathscr{P}$.

Proof. In the isomorphism $P \to A \times A^*$ described in Section 1 we have

$$u \to (u(A), u(A^*)), \tag{8}$$

$$y \to (y(A), y(A^*)) = (u(A), y(A^*)), \tag{9}$$

$$x \to (x(A), x(A^*)) = (x(A), u(A^*)), \tag{10}$$

$$v \to (v(A), v(A^*)) = (x(A), y(A^*)). \tag{11}$$

From (8), (9) and from $u < y$ we infer that $u(A^*) < y(A^*)$ and that the intervals $[u, y]$ and $[u(A^*), y(A^*)]$ are isomorphic. Combining this with (10) and (11) we infer that $x < v$, $[u, y]$ and $[x, v]$ are isomorphic. Thus $[x, v] \in \mathscr{P}$.

2.7. *If assumptions of* 2.5 *are valid, and* q_1 *and* q_2 *are the intervals with the end points* x, v *and* y, v, *respectively, then* $q_1, q_2 \in \mathscr{P}$.

This follows from 2.5 and 2.6.

Let $x \in P$, $x \neq x_0$. Since P is connected and almost discrete, there is a simple line $A = \{a_0, \ldots, a_n\}$ connecting x_0 and x. Let p_i be the interval with the end points a_{i-1}, a_i.

We denote $P_0 = \{p_1, \ldots, p_n\}$, $P_1 = \{p_1 \in P_0\colon a_{i-1} \equiv a_i(R(A))$ for each $A \in \mathscr{F}(x_0, p)\}$, and $P_2 = P_0 \setminus P_1$.

If $P_2 = \{p_{i_1}, \dots, p_{i_k}\}$, then there are $A_j \in \mathscr{F}(x_0, p)$ such that $a_{i_{j-1}} \equiv a_{i_j}(R(A_j^*))$. Therefore $a_{i_{j-1}} \equiv a_{i_j}(R(A_0^*))$, where $A_0 = A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_k}$. According to 2.3, $A_0 \in \mathscr{F}(x_0, p)$.

Now, using induction with respect to n, we obtain from 2.5 and 2.7:

2.8. *Let $x_0, x \in P$, $x \neq x_0$, and $p \in \mathscr{P}$. Then there are elements x^1 and x^2 such that*

$$x_0 \equiv x^1(R(A)) \quad \text{for each } A \in \mathscr{F}(x_0, p), \tag{12}$$

$$x \equiv x^1(R(A_0^*)) \quad \text{for some } A_0 \in \mathscr{F}(x_0, p), \tag{13}$$

$$x \equiv x^2(R(A)) \quad \text{for each } A \in \mathscr{F}(x_0, p), \tag{14}$$

$$x_0 \equiv x^2(R(A_0^*)) \quad \text{for some } A_0 \in \mathscr{F}(x_0, p). \tag{15}$$

For $x = x^0$ we set $x^1 = x^2 = x_0$.

2.9. *Given $x_0, x \in P$ and $p \in \mathscr{P}$, the elements x^1 and x^2 are uniquelly determined.*

Proof. Assume that $x_0 \equiv z(R(A))$ for each $A \in \mathscr{F}(x_0, p)$, and $x \equiv z(R(A_1^*))$ for some $A_1 \in \mathscr{F}(x_0, p)$.

Then $z = x(A_1)$ and $z \in A$ for each $A \in \mathscr{F}(x_0)$. Analogously, by (12) and (13) we get $x^1 = x(A_0)$ and $x^1 \in A$ for each $A \in \mathscr{F}(x_0)$, Then, by 1.13, we have $z = z(A_0) = x(A_1)(A_0) = x(A_0)(A_1) = x^1(A_1) = x^1$.

The proof for x^2 is similar.

2.10. *If x_0, x, p, A_0 are as in* 2.8, *then $x^1 = x(A)$ and $x^2 = x(A^*)$ for each $A \in \mathscr{F}(x_0, p)$ with $A \subset A_0$.*

Proof. Let $A \subset A_0$. Then $A_0^* \subset A^*$ and thus it follows from (12) and (13) that $x_0 \equiv x^1(R(A))$ and $x \equiv x^1(R(A^*))$; therefore $x(A) = x^1$. Analogously, by (14) and (15), we get $x(A^*) = x^2$.

2.11. *Let $x, y \in P$. Then there is $A_2 \in \mathscr{F}(x_0, p)$ such that $x^1 = x(A_2)$, $x^2 = x(A_2^*)$, $y^1 = y(A_2)$ and $y^2 = y(A_2^*)$.*

Proof. There is $A_1 \in \mathscr{F}(x_0, p)$ such that the relations analogous to (12)-(15) hold for x_0, y and A_1. Then $A_2 = A_0 \cap A_1$ belongs to $\mathscr{F}(x_0, p)$ and now it suffices to apply 2.10 with $A = A_2$.

3. Factors A^p and A^{p*}. Under the same assumptions and denotation as in Section 2 we put

$$A^p = \{x^1 \colon x \in P\} \quad \text{and} \quad A^{p*} = \{x^2 \colon x \in P\}.$$

From 2.10 it follows that $A^p \cap A^{p*} = \{x_0\}$. From the definition of x^1 (x^2) we get immediately

$$x \in A^p \Leftrightarrow x^1 = x \Leftrightarrow x^2 = x_0,$$

$$x \in A^{p*} \Leftrightarrow x^2 = x \Leftrightarrow x^1 = x_0.$$

Moreover, from 2.11 it follows that, for $x, y \in P$, we have $x \leqslant y$ if and only if $x^1 \leqslant y^1$ and $x^2 \leqslant y^2$. Let $a \in A^p$, $b \in A^{p*}$. By virtue of 2.8, there is $x \in P$ such that

$$a \equiv x(R(A_0)) \quad \text{for some } A_0 \in \mathscr{F}(x_0, p),$$
$$b \equiv x(R(A)) \quad \text{for each } A \in \mathscr{F}(x_0, p).$$

From the definition of x^1 and x^2 we get $x^1 = a$ and $x^2 = b$. Therefore the mapping $x \to (x^1, x^2)$ is an isomorphism of P onto $A^p \times A^{p*}$. Thus we have shown that conditions (i)-(iii) from Section 1 hold true, whence

3.1. *$P = [A^p \times A^{p*}]$ for each $p \in \mathscr{P}$.*

Clearly, for any $x \in P$, $x^1 = x(A^p)$ and $x^2 = x(A^{p*})$.

3.2. *$A^p \in \mathscr{F}(x_0, p)$ and* $\operatorname{card} A^p > 1$.

Proof. Let $p = [x, y]$. For any $A \in \mathscr{F}(x_0, p)$ we have $x \equiv y(R(A))$, whence $x^2 \equiv y^2(R(A))$; and so $x^2 = y^2$. Therefore $p \| A^p$. Thus the mapping $t \to t^1$ is an isomorphism of $[x, y]$ onto $[x^1, y^1] \subset A^p$; whence $\operatorname{card} A^p \geqslant \operatorname{card}[x, y] \geqslant 2$.

3.3. *A^p is irreducible.*

Proof. Assume that $A^p = [C \times D]$. Then either $C \in \mathscr{F}(x_0, p)$ or $D \in \mathscr{F}(x_0, p)$. Let $C \in \mathscr{F}(x_0, p)$. Thus $P = [C \times C^*]$, $D \subset C^*$. Let $d \in D$. Then $d \in C^*$, and therefore $d \equiv x_0(R(C^*))$.

Hence $d \in A^{p*}$. At the same time $d \in A^p$, therefore $d = x_0$, and so $D = \{x_0\}$.

3.4. *The partially ordered set $\mathscr{F}(x_0)$ is atomic. The system $\{A^p\colon p \in P\}$ is the set of all atoms of $\mathscr{F}(x_0)$.*

Proof. In view of 3.3, each A^p is an atom of $\mathscr{F}(x_0)$. Let $A \in \mathscr{F}(x_0)$' $A \neq \{x_0\}$. There is $a \in A$, $a \neq x_0$. From 2.4 it follows that there is $x_1 \in A$' $x_1 \neq x_0$, such that x_1 and x_0 are comparable and the interval p with the end points x_0, x_1 belongs to $\mathscr{P}$. Then we have $p \| A$, and hence $A^p \subset A$. This shows that $\mathscr{F}(x_0)$ is atomic and that each atom of $\mathscr{F}(x_0)$ belongs to the set $\{A^p\colon p \in P\}$.

3.5. *Let $x, y \in P$, $x \neq y$. Then there is $p \in \mathscr{P}$ such that $x(A^p) \neq y(A^p)$.*

Proof. Assume (to the contrary) that $x(A^p) = y(A^p)$ for each $p \in \mathscr{P}$. Then $x \equiv y(R(A^{p*}))$ for each $p \in \mathscr{P}$. According to 2.4, there is $x_1 \in P$ such that $x_1 \neq x$, x_1 is comparable with x, the interval p_0 with the end points x, x_1 belongs to $\mathscr{P}$ and $x \equiv x_1(R(A^{p*}))$ for each $p \in \mathscr{P}$. By 3.2, $p_0 \| A^{p_0}$, thus $x \equiv x_1(R(A^{p_0}))$. Therefore, $x = x_1$; a contradiction.

For $p_1, p_2 \in \mathscr{P}$ we put $p_1 \sim p_2$ if $A^{p_1} = A^{p_2}$. Then $\sim$ is an equivalence on $\mathscr{P}$. By the Axiom of Choice there is $\mathscr{P}_1 \subset \mathscr{P}$ such that (i) if $p_1, p_2 \in \mathscr{P}_1$, then $p_1 \sim p_2$ does not hold, and (ii) for any $p \in \mathscr{P}$, there is $p_1 \in \mathscr{P}_1$ such that $p \sim p_1$. Obviously, 3.4 and 3.5 remain valid if $\mathscr{P}$ is replaced by $\mathscr{P}_1$.

3.6. *If $p_1, p_2 \in \mathscr{P}_1$, $p_1 \neq p_2$, then $A^{p_1} \cap A^{p_2} = \{x_0\}$.*

This follows from the fact that A^{p_1} and A^{p_2} are distinct atoms of $\mathcal{F}(x_0)$.

3.7. *Let* $[x, y] = p \in \mathcal{P}$, $p \sim p_1 \in \mathcal{P}_1$. *Then* $x(A^{p_2}) = y(A^{p_2})$ *for each* $p_2 \in \mathcal{P}_1$, $p_2 \neq p_1$.

Proof. In view of 3.3 and 1.15, we have $A^{p_2} \subset A^{p^*}$, whence, by 1.17 and 1.13,

$$x(A^{p_2}) = x(A^{p_2} \cap A^{p^*}) = x(A^{p^*})(A^{p_2}) = y(A^{p^*})(A^{p_2}) = y(A^{p^*} \cap A^{p_2}) = y(A^{p_2}).$$

3.8. *Let* $x, y \in P$, $x \neq y$. *If* $\mathcal{P}_2 = \{p \in \mathcal{P}_1 : x(A^p) \neq y(A^p)\}$, *then the set* $\mathcal{P}_2$ *is finite.*

Proof. This follows by induction from 3.7 and from the existence of a minimal line connecting x and y.

In particular, we get from 3.8:

3.9. *Let* $x \in P$. *Then there is a finite subset* $\mathcal{P}_2 \subset \mathcal{P}_1$ *such that* $x(A^p) = x_0$ *for each* $p \in \mathcal{P}_1 \setminus \mathcal{P}_2$.

3.10. *Let* $p_1, \ldots, p_n$ *be distinct elements of* $\mathcal{P}_1$. *Then*

$$P = [A^{p_1} \times A^{p_2} \times \ldots \times A^{p_n} \times Q],$$

where $Q = A^{p_1^*} \cap \ldots \cap A^{p_n^*}$.

This follows from 3.1, 3.6 and 1.20.

3.11. *Let* $x, y \in P$. *If* $x(Q) \neq y(Q)$, *then there is* $p \in \mathcal{P}_1$, $p \neq p_i$ *for* $i = 1, \ldots, n$, *such that* $x(A^p) \neq y(A^p)$.

Proof. Put $u = x(Q)$ and $v = y(Q)$. Then $u(A^{p_i}) = v(A^{p_i}) = x_0$ for each $i = 1, \ldots, n$. Since $u \neq v$, there is, by 3.5, $u(A^p) \neq v(A^p)$ for some $p \in \mathcal{P}_1 \setminus \{p_1, \ldots, p_n\}$. Clearly, $A^p \subset Q$, and thus we have $u(A^p) = x(A^p)$ and $v(A^p) = y(A^p)$.

3.12. *Let* $x, y \in P$. *If* $x(A^p) \leqslant y(A^p)$ *for each* $p \in \mathcal{P}_1$, *then* $x \leqslant y$.

Proof. By 3.8 there are distinct elements $p_1, \ldots, p_n \in \mathcal{P}_1$ such that $x(A^p) = y(A^p)$ for each $p \in \mathcal{P}_1$ and $p \neq p_i$ for $i = 1, \ldots, n$. By 3.11, $x(Q) = y(Q)$ and hence, by 3.10, $x \leqslant y$.

3.13. *Let* $p_1, \ldots, p_n$ *be distinct elements of* $\mathcal{P}_1$, $y_i \in A^{p_i}$ *for* $i = 1, \ldots, n$. *Then there is* $x \in P$ *such that* $x(A^{p_i}) = y_i$, $i = 1, \ldots, n$, *and* $x(A^p) = x_0$ *for each* $p \in P_1$, $p \neq p_i$, $i = 1, \ldots, n$.

Proof. By 3.10, there is $x \in P$ such that $x(A^p) = y_i$ for $i = 1, \ldots, n$ and $x(Q) = x_0$. Moreover, from 3.10 it follows that for $p \neq p_i$ we have $A^p \subset Q$, therefore $x(A^p) = x_0$.

4. Weak products of partially ordered sets. Let I be a non-empty set and, for each $i \in I$, let P_i be a partially ordered set. The direct product $\prod P_i$ $(i \in I)$ is the set of all mappings $f: I \to \bigcup P_i$ such that $f(i) \in P_i$ for each $i \in I$, and the partial order on $\prod P_i$ is defined by the rule $f \leqslant g$ if $f(i) \leqslant g(i)$ for each $i \in I$. Let $\emptyset \neq S \subset \prod P_i$ such that

(i) if $x, y \in S$, then the set $\{i \in I: x(i) \neq y(i)\}$ is finite;

(ii) if $x \in S$, $z \in \prod P_i$ and the set $\{i \in I: x(i) \neq z(i)\}$ is finite, then $z \in S$.

Under these suppositions S is said to be a *weak product* of partially ordered sets P_i for $i \in I$ (cf. [3] and [6]).

4.1. THEOREM. *If P is a connected almost discrete partially ordered set, then P is isomorphic to a weak product of irreducible partially ordered sets.*

Proof. Since the case $\operatorname{card} P = 1$ is trivial, we have $\operatorname{card} P > 1$. Under the same notation as in Section 3 let $T = \prod A^p$ $(p \in \mathscr{P}_1)$ and let φ be the mapping of the set P into T such that $\varphi(x)(p) = x(A^p)$ for each $p \in \mathscr{P}_1$. Put $S = \varphi(P)$. From 3.1, 3.5 and 3.12 it follows that φ is an isomorphism of P onto S. By 3.11, S satisfies (i). According to 3.13, S fulfils (ii). The proof is complete.

Let $G = (G; +, \leqslant)$ be a partially ordered group and let $\bar{G} = (G; \leqslant)$ be the corresponding partially ordered set. Assume that $\bar{G}$ is connected and almost discrete. It is well known that a connected partially ordered group is directed. Put $x_0 = 0$ and construct the sets A^p for $p \in \mathscr{P}_1$ as in Section 3. From 3.1 and [5] we infer that A^p and A^{p^*} are subgroups of the group $(G; +)$ and that the partially ordered group G is a direct product of its partially ordered subgroups A^p and A^{p^*}, the projection of any element x into A^p or A^{p^*} being the element $x(A^p)$ or $x(A^{p^*})$, respectively. Moreover, from [5] and 3.10 it follows that G is a direct product of its partially ordered subgroups $A^{p_1}, \ldots, A^{p_n}$ and Q.

Obviously, if G is a partially ordered group such that the partially ordered set $\bar{G}$ is irreducible, then G is irreducible as well. Therefore, by virtue of 4.1, we have

4.2. THEOREM. *Let G be a partially ordered group such that the corresponding partially ordered set is connected and almost discrete. Then G is a (restricted) direct product of irreducible partially ordered groups.*

We conclude with the following remarks and simple examples.

4.3. *If G is a partially ordered group such that the corresponding partially ordered set $\bar{G}$ is almost discrete, then G need not be discrete.*

Example. Let G be the additive group of all reals with the natural order. G is not discrete and, being linearly ordered, it is almost discrete.

Another example of this kind is furnished by the 1-group of all continuous real function defined on $[0, 1]$.

4.4. *If an interval $[x, y]$ has only two elements, then it is irreducible; thus any discrete partially ordered set is almost discrete.*

4.5. *The set $\mathscr{P}_1$ can be chosen in a way such that, for each $[x, y] \in \mathscr{P}_1$, we have either $x = x_0$ or $y = y_0$.*

Proof. We have to verify that for each $p = [x, y] \in \mathscr{P}_1$ there is $p_1 = [x_1, y_1] \in \mathscr{P}$ with $[x, y] \sim [x_1, y_1]$ such that either $x_1 = x_0$ or $y_1 = y_0$. Then the interval $[x, y]$ in $\mathscr{P}_1$ can be replaced by $[x_1, y_1]$. Since we have shown that $\operatorname{card} A^p > 1$, there is $a \in A^p$ with $a \neq x_0$ and, clearly, $x_0 \in A^p$. From 3.4 it follows that there is a simple line $z_0, z_1, \ldots, z_n$ connecting x_0 and a and such that $z_i \in A^p$ for $i = 0, \ldots, n$. Let p_1 be the interval with the end points $x_0 = z_0$ and z_1. Clearly, $p_1 \sim p$.

4.6. For the concept of a direct decomposition of a graph cf. [7]. Let P be a partially ordered set with four elements x_1, x_2, y_1, y_2 such that $x_i < y_j$, $i, j = 1, 2$, and that the elements x_1 and x_2 (y_1 and y_2) are incomparable. Let $G(P)$ be the corresponding graph. Then $G(P)$ can be written as a non-trivial direct product while P is irreducible. Hence there does not exist a one-to-one correspondence between the weak product decompositions of $G(P)$ and the weak product decompositions of P.

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