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Convergence of multistep methods for Volterra integro-differential equations

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Abstract. The convergence result for a general quasilinear multistep method under Perron type conditions with a nondecreasing comparison function is stated. The Lipschitz-continuity case is also discussed. The result is an extension of a recent result due to K. Taubert.

1. Introduction. Consider the initial-value problem for a Volterra integro-differential equation of the form

$$(1)\qquad \begin{aligned} &y'(x) = F\big(x, y(x), z(x)\big), \qquad x \in I := [x_0, x_0+a],\\ &y(x_0) = y_0, \end{aligned}$$

where

$$(2)\qquad z(x) = \int_{x_0}^{x} K\big(x, t, y(t)\big)dt.$$

It is assumed that the functions F and K are continuous on T and S, respectively, where

$$T := \{(x, y, z)\colon x \in I,\ |y| < \infty,\ |z| < \infty\},$$
$$S := \{(x, t, u)\colon x_0 \leqslant t \leqslant x \leqslant x_0+a,\ |u| < \infty\}.$$

For computing a numerical approximation to a solution of problem (1) a uniform step size h is used. The approximate solution is denoted by $\{y_i^h\}_{i=0}^N$, where y_i^h is an approximation to $Y_i^h = Y(x_i^h)$, Y is the solution of problem (1), $x_i^h = x_0+ih$ for $i = 0, 1, \ldots, N$, $Nh = a$.

Put $\mathcal{N} = \{0, 1, \ldots\}$, $I_{h_0} := [0, h_0]$, $h_0 > 0$. Let functions $\Phi_i\colon I^{k+1} \times R^{k+1} \times R^{k+1} \times I_{h_0} \to R$ for $i \in \mathcal{N}$ and $\alpha_s\colon \mathcal{N} \to R$ for $s = 0, 1, \ldots, k$, be given, and let $\alpha_k(i) \equiv 1$, $\alpha_0(i) \neq 0$ for $i \in \mathcal{N}$. Let $w_{i,s}^h \in R$ for $i = 0, 1, \ldots, N$, $s = 0, 1, \ldots, i$, and $|w_{i,s}^h| \leqslant W < \infty$ for a certain $W > 0$. It is assumed that Φ_i are continuous in all variables uniformly with respect to i.

The aim of this paper is to discuss the convergence problem for the quasilinear multistep (k-step) method of the form

$$(3)\qquad \sum_{s=0}^{k} \alpha_s(i)\, y_{i+s}^h = h\Phi_i(x_{i+k}^h, \ldots, x_i^h, y_{i+k}^h, \ldots, y_i^h, z_{i+k}^h, \ldots, z_i^h, h)$$

$i = 0, 1, \ldots, N-k$, where z_i^h are given by the following linear quadrature for (2):

$$z_i^h = h \sum_{s=0}^{i} w_{i,s}^h K(x_i^h, x_s^h, y_s^h), \quad i = 0, 1, \ldots, N, \tag{4}$$

and y_i^h for $i = 0, 1, \ldots, k-1$ are given. These starting values may be obtained by other methods, e.g. one-step methods [3], step-by-step methods [9], block methods [6] or by methods considered in [1]. Other starting procedures are given in [2].

Special cases of the methods of type (3) are:

quasilinear multistep methods with $\alpha_s(\cdot)$, $s = 0, 1, \ldots, k$, and Φ_i constant with respect to i:

$$\sum_{s=0}^{k} \alpha_s y_{i+s}^h = h\Phi(x_{i+k}^h, \ldots, x_i^h, y_{i+k}^h, \ldots, y_i^h, z_{i+k}^h, \ldots, z_i^h, h), \quad i = 0, 1, \ldots, N-k; \tag{5}$$

nonstationary linear methods of the form

$$\sum_{s=0}^{k} \alpha_s(i) y_{i+s}^h = h \sum_{s=0}^{k} \beta_s(i) F_{i+s}^h, \quad i = 0, 1, \ldots, N-k, \tag{6}$$

where $F_i^h = F(x_i^h, y_i^h, z_i^h)$, and, of course,

linear multistep methods with constant coefficients:

$$\sum_{s=0}^{k} \alpha_s y_{i+s}^h = h \sum_{s=0}^{k} \beta_s F_{i+s}^h, \quad i = 0, 1, \ldots, N-k. \tag{7}$$

Observe that the class of one-step methods discussed in [3] is a special case of (3), but not of (6). The class of methods of the form (3) seems to be sufficiently large to unify the convergence discussion concerning one-step and k-step methods.

The class of methods of type (7) has been studied in [2], [6], [8], under the assumption that F and K are Lipschitz-continuous in (y, z) and u, respectively. It was proved that both constistency and stability imply convergence. The order of methods of that type has been studied in [2].

It is the purpose of this paper to examine the convergence of the methods of type (3) under the Lipschitz-continuity assumption on F and K and also in the case where the only conditions imposed on the functions F and K are Perron type conditions with a nondecreasing comparison function.

Recently Taubert [10] proved that a result of this type holds for ordinary differential equations

$$y'(x) = f(x, y(x)), \quad x \in I,$$
$$y(x_0) = y_0,$$

and for the methods of type (7) with F_i^h replaced by $f_i^h = f(x_i^h, y_i^h)$. This result was extended in [4] to methods of type (3) with Φ_i replaced by $\Phi_i = \Phi_i^f(x_{i+k}^h, \ldots, x_i^h, y_{i+k}^h, \ldots, y_i^h, h)$.

In the present paper a similar result is established for Volterra integro-differential equations by the method given in [4].

2. Definitions of convergence and consistency. Let $\{y_i^h\}_{i=0}^N$ be the sequence produced by the method (3). Put $Y_i^h = Y(x_i^h)$ where Y is the solution of problem (1).

DEFINITION. The method (3) is said to be *convergent* if

$$\lim_{h\to 0} \max \{|y_i^h - Y_i^h|:\ 0 \leqslant i \leqslant N\} = 0.$$

DEFINITION. The linear quadrature (4) is said to be *convergent* if for any continuous function y on I and any $x \in I$

$$\Big|\int_{x_0}^{x} y(t)\,dt - h\sum_{s=0}^{i} w_{i,s}^h\, y(x_s^h)\Big| = \zeta(x, h)$$

and $\lim_{h\to 0} \bar{\xi}(h) = 0$, where

$$\bar{\xi}(h) := \sup\{|\zeta(x, h)|:\ x \in I\}, \tag{8}$$

and $i = E(x/h)$ is the greatest integer not exceeding x/h.

Let us introduce the difference-integral operator $\mathscr{L}$ associated with the method (3):

$$\mathscr{L}\big(Y(x), h, i\big) = \sum_{s=0}^{k} \alpha_s(i)\, Y(x+sh) - \\ - h\Phi_i\big(x+kh, \ldots, x,\ Y(x+kh), \ldots, Y(x),\ Z(x+kh), \ldots, Z(x),\ h\big), \tag{9}$$

where

$$Z(x) = \int_{x_0}^{x} K\big(x, s, Y(s)\big)\,ds.$$

DEFINITION. The method (3) is said to be *consistent with the problem* (1) *on the solution* Y if:

(A) $\mathscr{L}\big(Y(x), h, i\big) = h\eta(x, h, i)$ and $\lim_{h\to 0} \bar{\eta}(h) = 0$, where

$$\bar{\eta}(h) = \sup\{|\eta(x, h, i)|:\ x_0 \leqslant x \leqslant x_0 + a - kh,\ 0 \leqslant i \leqslant N-k\}. \tag{10}$$

(B) The linear quadrature (4) is convergent.

We have the following

THEOREM 1. *Under the assumption that* $Y \not\equiv 0$ *and that* $\alpha_s(\cdot)$,

$s = 0, 1, \ldots, k$, *are bounded, the method* (3) *is consistent with the problem* (1) *on* Y *if and only if*

(a) $\sum_{s=0}^{k} \alpha_s(i) = 0, \; i \in A'$,

(b) $\sum_{s=0}^{k} s\alpha_s(i) \, F(x, Y(x), Z(x))$

$$= \Phi_i(x, \ldots, x, Y(x), \ldots, Y(x), Z(x), \ldots, Z(x), 0), \; i \in A', \; x \in I,$$

(c) *the linear quadrature* (4) *is convergent.*

Proof. By Taylor's formula for Y we have $Y(x+sh) = Y(x) + Y'(x)sh + he(x, sh)$ for $s = 0, 1, \ldots, k$, $x \in [x_0, x_0+a-sh]$ and $\sup\{|e(x, sh)|: x_0 \leqslant x \leqslant x_0+a-sh\} \to 0$ as $h \to 0$. Accordingly, we get

$$\mathscr{L}(Y(x), h, i) = Y(x) \sum_{s=0}^{k} \alpha_s(i) + h\Big(Y'(x) \sum_{s=0}^{k} s\alpha_s(i) - \Phi_i(x+kh, \ldots, x, Y(x+kh), \ldots, Y(x), Z(x+kh), \ldots, Z(x), h)\Big) + h \sum_{s=0}^{k} \alpha_s(i) \, e(x, sh), \tag{11}$$

where $Y'(x) = F(x, Y(x), Z(x))$. If we assume consistency, then $\mathscr{L}(Y(x), h, i) = h\eta(x, h, i)$ and $\lim_{h \to 0} \bar{\eta}(h) = 0$, where $\bar{\eta}(h)$ is given by (10). Now (a) follows immediately if we let $h \to 0$ in (11). Consequently, we obtain

$$h\Big(Y'(x) \sum_{s=0}^{k} s\alpha_s(i) - \Phi_i(x+kh, \ldots, x, Y(x+kh), \ldots, Y(x), Z(x+kh), \ldots, Z(x), h)\Big) + h \sum_{s=0}^{k} \alpha_s(i) \, e(x, sh) = h\eta(x, h, i).$$

Dividing by h and passing with h to zero, we arrive at (b). Condition (c) is fulfilled obviously.

Now we assume (a), (b) and (c). As a consequence of the uniform continuity of the functions Φ_i with respect to i we have the relation

$$\Phi_i(x+kh, \ldots, x, Y(x+kh), \ldots, Y(x), Z(x+kh), \ldots, Z(x), h) = \Phi_i(x, \ldots, x, Y(x), \ldots, Y(x), Z(x), \ldots, Z(x), 0) + \varphi(x, h, i), \quad i = 0, 1, \ldots, N-k, \tag{12}$$

and $\lim_{h \to 0} \bar{\varphi}(h) = 0$, where

$$\bar{\varphi}(h) = \sup\{|\varphi(x, h, i)|: x_0 \leqslant x \leqslant x_0+a-kh, \; 0 \leqslant i \leqslant N-k\}.$$

In view of (12) and of the boundedness of $\alpha_s(\cdot)$, $s = 0, 1, \ldots, k$, we obtain consistency.

Remark. Note that the consistency conditions (a), (b) and (c) for the methods (6) take the form

(a$_1$) $\sum_{s=0}^{k} \alpha_s(i) = 0, \ i \in \mathcal{N}$,

(b$_1$) $\sum_{s=0}^{k} s\alpha_s(i) = \sum_{s=0}^{k} \beta_s(i), \ i \in \mathcal{N}$,

(c$_1$) the linear quadrature is convergent,

and similarly for the methods of type (7) (see [6]).

Let us now introduce the difference operator $\mathcal{M}$ associated with the method (3):

$$(13)\qquad \mathcal{M}(Y(x_i^h), h) = \sum_{s=0}^{k} \alpha_s(i)\, Y(x_{i+s}^h) - \\ - h\Phi_i(x_{i+k}^h, \ldots, x_i^h, Y(x_{i+k}^h), \ldots, Y(x_i^h), Z_{i+k}^h, \ldots, Z_i^h, h),$$

$i = 0, 1, \ldots, N-k$, where

$$Z_i^h = h \sum_{s=0}^{i} w_{i,s}^h K(x_i^h, x_s^h, Y(x_s^h)), \qquad i = 0, 1, \ldots, N.$$

It is easy to prove the following

THEOREM 2. *If the method* (3) *is consistent with the problem* (1) *on the solution* $Y \not\equiv 0$ *and if* $\alpha_s(\cdot)$, $s = 0, 1, \ldots, k$, *are bounded, then*

$$\mathcal{M}(Y(x_i^h), h) = h\mu(x_i^h, h)$$

and $\lim_{h\to 0} \bar{\mu}(h) = 0$, *where*

$$(14)\qquad \bar{\mu}(h) = \max\{|\mu(x_i^h, h)| : 0 \leqslant i \leqslant N-k\}.$$

Proof. Just as in the proof of Theorem 1 we obtain

$$\mathcal{M}(Y(x_i^h), h) = Y(x_i^h) \sum_{s=0}^{k} \alpha_s(i) + h\Big(F(x_i^h, Y(x_i^h), Z(x_i^h)) \sum_{s=0}^{k} s\alpha_s(i) - \\ - \Phi_i(x_{i+k}^h, \ldots, x_i^h, Y(x_{i+k}^h), \ldots, Y(x_i^h), Z_{i+k}^h, \ldots, Z_i^h, h)\Big) + \\ + h \sum_{s=0}^{k} \alpha_s(i)\, e(x_i^h, sh).$$

Theorem 1 and the uniform continuity of Φ_i imply that there exists $\theta(x_i^h, h)$ such that

$$\mathcal{M}(Y(x_i^h), h) = h\Big(F(x_i^h, Y(x_i^h), Z(x_i^h)) \sum_{s=0}^{k} s\alpha_s(i) - \\ - \Phi_i(x_{i+k}^h, \ldots, x_i^h, Y(x_{i+k}^h), \ldots, Y(x_i^h), Z(x_{i+k}^h) +$$

$$+\xi(x_{i+k}^h, h), \ldots, Z(x_i^h)+\xi(x_i^h, h), h)\big)+$$
$$+h \sum_{s=0}^{k} \alpha_s(i)\, e(x_i^h, sh)$$
$$= h\theta(x_i^h, h)+h \sum_{s=0}^{k} \alpha_s(i)\, e(x_i^h, sh)$$

and $\lim_{h\to 0} \bar{\theta}(h) = 0$, where

$$\bar{\theta}(h) = \max\{|\theta(x_i^h, h)|:\ 0 \leqslant i \leqslant N-k\}.$$

Now we put $\mu(x_i^h, h) = \theta(x_i^h, h)+\sum_{s=0}^{k} \alpha_s(i)\, e(x_i^h, sh)$ ending the proof of the theorem.

Remark. Note that if Φ_i, $i\in\mathcal{N}$, are of class C^1 with respect to Z_i, $i = 0, 1, \ldots, k$, then Theorem 2 is obvious. This is implied by the following relation which holds between the operators $\mathcal{M}$ and $\mathcal{L}$:

$$\mathcal{M}\big(Y(x_i^h), h\big) = \mathcal{L}\big(Y(x_i^h), h, i\big)+$$
$$+h \sum_{s=0}^{k} \frac{\partial\Phi_i\big(x_{i+k}^h, \ldots, x_i^h, Y(x_{i+k}^h), \ldots, Y(x_i^h), Z_{i+k}^{*h}, \ldots, Z_i^{*h}, h\big)}{\partial Z_{i+k}}\times$$
$$\times\big(Z_{i+s}^h - Z(x_{i+s}^h)\big),$$

where Z_{i+s}^{*h} lies between Z_{i+s}^h and $Z(x_{i+s}^h)$. Note that in this case, if $\bar{\eta}(h) = O(h^q)$ and $\bar{\xi}(h) = O(h^q)$, then $\bar{\mu}(h) = O(h^p)$, where $p = \min(q, q^*)$.

3. On recurrent systems of equations. It is well known that in investigations of multistep methods certain facts concerning recurrent equations are essential. We now quote certain facts of that theory (see [7]).

Consider the systems

$$x_{i+1} = A(i)\, x_i + g_i, \quad i\in\mathcal{N}, \tag{15}$$

$$x_{i+1} = A(i)\, x_i, \quad i\in\mathcal{N}, \tag{16}$$

where $A(i)$, $i\in\mathcal{N}$, are $k\times k$-matrices and g_i, $i\in\mathcal{N}$, are k-vectors, x_i, $i\in\mathcal{N}$, being the unknown k-vectors.

Let $\{x_i(i_0, u, g)\}_{i=i_0}^{\infty}$, where $g = (g_0, g_1, \ldots)$, $u\in R^k$, and i_0 is a fixed natural number, denote the solution of (15) satisfying the condition $x_{i_0}(i_0, u, g) = u$.

We introduce

Definition. The trivial solution of (16) is *stable* if for every $\varepsilon > 0$ and every $i_0\in\mathcal{N}$ there exists $\delta(i_0, \varepsilon)$ such that inequality $\|u\| < \delta(i_0, \varepsilon)$ implies $\|x_i(i_0, u)\| < \varepsilon$ for $i \geqslant i_0$ ($\|\cdot\|$ denotes a norm in R^k). If $\delta(i_0, \varepsilon)$ does not depend on i_0, then the stability is said to be *uniform*.

The following facts are obvious.

LEMMA 1. *Any solution of* (15) *has the form*

$$x_i(i_0, u, g) = \prod_{s=i_0}^{i-1} A(i-1+i_0-s)+\sum_{k=i_0}^{i-1} \Big(\prod_{s=k+1}^{i-1} A(i+k-s)\Big)g_k,$$

for $i = i_0, i_0+1, \ldots$ *By* $\prod_{s=i_0}^{i_0-1}$ *we mean I, the unit matrix, and by* $\sum_{s=i_0}^{i_0-1}$ *the zero vector.*

The trivial solution of (16) is stable if and only if $\prod_{s=i_0}^{i-1} A(i-1+i_0-s)$ is bounded for $i = i_0, i_0+1, \ldots$, i.e., there exists a constant $K_0(i_0)$ such that $\|\prod_{s=i_0}^{i-1} A(i-1+i_0-s)\| \leqslant K_0(i_0)$ for $i = i_0, i_0+1, \ldots$ The trivial solution of (16) is uniformly stable if and only if K_0 does not depend on i_0.

Remark. Let X be the fundamental matrix of equation (16), i.e. the matrix function $i \to X(i)$, $i \in \mathcal{N}$, with the properties: $X(i+1) = A(i)\, X(i)$, $X(0) = I$. If $\det A(i) \neq 0$ for $i \in \mathcal{N}$, Lemma 1 takes the form

LEMMA 2. *If* $\det A(i) \neq 0$ *for* $i \in \mathcal{N}$ *and X is the fundamental matrix of equation* (16), *then the solution of* (15) *has the form*

$$x_i(i_0, u, g) = X(i)\, X^{-1}(i_0)\, u+ \sum_{s=i_0}^{i-1} X(i)\, X^{-1}(s+1)\, g_s,$$

for $i = i_0, i_0+1, \ldots$ *The trivial solution of* (16) *is stable if and only if* $X(i)$ *is bounded for* $i = i_0, i_0+1, \ldots$, *i.e., there exists a constant* $K_0(i_0)$ *such that* $\|X(i)\| \leqslant K_0(i_0)$ *for* $i = i_0, i_0+1, \ldots$ *The trivial solution of* (16) *is uniformly stable if and only if* $X(i)\, X^{-1}(i_0)$ *is bounded, i.e., there exists a constant K not depending on* i_0 *such that* $\|X(i)\, X^{-1}(i_0)\| \leqslant K$ *for every* $i_0 \in \mathcal{N}$ *and* $i = i_0, i_0+1, \ldots$

LEMMA 3. *If the trivial solution of* (16) *is uniformly stable, then there exists a constant* $C > 1$ *such that*

$$\|x_i(i_0, u, g)\| \leqslant C\Big(\|u\| + \sum_{s=i_0}^{i-1} \|g_s\|\Big)$$

for $i = i_0, i_0+1, \ldots$

DEFINITION. A $k \times k$-matrix A is of *class* $\mathcal{M}$ if for every eigenvalue λ such that $|\lambda| = \varrho(A)$ every Jordan block associated with λ is 1×1 ($\varrho(A)$ denotes the spectral radius of A).

LEMMA 4. *The trivial solution of* (16) *with a constant matrix* $A(i) \equiv A$, $i \in \mathcal{N}$, *is uniformly stable if and only if* $\varrho(A) \leqslant 1$ *and if* $\varrho(A) = 1$ *implies that A is of class* $\mathcal{M}$.

Let us now consider the kth order linear recurrent equations of the form

$$\sum_{s=0}^{k} \alpha_s(i) z_{i+s} = h_i, \quad i \in \mathcal{N}, \tag{17}$$

$$\sum_{s=0}^{k} \alpha_s(i) z_{i+s} = 0, \quad i \in \mathcal{N}, \tag{18}$$

where $\alpha_s(\cdot)$, $s = 0, 1, \ldots, k-1$, are the coefficients which appear in (3). The notion of the stability and uniform stability of the trivial solution of equation (17) is now introduced by reducing (17) to the corresponding first order system of recurrent equations. Indeed, to write (17) in the form (15) it suffices to put

$$A(i) = \begin{bmatrix} -\alpha_{k-1}(i) & -\alpha_{k-2}(i) & \ldots & -\alpha_1(i) & -\alpha_0(i) \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & & 1 & 0 \end{bmatrix},$$

$$x_i = (z_{i+k-1}, z_{i+k-2}, \ldots, z_{i+1}, z_i)^T, \qquad g_i = (h_i, 0, \ldots, 0, 0)^T.$$

LEMMA 5. *The trivial solution of the k-th order recurrent equation with constant coefficients*

$$\sum_{s=0}^{k} \alpha_s y_{i+s} = 0, \quad i \in \mathcal{N},$$

is uniformly stable if and only if no root of the polynomial

$$p(\lambda) = \sum_{s=0}^{k} \alpha_s \lambda^s$$

has modulus greater than one, and if every root with modulus one is simple.

Taking the norm $\|w\|_{\mathcal{X}} = \max_{1 \leqslant i \leqslant k} |w_i|$, where $w = (w_1, \ldots, w_k) \in R^k$, we derive from Lemma 3

LEMMA 6. *If the trivial solution of the homogeneous equation* (18) *is uniformly stable, then there exists a constant* $C > 1$ *such that every solution of* (17) *satisfies the inequality*

$$\max_{0 \leqslant s \leqslant k-1} |z_{i+s}| \leqslant C\Big(\max_{0 \leqslant s \leqslant k-1} |z_s| + \sum_{s=0}^{k} |h_s|\Big), \quad i \in \mathcal{N}.$$

DEFINITION. The method (3) is said to be *stable* if the trivial solution of the linear homogeneous equation associated with the method (3) is uniformly stable.

4. The convergence of the method. The Lipschitz-continuity case. We have the following

THEOREM 3. *Suppose that:*

(i) *There exists constants* L_s, $N_s \in R$, $s = 0, 1, \ldots, k$, *such that for every* $s_j \in I$, u_j, $\bar{u}_j$, v_j, $\bar{v}_j \in R$, $h \in I_{h_0}$, $j = 0, 1, \ldots, k$, *and* $i \in \mathcal{N}$

$$|\Phi_i(s_0, \ldots, s_k, u_0, \ldots, u_k, v_0, \ldots, v_k, h) - \Phi_i(s_0, \ldots, s_k, \bar{u}_0, \ldots, \bar{u}_k, \bar{v}_0, \ldots, \bar{v}_k, h)| \leqslant \sum_{s=0}^{k} L_s |u_s - \bar{u}_s| + \sum_{s=0}^{k} N_s |v_s - \bar{v}_s|.$$

(ii) *There exists a constant* $D \in R$ *such that for every* $x \in I$, $t \in [x_0, x]$, u, $\bar{u} \in R$

$$|K(x, t, u) - K(x, t, \bar{u})| \leqslant D|u - \bar{u}|.$$

(iii) *There exists* $i_0 \in \mathcal{N}$ *such that* $\sum_{s=0}^{k} s\alpha_s(i_0) \neq 0$.

(iv) *The method* (3) *is stable and consistent with problem* (1) *on the solution* Y.

(v) $\lim_{h \to 0} y_i^h = y_0$ *for* $i = 0, 1, \ldots, k-1$.

Then the method (3) *is convergent to the solution* Y *of problem* (1).

Proof. First of all we note that the assumptions of this theorem ensure the existence in I and uniqueness of the solution of problem (1). Indeed, in this case, in view of (iii) and condition (b) of consistency (see Theorem 1), F is Lipschitz-continuous with respect to (y, z). Now, the existence and uniqueness is implied by Lipschitz-continuity of F and K. Next observe that the sequence $\{y_i^h\}_{i=0}^N$ is well defined by formula (3) for all sufficiently small h. This follows from assumption (i) and the Banach contraction principle.

Put $\varepsilon_i^h = y_i^h - Y_i^h$, $i = 0, 1, \ldots, N$. By consistency we have

$$(19) \qquad \sum_{s=0}^{k} \alpha_s(i)\, Y_{i+s}^h = h\Phi_i(x_{i+k}^h, \ldots, x_i^h, Y_{i+k}^h, \ldots, Y_i^h, Z_{i+k}^h, \ldots, Z_i^h, h) + h\mu(x_i^h, h),$$

$i = 0, 1, \ldots, N-k$, and $\lim_{h \to 0} \bar{\mu}(h) = 0$, where $\bar{\mu}(h)$ is given by (14). Subtracting (19) from (3) we obtain

$$(20) \qquad \sum_{s=0}^{k} \alpha_s(i)\, \varepsilon_{i+s}^h = h\gamma_i - h\mu(x_i^h, h), \qquad i = 0, 1, \ldots, N-k,$$

where

$$(21) \qquad \gamma_i = \Phi_i(x_{i+k}^h, \ldots, x_i^h, y_{i+k}^h, \ldots, y_i^h, z_{i+k}^h, \ldots, z_i^h, h) - \Phi_i(x_{i+k}^h, \ldots, x_i^h, Y_{i+k}^h, \ldots, Y_i^h, Z_{i+k}^h, \ldots, Z_i^h, h).$$

By the stability of the method (3) we have (see Lemma 6)

$$e_i^h \leqslant C\left(e_0^h + h\sum_{s=0}^{i-1}|\gamma_s| + h\sum_{s=0}^{i-1}|\mu(x_s^h, h)|\right), \tag{22}$$

$i = 0, 1, \ldots, N-k+1$, where $e_i^h := \max_{0\leqslant s\leqslant k-1} |\varepsilon_{i+s}^h|$. It is obvious that

$$|\varepsilon_{i+j}^h| \leqslant e_i^h, \quad |\varepsilon_{i+k}^h| \leqslant e_{i+1}^h, \tag{23}$$

for $j = 0, 1, \ldots, k-1$, $i = 0, 1, \ldots, N-k$. From assumption (i) we get the estimates

$$|\gamma_s| \leqslant \sum_{j=0}^{k} L_j|\varepsilon_{s+j}^h| + \sum_{j=0}^{k} N_j\delta_{s+j}^h,$$

$s = 0, 1, \ldots, N-k$, where $\delta_i^h = |z_i^h - Z_i^h|$. We have the following estimation for δ_{s+j}^h, $s = 0, 1, \ldots, N-k$, $j = 0, 1, \ldots, k$:

$$\delta_{s+j}^h \leqslant h\sum_{n=0}^{s+j}|w_{s+j,n}^h|\,|K(x_{s+j}^h, x_n^h, y_n^h) - K(x_{s+j}^h, x_n^h, Y_n^h)|$$
$$\leqslant hWD\sum_{n=0}^{s+j}|\varepsilon_n|.$$

From (22) we obtain

$$e_i^h \leqslant C\left(e_0^h + h\sum_{s=0}^{i-1}\sum_{j=0}^{k} L_j|\varepsilon_{s+j}^h| + h^2 WD\sum_{s=0}^{i-1}\sum_{j=0}^{k} N_j\sum_{n=0}^{s+j}|\varepsilon_n^h| + h\sum_{s=0}^{i-1}|\mu(x_s^h, h)|\right). \tag{24}$$

Now we evaluate $\sum_{s=0}^{i-1}\sum_{j=0}^{k} L_j|\varepsilon_{s+j}^h|$ and $\sum_{s=0}^{i-1}\sum_{j=0}^{k} N_j\sum_{n=0}^{s+j}|\varepsilon_n^h|$. According to (24) we have

$$\sum_{s=0}^{i-1}\sum_{j=0}^{k} L_j|\varepsilon_{s+j}^h| = \sum_{s=0}^{i-1}\left(\sum_{j=0}^{k-1} L_j|\varepsilon_{s+j}^h| + L_k|\varepsilon_{s+k}^h|\right)$$
$$\leqslant \sum_{s=0}^{i-1}\left(\sum_{j=0}^{k-1} L_j e_s^h + L_k e_{s+1}^h\right) = L\sum_{s=0}^{i-1} e_s^h + L_k\sum_{s=0}^{i-2} e_{s+1}^h + L_k e_i^h$$
$$\leqslant (L+L_k)\sum_{s=0}^{i-1} e_s^h + L_k e_i^h,$$

where $L := \sum_{j=0}^{k-1} L_j$. Similarly we get

$$\sum_{s=0}^{i-1}\sum_{j=0}^{k} N_j\sum_{n=0}^{s+j}|\varepsilon_n^h| \leqslant \sum_{s=0}^{i-1}\sum_{j=0}^{k} N_j\sum_{n=0}^{s+k}|\varepsilon_n^h|$$

$$\leqslant N \sum_{s=0}^{i-1} \Big(\sum_{n=0}^{k-1} |\varepsilon_n^h| + \sum_{n=k}^{s+k-1} |\varepsilon_n^h| + |\varepsilon_{s+k}^h| \Big)$$

$$\leqslant N \sum_{s=0}^{i-1} \Big(\sum_{n=0}^{k-1} e_0^h + \sum_{n=k}^{s+k-1} e_{n-k+1}^h + e_{s+1}^h \Big)$$

$$\leqslant kN \sum_{s=0}^{i-1} \Big(\sum_{j=0}^{s} e_j^h + e_{s+1}^h \Big) \leqslant kN \Big(\sum_{s=0}^{i-1} \sum_{j=0}^{s} e_j^h + \sum_{s=0}^{i} e_s^h \Big),$$

where $N := \sum_{j=0}^{k} N_j$. Substituting these evaluations to (24) we obtain

$$e_i^h \leqslant C \Big(e_0^h + h(L+L_k) \sum_{s=0}^{i-1} e_s^h + hL_k e_i^h +$$
$$+ h^2 kWDN \Big(\sum_{s=0}^{i-1} \sum_{j=0}^{s} e_j^h + \sum_{s=0}^{i-1} e_s^h + e_i^h \Big) + h \sum_{s=0}^{i-1} |\mu(x_s^h, h)| \Big).$$

Put

$$\bar{e}_0^h = Ce_0^h/(1 - h_0 CL_k - h_0^2 kCWDN),$$
$$A = C(L+L_k)/(1 - h_0 CL_k - h_0^2 kCWDN),$$
$$B = 2kCWDN/(1 - h_0 CL_k - h_0^2 kCWDN),$$
$$E = Ca/(1 - h_0 CL_k - h_0^2 kCWDN),$$

and assume that $Ch_0(L_k + h_0 kWDN) < 1$. Under this assumption we have

$$e_i^h \leqslant \bar{e}_0^h + hA \sum_{s=0}^{i-1} e_s^h + h^2 B \sum_{s=0}^{i-1} \sum_{j=0}^{s} e_j^h + E\bar{\mu}(h), \tag{25}$$

$i = 0, 1, \ldots, N-k+1$, where $\bar{\mu}(h)$ is given by (14). Denote the right-hand side of inequality (25) by v_i^h. Now

$$v_{i+1}^h - v_i^h = hAe_i^h + h^2 B \sum_{j=0}^{i} e_j^h \leqslant hAv_i^h + h^2 B \sum_{j=0}^{i} v_j^h,$$

and

$$v_{i+1}^h \leqslant (1+hA) v_i^h + h^2 B \sum_{j=0}^{i} v_j^h, \quad i = 0, 1, \ldots, N-k.$$

It is obvious that the sequence $\{v_i^h\}_{i=0}^{N-k+1}$ is nondecreasing. From this we have

$$v_{i+1}^h \leqslant \big(1 + (A+Ba)h\big) v_i^h,$$

$i = 0, 1, \ldots, N-k$. We show by induction that

$$v_i^h \leqslant (1+Gh)^i v_0^h \leqslant v_0^h \exp G(x_i^h - x_0^h) \leqslant v_0^h \exp Ga, \tag{26}$$

where $G = A+Ba$. Now the assertion of theorem follows in view of the relation $\lim_{h\to 0} v_0^h = 0$.

Remark. Note that if $\bar{\mu}(h) = O(h^p)$ and $e_0^h = O(h^r)$, then $v_i^h = O(h^q)$, where $q = \min(p, r)$. Finally, in view of the inequality $|\varepsilon_{i+k-1}^h| \leqslant e_i^h \leqslant v_i^h$, we get $|\varepsilon_i^h| = O(h^q)$, $i = 0, 1, \ldots, N$.

5. The convergence of the method. The general case. We have the following

THEOREM 4. *Suppose that:*

(i) *There exists* $\omega_1\colon I^{k+1}\times R^{k+1}\times R^{k+1}\times I_{h_0}\to R$ *such that for every* $s_j\in I$, $u_j, \bar{u}_j, v_j, \bar{v}_j\in R$, $h\in I_{h_0}$, $j = 0, 1, \ldots, k$,

$$|\Phi_i(s_0, \ldots, s_k, u_0, \ldots, u_k, v_0, \ldots, v_k, h) - \Phi_i(s_0, \ldots, s_k, \bar{u}_0, \ldots, \bar{u}_k, \bar{v}_0, \ldots, \bar{v}_k, h)| \leqslant \omega_1(s_0, \ldots, s_k, |u_0-\bar{u}_0|, \ldots, |u_k-\bar{u}_k|, |v_0-\bar{v}_0|, \ldots, |v_k-\bar{v}_k|, h).$$

(ii) ω_1 *is continuous, bounded and nondecreasing with respect to* u_i, v_i, $i = 0, 1, \ldots, k$, *and, moreover,*

$$\omega_1(s_0, \ldots, s_k, 0, \ldots, 0, 0, \ldots, 0, 0) = 0.$$

(iii) *There exists* $\omega_2\colon S\to R$ *such that* ω_2 *is continuous, bounded, nondecreasing with respect to the last argument, and*

$$|K(x, t, u)-K(x, t, \bar{u})| \leqslant \omega_2(x, t, |u-\bar{u}|)$$

for every $x\in I$, $t\in[x_0, x]$, $u, \bar{u}\in R$.

(iv) *For any* $p\geqslant 1$, $q\geqslant 1$ *the problem*

$$u'(x) = p\omega_1\Big(x, \ldots, x, u(x), \ldots, u(x), q\int_{x_0}^{x}\omega_2(x, t, u(t))dt, \ldots, q\int_{x_0}^{x}\omega_2(x, t, u(t))dt, 0\Big),$$

$$u(x_0) = 0,$$

has in I *only the trivial solution.*

(v) *The method* (3) *is stable and consistent with the problem* (1) *on the solution* Y.

(vi) *There exists* $i_0\in\mathcal{N}$ *such that* $\sum_{s=0}^{k} s\alpha_s(i_0) \neq 0$.

(vii) $\lim_{h\to 0} y_i^h = y_0$ *for* $i = 0, 1, \ldots, k-1$.

Then the method (3) *is convergent to the solution* Y *of the problem* (1).

Proof. Note that the assumptions of the theorem ensure the existence and uniqueness of the solution of problem (1). Indeed, by assumptions (i), (vi) and condition (b) of the consistency (see Theorem 1) we have the following estimate for F:

$$|F(x, y, z)-F(x, \bar{y}, \bar{z})|$$
$$\leqslant \Big(1/\big(\sum_{s=0}^{k} s\alpha_s(i_0)\big)\Big)\, \omega_1(x, \ldots, x, |y-\bar{y}|, \ldots, |y-\bar{y}|, |z-\bar{z}|, \ldots, |z-\bar{z}|, 0).$$

This implies that F is bounded. It is clear that the operator

$$\mathscr{K} y(x) := y_0 + \int_{x_0}^{x} F\big(t, y(t), z(t)\big)dt$$

with $z(t)$ defined by (2) is compact. The existence of the solution of problem (1) is now a consequence of the Schauder fixed point theorem. The uniqueness of the solution of problem (1) is implied by the theory of integral inequalities (see [5]).

Observe that the sequence $\{y_i^h\}_{i=0}^N$ is well defined by formula (3). This is a consequence of the boundedness of Φ_i for any fixed i.

In this section we use the notations introduced in Section 4. As in the proof of Theorem 3 we have

$$e_i^h \leqslant C\big(e_0^h + h\sum_{s=0}^{i-1} |\gamma_s| + h\sum_{s=0}^{i-1} |\mu(x_s^h, h)|\big), \tag{27}$$

$i = 0, 1, \ldots, N-k+1$. Taking r_i^h to be equal to the right-hand side of inequality (27) we obtain $e_i^h \leqslant r_i^h$ for $i = 0, 1, \ldots, N-k+1$ and

$$r_{i+1}^h - r_i^h = Ch|\gamma_i| + Ch|\mu(x_i^h, h)|.$$

From assumption (i) we have the following inequality:

$$|\gamma_i| \leqslant \omega_1(x_{i+k}^h, \ldots, x_i^h, |\varepsilon_{i+k}^h|, \ldots, |\varepsilon_i^h|, \delta_{i+k}^h, \ldots, \delta_i^h, h),$$

$i = 0, 1, \ldots, N-k$. It is obvious that

$$|\varepsilon_{i+j}^h| \leqslant e_i^h \leqslant r_i^h, \qquad |\varepsilon_{i+k}^h| \leqslant e_{i+1}^h \leqslant r_{i+1}^h \tag{28}$$

for $j = 0, 1, \ldots, k$, $i = 0, 1, \ldots, N-k$. Let us now estimate δ_s^h for $s = i, i+1, \ldots, i+k$, $i = 0, 1, \ldots, N-k$. We have

$$\delta_{i+k}^h = |z_{i+k}^h - Z_{i+k}^h| = \big|h\sum_{s=0}^{i+k} w_{i+k,s}^h\big(K(x_{i+k}^h, x_s^h, y_s^h) - K(x_{i+k}^h, x_s^h, Y_s^h)\big)\big|$$
$$\leqslant hW\sum_{s=0}^{i+k} \omega_2(x_{i+k}^h, x_s^h, |\varepsilon_s^h|)$$
$$= hW\sum_{s=0}^{k-1} \omega_2(x_{i+k}^h, x_s^h, |\varepsilon_s^h|) + hW\sum_{s=k}^{i+k} \omega_2(x_{i+k}^h, x_s^h, |\varepsilon_s^h|)$$

$$\leqslant hW\sum_{s=0}^{k-1}\left(\omega_2(x_{i+k}^h, x_0^h, e_0^h)+A(h)\right)+$$
$$+hW\sum_{s=k}^{i+k}\left(\omega_2(x_{i+k}^h, x_{s-k+1}^h, e_{s-k+1}^h)+A(h)\right)$$
$$\leqslant khW\sum_{s=0}^{i+1}\omega_2(x_{i+k}^h, x_s^h, e_s^h)+kaWA(h),$$

where

$$A(h)=\sup\{|\omega_2(x,t,u)-\omega_2(x,\bar{t},u)|:\ |t-\bar{t}|\leqslant(k-1)h,\ x\in I,\ u\in R\}.$$

Similarly we get

$$\delta_{i+j}^h\leqslant khW\sum_{s=0}^{i}\omega_2(x_{i+j}^h, x_s^h, e_s^h)+kWaA(h),\qquad j=0,1,\ldots,k-1.$$

From these and in view of the monotonicity of ω_1 and ω_2 we have

(29) $$r_{i+1}^h\leqslant r_i^h+Ch\omega_1\Big(x_{i+k}^h,\ldots,x_i^h,r_{i+1}^h,\ldots,r_i^h,$$
$$khW\sum_{s=0}^{i+1}\omega_2(x_{i+k}^h, x_s^h, r_s^h)+kWaA(h),$$
$$khW\sum_{s=0}^{i}\omega_2(x_{i+k-1}^h, x_s^h, r_s^h)+kWaA(h),\ldots$$
$$\ldots,khW\sum_{s=0}^{i}\omega_2(x_i^h, x_s^h, r_s^h)+kWaA(h),h\Big)+Ch|\mu(x_i^h,h)|$$

for $i=0,1,\ldots,N-k$. It is clear, in view of the boundedness of ω_1, that there exists a constant D^* such that

$$0\leqslant r_{i+1}^h-r_i^h\leqslant hD^*.$$

But

$$\sum_{s=0}^{i+1}\omega_2(x_{i+k}^h, x_s^h, r_s^h)=\sum_{s=0}^{i}\omega_2(x_{i+k}^h, x_s^h, r_s^h)+\omega_2(x_{i+k}^h, x_{i+1}^h, r_{i+1}^h)$$
$$\leqslant\sum_{s=0}^{i}\omega_2(x_{i+k}^h, x_s^h, r_s^h)+\omega_2(x_{i+k}^h, x_i^h, r_i^h)+A(h)+B(h)$$
$$\leqslant 2\sum_{s=0}^{i}\omega_2(x_{i+k}^h, x_s^h, r_s^h)+A(h)+B(h),$$

where

$$B(h)=\sup\{|\omega_2(x,t,u)-\omega_2(x,t,\bar{u})|:\ x\in I,\ t\in[0,x],\ |u-\bar{u}|\leqslant hD^*\}.$$

Now consider the initial-value problem

$$\lambda'(x) = C\omega_1\Big(x, \ldots, x, \lambda(x), \ldots, \lambda(x), M\int_{x_0}^{x} \omega_2(x, s, \lambda(s))ds, \ldots$$
$$\ldots, M\int_{x_0}^{x} \omega_2(x, s, \lambda(s))ds, 0\Big)+q,$$
$$\lambda(x_0) = r_0^h,$$

where $q\in[0, Q]$, and Q is a fixed positive constant. Denote by Λ_q the set of all solutions of this problem and put $\Lambda = \bigcup_{q\in[0,Q]} \Lambda_q$. The set Λ is compact in view of the Ascoli Arzelà theorem. Let $\bar{\lambda}^h$ be the maximal solution of the above problem for $q = Q$. Write

$$S = \sup\{\bar{\lambda}^h(x):\ x\in I\},\qquad S^* = \sup\Big\{\int_{x_0}^{x} \omega_2(x, s, \bar{\lambda}^h(s))ds:\ x\in I\Big\}.$$

The set Λ and the constants S and S^* will occur in the definition of certain quantities introduced in the sequel.

Now, relation (29) is rewritten as follows:

$$r_{i+1}^h \leqslant r_i^h+Ch\omega_1\Big(x_{i+k}^h, \ldots, x_i^h, r_i^h, \ldots, r_i^h,$$
$$2hkW\sum_{s=0}^{i} \omega_2(x_{i+k}^h, x_s^h, r_s^h)+khW(A(h)+B(h))+kWaA(h),$$
$$2hkW\sum_{s=0}^{i} \omega_2(x_{i+k-1}^h, x_s^h, r_s^h)+kWaA(h), \ldots$$
$$\ldots, 2hkW\sum_{s=0}^{i} \omega_2(x_i^h, x_s^h, r_s^h)+kWaA(h), h\Big)+ChG(h)+Ch\bar{\mu}(h),$$

where

$$G(h) = \sup\{|\omega_1(x, \ldots, x, a, b, \ldots, b, c, \ldots, c, h)-$$
$$-\omega_1(x, \ldots, x, \bar{a}, b, \ldots, b, c, \ldots, c, h)|:$$
$$x\in I,\ |a-\bar{a}| \leqslant hD^*,\ |b| \leqslant S,\ |c| \leqslant S^*\},$$

and $\bar{\mu}(h)$ is given by (14). Put

$$D(h) = \sup\{|\omega_2(x, t, u)-\omega_2(\bar{x}, t, u)|:\ t\in I,\ |x-\bar{x}| \leqslant kh,\ |u| \leqslant S^*\},$$

$$f_i^h = \sum_{s=0}^{k} \omega_2(x_{i+k}^h, x_s^h, r_s^h),\qquad i = 0, 1, \ldots, N-k,$$

$$M = 2kW.$$

In view of the continuity of the function ω_1 there exists $E(h)$ such that $\lim_{h\to 0} E(h) = 0$ and

$$\omega_1\big(x_{i+k}^h, \ldots, x_i^h, r_i^h, \ldots, r_i^h, hMf_i^h + khW(A(h)+B(h)) + kWaA(h), \\ hMf_i^h + kWaA(h) + 2hkWD(h), \ldots, hMf_i^h + kWaA(h) + 2hkWD(h), h\big) \\ = \omega_1(x_{i+k}^h, \ldots, x_i^h, r_i^h, \ldots, r_i^h, hMf_i^h, \ldots, hMf_i^h, h) + E(h).$$

Finally, we obtain

$$r_{i+1}^h \leqslant r_i^h + Ch\omega_1(x_{i+k}^h, \ldots, x_i^h, r_i^h, \ldots, r_i^h, hMf_i^h, \ldots, hMf_i^h, h) + hF(h)$$

for $i = 0, 1, \ldots, N-k$, where

$$F(h) = C\big(\bar{\mu}(h) + G(h) + E(h)\big).$$

Let us now consider another initial-value problem:

$$(30)\quad \lambda'(x) = C\omega_1\Big(x+kh, \ldots, x, \lambda(x), \ldots, \lambda(x), \\ M\Big[\int_{x_0}^{x}\big(\omega_2(x, t, \lambda(t)) + D(h) + T(h)\big)dt + hD(h) + hT(h) + P(h)\Big], \ldots \\ \ldots, M\Big[\int_{x_0}^{x}\big(\omega_2(x, t, \lambda(t)) + D(h) + T(h)\big)dt + hD(h) + hT(h) + P(h)\Big], h\Big) + \\ + CQ(h) + F(h), \quad x \in [x_0, x_0 + a - kh], \\ \lambda(x_0) = r_0^h,$$

where

$$P(h) = \sup\Big\{\Big|\int_{x_0}^{x}\omega_2(x, t, z(t))dt - \int_{x_0}^{\bar{x}}\omega_2(x, t, z(t))dt\Big|: \\ |x-\bar{x}| \leqslant h,\ x_0 \leqslant t \leqslant x,\ \bar{x} \leqslant x_0 + a,\ z \in \Lambda\Big\},$$

$$Q(h) = \sup\{|\omega_1(x+k\ \ , \ldots, x, a, \ldots, a, b, \ldots, b, h) - \\ - \omega_1(\bar{x}+kh, \ldots, \bar{x}, a, \ldots, a, b, \ldots, b, h)|: \\ x, \bar{x} \in I,\ |x-\bar{x}| \leqslant h,\ |a| \leqslant S,\ |b| \leqslant S^*\},$$

$$T(h) = \sup\{|\omega_2(x, t, u) - \omega_2(x, \bar{t}, u)|: x_0 \leqslant t, \bar{t} \leqslant x \leqslant x_0 + a, \\ |t - \bar{t}| \leqslant h,\ |u| \leqslant S^*\}.$$

The solution λ^h of this problem is a nondecreasing function. We shall prove that

$$\lambda^h(x_i^h) \geqslant r_i^h, \quad i = 0, 1, \ldots, N-k.$$

This relation holds for $i = 0$. Assuming that it holds for any fixed i and integrating (30) from x_i^h to x_{i+1}^h, we get

$$\lambda^h(x_{i+1}^h) = \lambda^h(x_i^h) + C \int_{x_i^h}^{x_{i+1}^h} \Big(\omega_1\big(x+kh, \ldots, x, \lambda^h(x), \ldots, \lambda^h(x),$$

$$M\Big[\int_{x_0}^{x} \big(\omega_2(x, t, \lambda^h(t)) + D(h) + T(h)\big)dt + hD(h) + hT(h) + P(h)\Big], \ldots$$

$$\ldots, M\Big[\int_{x_0}^{x} \big(\omega_2(x, t, \lambda^h(t)) + D(h) + T(h)\big)dt + hD(h) +$$

$$+ hT(h) + P(h)\Big], h\big) + Q(h)\Big)dx + hF(h)$$

$$\geqslant r_i^h + Ch\omega_1\big(x_{i+k}^h, \ldots, x_i^h, r_i^h, \ldots, r_i^h,$$

$$M \int_{x_0}^{x_{i+1}^h} \big(\omega_2(x_{i+1}^h, t, \lambda^h(t)) + D(h) + T(h)\big)dt, \ldots$$

$$\ldots, M \int_{x_0}^{x_{i+1}^h} \big(\omega_2(x_{i+1}^h, t, \lambda^h(t)) + D(h) + T(h)\big)dt, h\big) + hF(h)$$

$$\geqslant r_i^h + Ch\omega_1\big(x_{i+1}^h, \ldots, x_i^h, r_i^h, \ldots, r_i^h,$$

$$M \sum_{s=0}^{i} \int_{x_s^h}^{x_{s+1}^h} \big(\omega_2(x_{i+k}^h, t, \lambda^h(t)) + T(h)\big)dt, \ldots$$

$$\ldots, M \sum_{s=0}^{i} \int_{x_s^h}^{x_{s+1}^h} \big(\omega_2(x_{i+k}^h, t, \lambda^h(t)) + T(h)\big)dt, h\big) + hF(h)$$

$$\geqslant r_i^h + Ch\omega_1(x_{i+k}^h, \ldots, x_i^h, r_i^h, \ldots, r_i^h, hMf_i^h, \ldots, hMf_i^h, h) + hF(h)$$

$$\geqslant r_{i+1}^h.$$

By (28) we have also

$$\max_{0 \leqslant s \leqslant k-1} |y_{N-k+1+s}^h - Y_{N-k+1+s}^h| = e_{N-k+1}^h \leqslant r_{N-k+1}^h$$

$$\leqslant r_{N-k}^h + Ch\omega_1(x_N^h, \ldots, x_{N-k}^h, r_{N-k}^h, \ldots, r_{N-k}^h, hMf_{N-k}^h, \ldots, hMf_{N-k}^h, h) + hF(h)$$

$$\leqslant r_{N-k}^h + Ch\omega_1\big(x_N^h, \ldots, x_{N-k}^h, \lambda^h(x_{N-k}^h), \ldots, \lambda^h(x_{N-k}^h), hM\bar{f}_{N-k}^h, \ldots$$

$$\ldots, hM\bar{f}_{N-k}^h, h\big) + hF(h) := \lambda^h(x_{N-k+1}^h),$$

where

$$\bar{f}_{N-k}^h = \sum_{s=0}^{N-k} \omega_2\big(x_N^h, x_s^h, \lambda^h(x_s^h)\big).$$

Finally, we obtain

$$(31) \qquad e_i^h \leqslant r_i^h \leqslant \lambda^h(x_i^h), \qquad i = 0, 1, \ldots, N-k+1.$$

According to the theorem on the continuous dependence of the solution of problem (30) on parameters and initial data we have

$$\lim_{h \to 0} \sup \{\lambda^h(x) \colon x \in [x_0, x_0 + a - kh]\} = 0$$

and by (31) $\lim_{h\to 0} e_i^h = 0$ for $i = 0, 1, \ldots, N-k+1$. Thus the proof of the theorem is complete.

6. Some remarks. a) Note that assumption (iv) in Theorem 4 may be weakened. It suffices only to assume that the problem

$$u'(x) = C\omega_1\Big(x, \ldots, x, u(x), \ldots, u(x), M\int_{x_0}^{x}\omega_2(x, t, u(t))dt, \ldots$$
$$\ldots, M\int_{x_0}^{x}\omega_2(x, t, u(t))dt, 0\Big),$$
$$u(x_0) = 0$$

with C defined by Lemma 6 and $M = 2kW$, where W is a bound for the weights in the linear quadrature (4), has in I only the trivial solution.

b) In the proof of Theorems 3 and 4 we found the effective error evaluations given by (26) and (31), respectively.

c) If we consider the explicit method given by

$$\sum_{s=0}^{k}\alpha_s(i)\, y_{i+s}^h = h\Phi_i(x_{i+k}^h, \ldots, x_i^h, y_{i+k-1}^h, \ldots, y_i^h, z_{i+k-1}^h, \ldots, z_i^h, h)$$

then the boundedness of ω_1 assumed in (ii) can be dropped. But we have to assume that the solution Y of problem (1) exists.

d) Theorems 3 and 4 are also valid for systems of Volterra integro-differential equations.

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