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THE DAM WITH CONTENT-DEPENDENT INPUT AND OUTPUT

1. Introduction. We consider the infinite dam in which the content process alternatively increases (input phase) or decreases (output phase) at random time-intervals. The input and output rates depend on the state of the process. Such a model is a generalization of the model considered by Gaver and Miller [2] (see also [6]) with alternatively linear changes of the content process.

We are interested in two problems. The first one is to investigate the effect of some random deformations of both phases on the content and on the wet period. Such investigations were performed by Kopociński ([4], [5]) for the dam with compound renewal (compound Poisson) input and content-dependent output. In Section 3, making analogous assumptions and using the same methods as in [4] and [5], we give theorems about the expected increase of the contents at the initial moments of both phases and about the expected value of the wet period.

The second problem regards the stationary behaviour of the content process. In Section 4.1 we deal with the existence of invariant probability measures for Markov chains imbedded at the initial moments of both phases. The investigations base on the paper [1] by Çinlar. In Section 4.2 we assume that the extended Markov process, induced by the content process, is stationary and we investigate relations between the stationary distributions of the process and the invariant measures of imbedded Markov chains. Moreover, assuming that the output phases are exponential, we give the forms of stationary distributions. Here we apply the paper [3].

2. The model.

2.1. Content process. Let $(\Omega, \sigma, \Pr)$ be a probability space, $\mathscr{R}_0 = [0, \infty)$, $\mathscr{R}_+ = (0, \infty)$ and let $\mathscr{B}_0$, $\mathscr{B}_+$ stand for the σ-algebras of Borel subsets of these half-lines, respectively. Introduce the notation obligatory in the whole paper: $\mathbf{1} = 1(x)$, $x \in \mathscr{X}$, for the function equal to 1 on a space $\mathscr{X}$; I_A for the indicator of a set A; $f \in \mathrm{b}\mathscr{F}$ for a real bounded $\mathscr{F}$-measurable function and a σ-algebra $\mathscr{F}$; γ^n for the nth iterate of an operator γ, $n = 0, 1, \ldots$, where $\gamma^0 f = f$ for $f \in \mathrm{b}\mathscr{F}$; $\mathrm{E}_A f(A) = \int_{-\infty}^{\infty} f(u)\, d\Pr\{A \leqslant u\}$ for a real function f and a random variable A; Y for a stochastic process $\{Y(t),\ t \geqslant 0\}$; $\boldsymbol{Y} = \{Y_n,\ n = 0, 1, \ldots\}$ and $\boldsymbol{Y}_n = \{Y_0, Y_1, \ldots, Y_n\}$ for a sequence of random variables $Y_0, Y_1, \ldots$

Suppose we are given the following:

(a) a random variable X_0^1: $\Omega \to \mathscr{R}_+$,

(b) sequences of moments τ_{2m+a}, $m = 0, 1, \ldots$, ($\tau_0 = 0$), such that the distances $t_m^a := \tau_{2m+2-a} - \tau_{2m+1-a}$ are positive independent random variables with common distribution function H_a and finite mean $1/\mu_a$, $a = 1, 0$; the sequences t^a, $a = 1, 0$, are also independent,

(c) real functions r_a, $a = 1, 0$, positive, monotone in $\mathscr{R}_+$ and such that $r_1(0) = 0$, $r_0(0) > 0$; $\int_0^{\infty} (1/r_0(u))\, du = \infty$.

Define the content process X for the distances t^a and for the intensity functions r_a, $a = 1, 0$. Assume that the content $X(t)$ decreases with intensity $r_1(X(t))$ or alternatively increases with intensity $r_0(X(t))$ with regard to whether it is in the output or input phase. Hence during the output phase the content process is submitted to the function $q_1 = q_1(c, t)$, $t \geqslant 0$, $c > 0$, fulfilling the differential equation

$$\frac{dq_1}{dt} = -r_1(q_1) \tag{1}$$

with initial value $q_1(c, 0) = c$; during the input phase the content process is submitted to the function $q_0 = q_0(c, t)$, $t \geqslant 0$, $c \geqslant 0$, fulfilling the differential equation

$$\frac{dq_0}{dt} = r_0(q_0) \tag{2}$$

with initial value $q_0(c, 0) = c$.

Let R_a denote the antiderivatives for $1/r_a$ such that $R_a(0) = 0$, provided there exist $R_a(0) < \infty$, $a = 1, 0$. Then the solutions of (1) and (2) can be presented as follows: if $\int_0^{c} (1/r_1(u))\, du < \infty$ then

$$q_1(c, t) = \begin{cases} R_1^{-1}(R_1(c) - t), & 0 \leqslant t \leqslant R_1(c), \\ 0, & t > R_1(c),\ c > 0; \end{cases} \tag{3}$$

if $\int_0^{c} (1/r_1(u))\, du = \infty$ then

$$q_1(c, t) = R_1^{-1}(R_1(c) - t), \quad t \geqslant 0,\ c > 0; \tag{4}$$

$$q_0(c, t) = R_0^{-1}(R_0(c) + t), \quad t \geqslant 0,\ c \geqslant 0. \tag{5}$$

It follows from (1) that $\int_0^c (1/r_1(u))\,du$ is the time to emptiness of the content $c > 0$ provided we are always in the output phase. If this time is finite then we have (3), otherwise we have (4). Similarly, from (2) we see that $\int_0^c (1/r_0(u))\,du$ is the time to obtain the content $c > 0$ starting from zero provided we are always in the input phase. If r_0 is positive, monotone in $\mathscr{R}_+$ and $r_0(0) > 0$ then this time is finite for each c, $0 \leqslant c < \infty$. Hence in (c) we have assumed that the content never reaches infinity in finite time.

Define two chains X^a, $a = 1, 0$, as follows: for each a let

(6) $X_0^0 = q_1(X_0^1, t_0^1), \qquad X_{m+1}^a = q_{1-a}(q_a(X_m^a, t_m^a), t_{m+1-a}^{1-a}), \qquad m = 0, 1, \ldots$

Then we have

$$X(t) = q_a(X_m^a, t-\tau_{2m+1-a}) \tag{7}$$

when $\tau_{2m+1-a} \leqslant t < \tau_{2m+2-a}$, $a = 1, 0$, $m = 0, 1, \ldots$

From the definition (7) we see that $X_m^a = X(\tau_{2m+1-a})$, i.e. the chain X^1 gives the contents at the initial moments of the output phases and the chain X^0 gives the contents at the initial moments of the input phases. Simultaneously, because the distances t_m^a are independent for $m = 0, 1, \ldots$, it follows from (6) that the chains X^a, $a = 1, 0$, are Markov chains.

The model described in this section is denoted by $\{t^1, t^0, r_1, r_0\}$.

2.2. Wet period. Consider the content process X in the model $\{t^1, t^0, r_1, r_0\}$ and for each $c > 0$ define the random variable $T(c)$ by the formula

$$T(c) = \inf_{t>0} \{t\colon X(t) = 0 \,|\, X(0) = c\}.$$

The random variable $T(c)$ is called the *wet period* (the time to emptiness). In this section we give the probability distribution of $T(c)$ and its expected value.

Assume that

$$\int_0^c (1/r_1(u))\,du < \infty \qquad \text{for some } c > 0. \tag{8}$$

Because r_1 is positive monotone in $\mathscr{R}_+$ then from (8) we have

$$\int_0^c (1/r_1(u))\,du < \infty \qquad \text{for each } c > 0.$$

For convenience we write sometimes $R(c)$ instead of $R_1(c)$, i.e. $R(c) = R_1(c)$.

For the process $T(c)$, $c > 0$, we have the relation (for each $\omega \in \Omega$)

$$T(c) = \begin{cases} R(c), & \text{if } \ t_0^1 > R(c), \\ t_0^1 + t_0^0 + T^*(q_0(q_1(c, t_0^1), t_0^0)), & \text{if } \ t_0^1 \leqslant R(c). \end{cases} \tag{9}$$

where $T^*(c)$, $c > 0$, is the wet period process defined for the sequence $t^* = t_1^1, t_1^0, t_2^1, t_2^0, \ldots$ and for the functions r_a, $a = 1, 0$.

Let

$$g(c, t) = I_{[0,t]}(R(c))\big(1 - H_1(R(c))\big),$$

$$A(c, t) = \{(v, u):\ 0 < v+u < t,\ 0 < v,\ 0 < u < R(c)\}, \qquad t \geq 0, c > 0.$$

Define the operator ψ by the formula

$$\psi f(c, t) = \iint_{A(c,t)} f\big(q_0(q_1(c, u), v),\ t-v-u\big)\, dH_0(v)\, dH_1(u), \tag{10}$$

where $f \in \mathrm{b}\mathscr{B}_+ \times \mathscr{B}_0$.

PROPOSITION 1. *Let* $F(c, t) = \Pr\{T(c) \leq t\}$. *Then* F *fulfils the equation*

$$F = g + \psi F, \tag{11}$$

which has a unique solution of the form

$$F = \sum_{n=0}^{\infty} \psi^n g. \tag{12}$$

Proof. The equation (11) follows easily from (9) and from the fact that the sequence t^* does not depend on t_0^1 and t_0^0.

We show the existence and uniqueness of the solution of (11) in a standard way (see [1], Theorem (2.19) or Theorem (3.7)). From (10) it is easy to verify that

$$0 \leq g(c, t) \leq 1 - H_1\big(R(c)\big) \leq 1(c, t) - \psi 1(c, t).$$

The operator ψ is nonnegative, thus we have for each N

$$0 \leq \sum_{n=0}^{N} \psi^n g \leq \sum_{n=0}^{N} \psi^n (1 - \psi \mathbf{1}) = 1 - \psi^{N+1} \mathbf{1} \leq 1.$$

Hence the series in (12) is convergent and it is the solution of (11). For the uniqueness of the solution it is sufficient to show that $\lim_{n\to\infty} \psi^n F = 0$. Because $0 \leq F \leq 1$ and ψ is nonnegative, we have

$$0 \leq \psi^n F \leq \psi^n \mathbf{1}. \tag{13}$$

Using (10), it is not difficult to verify that

$$\psi^n 1(c, t) \leq \Pr\{t_0^1 + t_0^0 + \ldots + t_{n-1}^1 + t_{n-1}^0 \leq t\}. \tag{14}$$

The right-hand side of (14) tends to zero when $n \to \infty$ by the assumption that t_m^a, $m = 0, 1, \ldots$, are positive, independent and identically distributed. Hence and from (13) and (14) we have $\lim_{n\to\infty} \psi^n F = 0$. ■

The random variable $T(c)$ may be improper. In order to find the probability $\Pr\{T(c) < \infty\}$ we introduce the random variables $M(c)$ and $S(c)$ by the formulas

$$M(c) = \min_{m \geq 0}\{m:\ X_m^0 = 0 \mid X(0) = c\}, \qquad S(c) = \tau_{2M(c)}, \qquad \text{for every } \omega \in \Omega.$$

The random variable $M(c)$ is the index m of τ_{2m+1} such that $X(\tau_{2m+1}) = 0$ and the random variable $S(c)$ is the initial moment of the output phase being last before $T(c)$. Because τ_{2m+a} and t_m^a are finite for almost all ω, one can verify that

(15) $$\Pr\{T(c) < \infty\} = \Pr\{S(c) < \infty\} = \Pr\{M(c) < \infty\}.$$

Define the Markov chain Y^0 by killing the Markov chain X^0 at the $M(c)$-th step. The transition kernel P of Y^0 is given as follows:

$$P(x, A) = \int_0^\infty dH_0(v) \int_0^{R(q_0(x,v))} dH_1(u) I_A\big(q_1(q_0(x, v), u)\big), \qquad x \geq 0,\ A \in \mathscr{B}_0.$$

Denote by φ the operator on $f \in \mathrm{b}\mathscr{B}_0$ such that

$$\varphi f(x) = \int_{0-}^\infty f(y) P(x, dy).$$

Proposition 2. *Let $h \in \mathrm{b}\mathscr{B}_0$ be the maximal solution of the equation*

(16) $$f = \varphi f, \qquad 0 \leq f \leq 1.$$

Then we have

(17) $$\Pr\{T(c) < \infty\} = 1 - h(c).$$

Proof. Let $h = \lim_{n\to\infty} \varphi^n \mathbf{1}$. Because $1 \geq \varphi\mathbf{1} \geq \varphi^2\,\mathbf{1} \geq \ldots$, h is the solution of (16) and $0 \leq h \leq 1$. If f is any other solution of (16) then $f = \varphi^n f \leq \varphi^n \mathbf{1}$ for all $n \geq 1$; therefore $f \leq h$. For each n we have

$$\Pr\{M(c) \geq n\} = \varphi^n 1(c),$$

whence and from (15) we obtain (17). ■

Assume that $T(c)$ is proper and there exists the expected value $\Theta(c) = \mathrm{E}T(c) < \infty$. Introduce the function $\bar{g}$ and the operator $\bar{\psi}$ by the formulas

$$\bar{g}(c) = \int_0^{R(c)} \big(1 - H_1(u)\big) du + (1/\mu_0) H_1\big(R(c)\big), \qquad c > 0,$$

$$\bar{\psi}\bar{f}(c) = \int_0^{R(c)} dH_1(u) \int_0^\infty dH_0(v) \bar{f}\big(q_0(q_1(c, u), v)\big)$$

for a $\mathscr{B}_+$-measurable function $\bar{f}$. In view of Proposition 1 we can formulate a similar one for the expected value $\Theta(c)$.

Proposition 3. *The function Θ fulfils the equation*

(18) $$\Theta(c) = \bar{g}(c) + \bar{\psi}\Theta(c), \qquad c > 0,$$

which gives the only expected value of $T(c)$ and

$$\Theta(c) = \sum_{n=0}^{\infty} \bar{\psi}^n \bar{g}(c), \quad c > 0. \tag{19}$$

Proof. By the assumption $\Theta(c) < \infty$ we have

$$\Theta(c) = \int_0^{\infty} \Pr\{T(c) > t\}\, dt.$$

From (11) the function $\bar{F}(c, t) = \Pr\{T(c) > t\}$ fulfils the equation

$$\bar{F} = \bar{g} + \psi\bar{F}, \tag{20}$$

where

$$\bar{g}(c, t) = I_{(t,\infty)}(R(c))\big(1 - H_1(R(c))\big) + H_1(R(c)) - \psi 1(c, t)$$

and the operator ψ is defined by (10). Hence we have

$$0 \leqslant \bar{g}(c, t) \leqslant 1(c, t) - \psi 1(c, t).$$

The same arguments as in the proof of Proposition 1 allow to show that the equation (20) has a unique solution of the form $\bar{F}(c, t) = \sum_{n=0}^{\infty} \psi^n \bar{g}(c, t)$.

Simultaneously, it is easy to verify that

$$\int_0^{\infty} \bar{g}(c, t)\, dt = \bar{g}(c),$$

$$\int_0^{\infty} \psi^n \bar{h}(c, t)\, dt = \bar{\psi}^n \bar{h}(c), \quad n = 1, 2, \ldots$$

for each $\bar{h}$ such that $\int_0^{\infty} \bar{h}(c, t)\, dt = \bar{h}(c)$. Hence $\Theta(c)$ fulfils the equation (18). From Lebesgue's theorem we obtain

$$\Theta(c) = \int_0^{\infty} \sum_{n=0}^{\infty} \psi^n \bar{g}(c, t)\, dt = \sum_{n=0}^{\infty} \bar{\psi}^n \bar{g}(c), \quad c > 0.$$

Thus $\Theta(c)$ is uniquely determined by the function $\bar{g}$ and the operator $\bar{\psi}$.

3. Random deformations.

3.1. Markov chains. Consider random deformations of the distances between consecutive initial moments of the phases, separately for output and input.

The sequence of independent random variables Δ is called (see [4]) *random deformation of the sequence of independent random variables U* if the pairs (U_n, Δ_n), $n = 0, 1, \ldots$ are independent, $U_n + \Delta_n \geqslant 0$ and $\mathrm{E}(\Delta_n \mid U_n) = 0$.

Let Δ^a be the random deformation of t^a, $a = 1, 0$. Because we consider the deformation separately for each a, then, as the result of such a deformation, we obtain new models

$$\{\tilde{t}^1, t^0, r_1, r_0\} \quad \text{and} \quad \{t^1, \tilde{t}^0, r_1, r_0\},$$

where

$$\tilde{t}^1_m = t^1_m + \Delta^1_m, \quad \tilde{t}^0_m = t^0_m + \Delta^0_m, \quad m = 0, 1, \ldots$$

The Markov chains defined analogously to (6) for these models are denoted by $\check{X}^1$, $\check{X}^0$ and $\tilde{X}^1$, $\tilde{X}^0$, respectively. Thus we have the relations

(21) $$\check{X}^1_0 = X^1_0, \quad \check{X}^1_{m+1} = q_0\big(q_1(\check{X}^1_m, \tilde{t}^1_m), t^0_m\big),$$

(22) $$\check{X}^0_0 = q_1(X^1_0, \tilde{t}^1_0), \quad \check{X}^0_{m+1} = q_1\big(q_0(\check{X}^0_m, t^0_m), \tilde{t}^1_{m+1}\big),$$

(23) $$\tilde{X}^1_0 = X^1_0, \quad \tilde{X}^1_{m+1} = q_0\big(q_1(\tilde{X}^1_m, t^1_m), \tilde{t}^0_m\big),$$

(24) $$\tilde{X}^0_0 = q_1(X^1_0, t^1_0), \quad \tilde{X}^0_{m+1} = q_1\big(q_0(\tilde{X}^0_m, \tilde{t}^0_m), t^1_{m+1}\big), \quad m = 0, 1, \ldots$$

Now we prove that the deformation of output or input increases the content in expectation.

THEOREM 1. *If the function r_1 is increasing concave and r_0 is convex then the random deformation of the output phase increases the content in expectation, i.e.*

(25) $$\mathrm{E}(\check{X}^a_{m+1} \mid t^1_{m+1-a}, t^0_m) \geqslant X^a_{m+1}, \quad a = 1, 0, \quad m = 0, 1, \ldots$$

THEOREM 2. *If the function r_1 is concave and r_0 is increasing convex then the random deformation of the input phase increases the content in expectation, i.e.*

(26) $$\mathrm{E}(\tilde{X}^a_{m+1} \mid t^1_{m+1-a}, t^0_m) \geqslant X^a_{m+1}, \quad a = 1, 0, \quad m = 0, 1, \ldots$$

For the proofs of these theorems simple properties concerning the convexity (concavity) of the functions q_a, $a = 1, 0$, with fixed c or t are useful. We formulate these properties in two lemmas. We write some dual results using square brackets.

LEMMA 1. (a) *If the function r_1 is increasing [decreasing] then q_1 for each c is convex with respect to $t > 0$ [concave on the set $\{t\colon q_1(c, t) > 0\}$].*

(b) *If the function r_0 is increasing [decreasing] then q_0 for each c is convex with respect to t on the set $\{t\colon q_0(c, t) < \infty\}$ [concave for $t \geqslant 0$].*

LEMMA 2. (a) *If the function r_1 is convex [concave] then q_1 for each t is concave with respect to c on the set $\{c\colon q_1(c, t) > 0\}$ [convex for $c > 0$].*

(b) *If the function r_0 is convex [concave] then q_0 for each t is convex with respect to c on the set $\{c\colon q_0(c, t) < \infty\}$ [concave for $c \geqslant 0$].*

Parts (a) of Lemmas 1 and 2 have been proved in [4], the proofs of parts (b) are analogous.

Using Lemmas 1 and 2 we can prove Theorems 1 and 2. We use mathematical induction and many times Jensen's inequality (J.i.).

Proof of Theorem 1. We prove the formula (25) for $a = 1$; the proof for $a = 0$ is similar. Using in sequence Lemma 2 (b) and J.i., Lemma 1(a) and J.i. and the monotonicity of q_0 in c, Lemma 2(a) and J.i. and the monotonicity of q_0 in c, the inductive assumption and the monotonicity of q_1 and q_0 in c, we have from (1)

$$\begin{aligned} \mathrm{E}(\check{X}^1_{m+1} \mid t^1_m, t^0_m) &= \mathrm{E}\big(q_0(q_1(\check{X}^1_m, \check{t}^1_m), t^0_m) \big| t^1_m, t^0_m\big) \\ &\geqslant q_0\big(\mathrm{E}(q_1(\check{X}^1_m, \check{t}^1_m) \mid t^1_m, t^0_m), t^0_m\big) \\ &\geqslant q_0\big(\mathrm{E}(q_1(\check{X}^1_m, t^1_m) \mid t^1_m, t^0_m), t^0_m\big) \\ &\geqslant q_0\big(q_1(\mathrm{E}(\check{X}^1_m \mid t^1_{m-1}, t^0_{m-1}), t^1_m), t^0_m\big) \\ &\geqslant q_0\big(q_1(X^1_m, t^1_m), t^0_m\big) = X^1_{m+1}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 2. We prove the formula (26) for $a = 0$; the proof for $a = 1$ is similar. Applying in turn Lemma 2(a) and J.i., Lemma 1(b) and J.i. and the monotonicity of q_1 in c, Lemma 2(b) and J.i. and the monotonicity of q_1 in c, lastly the inductive assumption and the monotonicity of q_0 and q_1 in c, we obtain in virtue of (4)

$$\begin{aligned} \mathrm{E}(\tilde{X}^0_{m+1} \mid t^1_{m+1}, t^0_m) &= \mathrm{E}\big(q_1(q_0(\tilde{X}^0_m, \tilde{t}^0_m), t^1_{m+1}) \big| t^1_{m+1}, t^0_m\big) \\ &\geqslant q_1\big(\mathrm{E}(q_0(\tilde{X}^0_m, \tilde{t}^0_m) \mid t^1_{m+1}, t^0_m), t^1_{m+1}\big) \\ &\geqslant q_1\big(\mathrm{E}(q_0(\tilde{X}^0_m, t^0_m) \mid t^1_{m+1}, t^0_m), t^1_{m+1}\big) \\ &\geqslant q_1\big(q_0(\mathrm{E}(\tilde{X}^0_m \mid t^1_m, t^0_{m-1}), t^0_m), t^1_{m+1}\big) \\ &\geqslant q_1\big(q_0(X^0_m, t^0_m), t^1_{m+1}\big) = X^0_{m+1}. \quad \blacksquare \end{aligned}$$

It follows from Theorems 1 and 2 that the contemporary deformation of output and input increases the content if the function r_0 is increasing convex and r_1 is increasing concave.

The assumptions of convexity or concavity of the functions r_a, $a = 1, 0$, in Theorems 1 and 2 are essential. In the following example we consider a function r_0, fulfilling the assumptions of Theorem 2 with the exception of convexity, and we give such an input deformation which decreases the content in expectation.

Example. Let us consider the model with deterministic distances $t^1_m = 1/2$, $t^0_m = 2$, $m = 0, 1, \ldots$ and with intensity functions of the form

$$r_1(t) = 2, \quad t > 0,$$

$$r_0(t) = \begin{cases} 1/4, & 0 \leqslant t < 1, \\ 1, & 1 \leqslant t. \end{cases}$$

Assuming $X_0^1 = 5/3$ we obtain from (6)

$$X_m^1 = 5/3, \quad m = 1, 2, \ldots, \quad X_m^0 = 2/3, \quad m = 0, 1, \ldots$$

Let us consider further the input deformation Δ^0 with the probability distribution

$$\Pr\{\Delta_m^0 = -1\} = 1/2 = \Pr\{\Delta_m^0 = 1\}, \quad m = 0, 1, \ldots$$

Applying (23) and (24) we calculate

$$\Pr\{\tilde{X}_1^1 = 11/12\} = \Pr\{\tilde{X}_1^1 = 17/12\} = \Pr\{\tilde{X}_1^0 = 0\} = \Pr\{\tilde{X}_1^0 = 5/12\} = 1/2,$$

$$\Pr\{\tilde{X}_2^1 = 1/4\} = \Pr\{\tilde{X}_2^1 = 2/3\} = \Pr\{\tilde{X}_2^1 = 3/4\} = \Pr\{\tilde{X}_2^1 = 7/6\} = 1/4,$$

$$\Pr\{\tilde{X}_2^0 = 0\} = 3/4, \quad \Pr\{\tilde{X}_2^0 = 1/6\} = 1/4,$$

$$\Pr\{\tilde{X}_3^1 = 5/12\} = \Pr\{\tilde{X}_3^1 = 11/12\} = 1/8,$$

$$\Pr\{\tilde{X}_3^1 = 1/4\} = \Pr\{\tilde{X}_3^1 = 3/4\} = 3/8,$$

$$\Pr\{\tilde{X}_m^0 = 0\} = 1, \quad m = 3, 4, \ldots,$$

$$\Pr\{\tilde{X}_m^1 = 1/4\} = \Pr\{\tilde{X}_m^1 = 3/4\} = 1/2, \quad m = 4, 5, \ldots$$

Hence it is easy to verify that $\mathrm{E}(\tilde{X}_m^1) < 5/3$ and $\mathrm{E}(\tilde{X}_m^0) < 2/3$ for $m = 1, 2, \ldots$

3.2. Wet period. Assume that the sequence t^1 has a common exponential distribution, the random variable $T(c)$ defined in Section 2.2 is proper and its mean $\Theta(c)$ is finite.

Let us consider the random deformation Δ^0 of t^0. Thus we have a new model $\{t^1, \tilde{t}^0, r_1, r_0\}$. We indicate by $\sim$ the characteristics of this new model.

THEOREM 3. *If the function r_1 is decreasing [increasing] and r_0 is increasing [decreasing] then the random deformation of the input phase increases [decreases] the expected value of the wet period, i.e.*

$$\tilde{\Theta}(c) \geqslant [\leqslant]\ \Theta(c), \quad c > 0. \tag{27}$$

We precede the proof of Theorem 3 with the following lemma in which we give some properties of the function Θ. Introduce the notation $1/v = 1/\mu_0 + 1/\mu_1$.

LEMMA 3. *If the function r_1 is decreasing [increasing] and r_0 is increasing [decreasing] then Θ is convex [concave].*

Proof. We prove the version without brackets; the proof of the version in brackets is similar. First, using Proposition 3 we show that Θ fulfils the integro-differential equation

$$\Theta'(c) = \frac{1}{r_1(c)}\left(1 + \mu_1/\mu_0 - \mu_1 \Theta(c) + \mu_1 \mathrm{E}_{t_0^0} \Theta(q_0(c, t_0^0))\right), \quad c > 0. \tag{28}$$

Indeed, substituting $H_1(u) = 1-\exp(-\mu_1 u)$, $u \geq 0$, in (18) we obtain

(29)
$$\Theta(c) = \big(1-\exp(-\mu_1 R(c))\big)/v + \int_0^{R(c)} \mathrm{E}_{t_0^0}\Theta\big(q_0(q_1(c, u), t_0^0)\big)\mu_1 \exp(-\mu_1 u)\,du.$$

From (1) and (2) we obtain

$$\partial q_a/\partial c = r_a(q_a)/r_a(c), \qquad a = 1, 0.$$

Using this and (29) we have

$$\Theta'(c) = \frac{1}{r_1(c)}\Big(\mu_1 \exp(-\mu_1 R(c))/v + \mathrm{E}_{t_0^0}\Theta\big(q_0(0, t_0^0)\big)\mu_1 \exp(-\mu_1 R(c))$$
$$-\int_0^{R(c)} \frac{\partial}{\partial u}\mathrm{E}_{t_0^0}\Theta\big(q_0(q_1(c, u), t_0^0)\big)\mu_1 \exp(-\mu_1 u)\,du\Big).$$

Integrating by parts and using again (29) we obtain (28).

Now let $D = \Theta'(c+b)-\Theta'(c)$ for $c > 0$ and $b > 0$. Define the random variables $T(c, d)$, $c > 0$, $d \geq 0$, $c > d$ as follows:

$$T(c, d) = \inf_{t>0}\{t\colon X(t) \leq d \mid X(0) = c\}$$

and let $\Theta(c, d) = \mathrm{E}T(c, d)$. Evidently $T(c, 0) = T(c)$. Because the distances t_m^1 are exponential, we have

$$T(c+d) \overset{\text{distr}}{=} T(c+d, c) + T(c).$$

Hence, from (28) and $r_1(c+b) \leq r_1(c)$ we obtain

(30)
$$D \geq \frac{\mu_1}{r_1(c)}\mathrm{E}_{t_0^0}\big(\Theta(q_0(c+b, t_0^0), c+b) - \Theta(q_0(c, t_0^0), c)\big).$$

Let $r_1^b(t) = r_1(t+b)$ for $t > 0$ and $r_1^b(0) = 0$ and let $r_0^b(t) = r_0(t+b)$ for $t \geq 0$, $b > 0$. Further let $T^b(c)$, $c > 0$, be the wet period process in the model $\{t^1, t^0, r_1^b, r_0^b\}$. We also indicate by b the other characteristics of this model. From the obvious equalities

$$\int_0^c \big(1/r_a^b(u)\big)\,du = \int_0^{c+b} \big(1/r_a(u)\big)\,du - \int_0^b \big(1/r_a(u)\big)\,du, \qquad a = 1, 0,$$

and from (3) and (5) we have

$$q_a^b(c, t) = q_a(c+b, t) - b, \qquad a = 1, 0.$$

Hence for arbitrary $c > 0$, $b > 0$, $t \geq 0$ we obtain

(31)
$$T^b\big(q_0^b(c, t), c\big) = T\big(q_0(c+b, t), c+b\big), \qquad \text{for each } \omega \in \Omega.$$

If r_1 is decreasing and r_0 is increasing then from (1) and (2) we have the inequalities

$$q_a^b(c, t) \geqslant q_a(c, t), \quad a = 1, 0.$$

Thus for arbitrary $c > 0$, $b > 0$, $t \geqslant 0$, we have

$$T^b(q_0^b(c, t), c) \geqslant T(q_0(c, t), c) \quad \text{for each } \omega \in \Omega. \tag{32}$$

Taking the expectations in (31) and (32) and substituting into (30) we see that $D \geqslant 0$ for every $c > 0$ and $b > 0$. ■

Proof of Theorem 3. We prove the version in brackets; the proof of the version without brackets is similar. Using Lemma 3, Lemma 1(b) and Jensen's inequality, we have

$$\begin{aligned} \mathrm{E}_{\tilde{t}_0^0} \tilde{\Theta}\left(q_0(q_1(c, u), \tilde{t}_0^0)\right) &= \mathrm{E}_{t_0^0} \mathrm{E}\left(\tilde{\Theta}\left(q_0(q_1(c, u), \tilde{t}_0^0)\right) \mid t_0^0\right) \\ &\leqslant \mathrm{E}_{t_0^0} \tilde{\Theta}\left(q_0(q_1(c, u), t_0^0)\right) \quad \text{for } c > 0,\ u > 0. \end{aligned}$$

Now from Proposition 3 we obtain (see also (29))

(33)

$$\begin{aligned} \tilde{\Theta}(c) &= \left(1 - \exp(-\mu_1 R(c))\right)/v + \int_0^{R(c)} \mathrm{E}_{\tilde{t}_0^0} \tilde{\Theta}\left(q_0(q_1(c, u), \tilde{t}_0^0)\right) \mu_1 \exp(-\mu_1 u)\, du \\ &\leqslant \left(1 - \exp(-\mu_1 R(c))\right)/v + \\ &\quad + \int_0^{R(c)} \mathrm{E}_{t_0^0} \tilde{\Theta}\left(q_0(q_1(c, u), t_0^0)\right) \mu_1 \exp(-\mu_1 u)\, du = \Theta(c). \end{aligned}$$

The last equality in (33) follows from Proposition 3, i.e. $\Theta(c)$ is uniquely determined by the function $\bar{g}$ and the operator $\bar{\psi}$.

4. Stationary behaviour.

4.1. Invariant measures. In this section we consider the model $\{t^1, t^0, r_1, r_0\}$ and we give a sufficient condition for the existence of invariant probability measures N_a^+ of the Markov chains X^a, $a = 1, 0$. The considerations are based on the paper [1].

Let σ_0 be the σ-algebra generated by X_0^1. Denote by P_x the conditional probability $\Pr\{\cdot \mid \sigma_0\}$ evaluated on $\{X_0^1 = x\}$ and by E_x the corresponding expectation.

Let Q^a be the operators defined by the transition probabilities for the Markov chains X^a, $a = 1, 0$. Thus for $f \in \mathrm{b}\mathscr{B}_0$ we have from (6)

$$Q^a f(x) = \int_0^\infty \int_0^\infty f\left(q_{1-a}(q_a(x, u), v)\right) dH_a(u)\, dH_{1-a}(v), \quad a = 1, 0. \tag{34}$$

Let $\beta_0 = \inf_{x \geqslant 0} r_0(x)$ and $\beta_1 = \sup_{x \geqslant 0} r_1(x)$.

THEOREM 4. *If the function r_1 is continuous non-decreasing in $\mathscr{R}_+$ and r_0 is continuous non-increasing in $\mathscr{R}_+$ and if $\beta_0/\mu_0 < \beta_1/\mu_1$ then there exist invariant measures N_a^+ such that $N_a^+(R_0) = 1$, $a = 1, 0$.*

Proof. We prove the existence of the invariant measure N_0^+. Then the existence of N_1^+ follows from (6).

Choose c_0 such that $\beta_0 < c_0 < \beta_1 \mu_0/\mu_1$ and define $\mathscr{D}_0 = \{x: r_0(x) < c_0\}$. Because r_0 is positive continuous non-increasing, the set $\mathscr{D}_0$ is non-empty and it is of the form (d_0, ∞). In addition, if $x \in \mathscr{D}_0$ then from (2) we have

$$q_0(x, t) < x + c_0 t, \quad t > 0. \tag{35}$$

Let us choose c_1 such that $c_0 \mu_1/\mu_0 < c_1 < \beta_1$ and define $\mathscr{D}_1 = \{x: r_1(x) > c_1\}$. The set $\mathscr{D}_1$ is also non-empty and it is of the form (d_1, ∞). In addition, if $q_1(x, t) \in \mathscr{D}_1$ then from (1) we have

$$q_1(x, t) < x - c_1 t, \quad t > 0. \tag{36}$$

Consider the set $\mathscr{D} = \mathscr{D}_0 \cap \mathscr{D}_1$. If $d = \max(d_0, d_1)$ then $\mathscr{D} = (d, \infty)$. Let us define the chain S by the formula

$$S_0 = X_0^0, \quad S_{m+1} = S_m + c_0 t_m^0 - c_1 t_{m+1}^1, \quad m = 0, 1, \ldots$$

Let also

$$U = \min_{m \geqslant 0} \{m: S_m \notin \mathscr{D}\}, \quad V = \min_{m \geqslant 0} \{m: X_m^0 \notin \mathscr{D}\}.$$

It follows from (35) and (36) that for fixed $\omega \in \Omega$ we have: if $X_0^0, X_1^0, \ldots, X_m^0 > d$ then $X_1^1, X_2^1, \ldots, X_{m+1}^1 > d$ and

$$X_m^0 < X_{m-1}^0 + c_0 t_{m-1}^0 - c_1 t_m^1.$$

Hence, if $X_0^0, X_1^0, \ldots, X_m^0 > d$ then $S_m > d$ and we obtain

$$\begin{aligned}\{V > m\} &= \{X_0^0 > d, X_1^0 > d, \ldots, X_m^0 > d\} \subset \{S_0 > d, S_1 > d, \ldots, S_m > d\} \\ &= \{U > m\}.\end{aligned} \tag{37}$$

Because the random variables t_m^a, $a = 1, 0$, $m = 0, 1, \ldots$ are independent then the chain S forms a random walk. The constants c_a, $a = 1, 0$, have been chosen such that

$$\mathrm{E}_x\{S_{m+1} - S_m\} = c_0/\mu_0 - c_1/\mu_1 < 0.$$

Therefore the random walk S drifts to $-\infty$ and for each $x \geqslant 0$ we have $\mathrm{E}_x(U) < \infty$.

Define the operator $\mathscr{G}$ such that $\mathscr{G}f(x) = Q^0(I_{\mathscr{D}}f)(x)$ for each $f \in \mathrm{b}\mathscr{B}_0$. From (37) we obtain

$$\sum_{n=0}^{\infty} \mathscr{G}^n 1(x) = \mathrm{E}_x(V) = \sum_{n=0}^{\infty} P_x\{V > n\} \leqslant \sum_{n=0}^{\infty} P_x\{U > n\} = \mathrm{E}_x(U).$$

Thus we have $\sum_{n=0}^{\infty} \mathcal{G}^n 1(x) < \infty$ for each $x \geqslant 0$. Now the same arguments as in the proof of Theorem (4.4) in [1] allow to prove the existence and finiteness of N_0^+.

4.2. Relations. Assume that the functions r_a are continuous in $\mathcal{R}_+$ and that the Markov chains X^a have the unique invariant probability measures N_a^+, $a = 1, 0$. Now we investigate the relations between the stationary distributions of the content process and the invariant measures N_a^+. If the distances t_m^1 or t_m^0 are exponential then these relations turn out to be simple. Moreover, if t_m^1 are exponential, we find forms of both stationary distributions and invariant measures.

In order to obtain the relations we use the paper [3] which deals with extended piecewise Markov processes. An attentive look at the definition of such processes given in [3] allows to find that the content process X defined by (7) is an extended piecewise Markov process with regenerative moments τ_{2m+a}. On the interval $[\tau_{2m+a}, \tau_{2m+a+1})$ (m, a fixed) it is a Markov process with transition probabilities P_a as follows:

(38) $$P_a(t, x, A) = I_A\big(q_{1-a}(x, t)\big), \quad t, x \geqslant 0, \; A \in \mathcal{B}_0.$$

Thus we have

PROPOSITION 4. *The content process X defined by (7) is an extended piecewise Markov process on $(\Omega, \sigma, \Pr)$ valued in $(\mathcal{R}_0, \mathcal{B}_0)$ whose trajectories are continuous.*

The process X is a simple example of an extended piecewise Markov process; the distributions of the distances t_m^a between regenerative moments depend only on a and not on the state $X(\tau_{2m+a})$. Also the transition probabilities P_a are simple and do not depend on $X(\tau_{2m+a})$. Further at the moments τ_{2m+a} there are no jumps of X and there are no contractions of the output phase.

The continuity of the trajectories of X follows from the continuity of q_a with respect to t and from the lack of jumps at the regenerative moments.

Let $\mathcal{B}_1$ denote the σ-algebra of all subsets of the set $\{0, 1\}$ and let $\bar{\mathcal{R}} = \mathcal{R}_0 \times \mathcal{R}_+ \times \{0, 1\}$, $\bar{\mathcal{B}} = \mathcal{B}_0 \times \mathcal{B}_+ \times \mathcal{B}_1$. Introduce the Markov process $\bar{X}$ by the formula

$$\bar{X}(t) = [X(t), Z(t), \alpha(t)], \quad t \geqslant 0,$$

valued in $(\bar{\mathcal{R}}, \bar{\mathcal{B}})$, where X is the content process, $Z(t) = \tau_{2m+a+1} - t$, $\alpha(t) = a$ when $t \in [\tau_{2m+a}, \tau_{2m+a+1})$, $a = 0, 1$, $m = 0, 1, \ldots$ The component $Z(t)$ is called *residual-time process* and the component $\alpha(t)$ is called *alternating process*. Since both the transition probabilities P_a and the distributions H_a do not depend on the state $X(\tau_{2m+a})$, the extended Markov process $\bar{X}$ is here simplified.

It is known that the transition probabilities $\bar{P}$ of the Markov process $\bar{X}$ induce the semi-group of contraction operators $\{\bar{\mathscr{P}}_t, t \geqslant 0\}$. Assume that $\bar{X}$ is stationary. Then its stationary probability distribution $\bar{N}$ is an invariant measure for $\{\bar{\mathscr{P}}_t, t \geqslant 0\}$. We assume that $\bar{N}$ is unique.

Introduce the notation: $\mathscr{A}(P_a)$ for the infinitesimal operator induced by the transition probabilities P_a,

$$N_a(A) = \bar{N}(A \times \mathscr{R}_+ \times \{a\}),$$

$$N_a(x) = N_a([0, x]), \qquad N_a^+(x) = N_a^+([0, x]), \qquad a = 1, 0.$$

Assume that there exist the derivatives

$$n_a(x) = \frac{d}{dx} N_a(x) \quad \text{and} \quad n_a^+(x) = \frac{d}{dx} N_a^+(x), \qquad x > 0,$$

and that $N_0^+(0) > 0$.

LEMMA 4. *The marginal measures N_a and the invariant probability measures N_a^+ fulfil the relations*

$$N_{1-a}(A) = v \int_{0-}^{\infty} N_a^+(du) \int_{0}^{\infty} P_{1-a}(t, u, A)\big(1 - H_a(t)\big)dt, \tag{39}$$

$$\mathscr{A}(P_a) N_a(A) = v\big(N_a^+(A) - N_{1-a}^+(A)\big), \tag{40}$$

$$r_{1-a}(x) n_a(x) = v\big(N_0^+(x) - N_1^+(x)\big), \qquad x > 0, \ A \in \mathscr{B}_0, \ a = 1, 0. \tag{41}$$

Proof. In virtue of Theorems 2 and 3 in [3] it suffices to verify that for each function f on $\mathscr{R}_0$ real continuous vanishing at infinity and for $a = 1, 0$

$$\lim_{t \searrow 0} \sup_{x \geqslant 0} \Big| \int_{0-}^{\infty} f(u) P_a(t, x, du) - f(x) \Big| = 0. \tag{42}$$

By (38) we have

$$\int_{0-}^{\infty} f(u) P_a(t, x, du) = f\big(q_{1-a}(x, t)\big). \tag{43}$$

Using (1) and (2) we obtain

$$q_a(x, t) = x + (-1)^a \int_0^t r_a\big(q_a(x, u)\big) du.$$

Hence $\lim_{t \searrow 0} q_a(x, t) = x$ and $\lim_{t \searrow 0} f\big(q_a(x, t)\big) = f(x)$ uniformly on $\mathscr{R}_0$. Thus by (43) we have (42).

Now using Theorem 2 (b) from [3] we obtain (39) and using Theorem 3 from [3] we obtain (40).

Since the transition probabilities P_a are simple, it is easy to compute

$$\mathscr{A}(P_a)N_a\big((x,\infty)\big) = (-1)^a r_{1-a}(x)\frac{d}{dx}N_a\big((x,\infty)\big).$$

Hence, substituting this into (40) for $A = (x, \infty)$ and passing to the $N_a(x)$, we obtain (41).

THEOREM 5. *The marginal measures* N_a *and the invariant probability measures* N_a^+ *fulfil the relations*

$$N_1(x) = v\int_{0-}^{x} N_0^+(du)\int_{0}^{R_0(x)-R_0(u)}(1-H_0(t))\,dt, \quad x \geqslant 0, \tag{44}$$

$$N_0(x) = v\int_{0}^{\infty} N_1^+(du)\int_{x}^{\infty}(1-H_1(t))\,dt, \quad x \geqslant 0, \tag{45}$$

$$n_1(x) = v\int_{x}^{\infty}\big(1-H_1(R_1(u)-R_1(x))\big)/r_0(x)\,dN_1^+(u), \quad x > 0, \tag{46}$$

$$n_0(x) = v\int_{0-}^{\infty}\big(1-H_0(R_0(x)-R_0(u))\big)/r_1(x)\,dN_0^+(u), \quad x > 0, \tag{47}$$

where $\varkappa = \max(0, R_1(u)-R_1(x))$.

Proof. Substituting (38) into (39) for $A = [0, x]$ we get (44) and (45). Further by (6) we have

$$N_1^+(x) = \Pr\{q_0(X_m^0, t_m^0) \leqslant x\} = \int_{0-}^{x} H_0(R_0(x)-R_0(u))\,dN_0^+(u),$$

$$N_0^+(x) = \Pr\{q_1(X_m^1, t_m^1) \leqslant x\} = \int_{0}^{\infty}\big(1-H_1(R_1(u)-R_1(x))\big)\,dN_1^+(u).$$

Substituting these relations into (41) we get (46) and (47). ■

Assuming that t_m^1 or t_m^0 are exponential, we obtain simple relations between N_0 and N_0^+ or between N_1 and N_1^+, respectively.

THEOREM 6. *For fixed* $a = 1, 0$, *if* $H_a(u) = 1-\exp(-\mu_a u)$, $u \geqslant 0$, *then*

$$N_{1-a}(x) = (v/\mu_a)\,N_{1-a}^+(x), \quad x \geqslant 0. \tag{48}$$

Proof. For exponentially distributed t_m^a we can transform (39) to the form

$$\begin{aligned} N_{1-a}(A) &= v\int_{0}^{\infty}\exp(-\mu_a t)\,dt\int_{0-}^{\infty} N_a^+(du)\,P_{1-a}(t,u,A) \\ &= v\,\mathscr{R}_{\mu_a}(P_{1-a})\,N_a^+(A), \end{aligned} \tag{49}$$

where $\mathscr{R}_\lambda(P_a)$, $\lambda > 0$, denotes the resolvent induced by P_a. It is known from the theory of semi-groups of contraction operators that $\mathscr{R}_\lambda(P_a) = (\lambda I - \mathscr{A}(P_a))^{-1}$, where I is the identity operator. Thus we have

$$(50)\qquad \mu_a \mathscr{R}_{\mu_a}(P_{1-a}) N_a^+(A) - \mathscr{A}(P_{1-a}) \mathscr{R}_{\mu_a}(P_{1-a}) N_a^+(A) = N_a^+(A).$$

From (50), (49) and (40) we get (48). ■

Relations (46)–(48) allow to find the forms of the distributions N_a^+ and N_a. For t_m^1 exponential we obtain the results formulated in Theorem 7.

Let us define the non-negative kernel K and its iterates

$$K(x, y) = \mu_1\big(1 - H_0(R_0(x) - R_0(y))\big)/r_1(x),$$

$$K^{*(n+1)}(x, y) = \int_y^x K^{*n}(x, u) K(u, y)\, du = \int_y^x K(x, u) K^{*n}(u, y)\, du,$$

$$0 \leqslant y < x < \infty, \qquad n = 1, 2, \ldots, \qquad \text{where } K^{*1} = K.$$

Using the obvious bound $K(x, y) \leqslant \mu_1/r_1(x)$, it is easy to verify that

$$(51)\qquad K^{*(n+1)}(x, y) \leqslant \big(\mu_1^{n+1}/(r_1(x)\, n!)\big)\big(R_1(x) - R_1(y)\big)^n$$

$$0 \leqslant y < x < \infty, \qquad n = 1, 2, \ldots$$

Thus the kernel $K^*(x, y) = \sum_{n=1}^{\infty} K^{*n}(x, y)$ is well defined and satisfies

$$K^*(x, y) \leqslant \mu_1 \exp\big(\mu_1 R_1(x) - R_1(y)\big)/r_1(x), \qquad 0 \leqslant y < x < \infty.$$

Moreover we introduce the notation

$$k = 1/(1 + \int_0^\infty K^*(u, 0)\, du),$$

$$K_0(x, y) = \mu_0\big(1 - H_0(R_0(x) - R_0(y))\big)/r_0(x).$$

THEOREM 7. *If* $H_1(u) = 1 - \exp(-\mu_1 u)$, $u \geqslant 0$, *and if* $\int_0^\infty K^*(u, 0)\, du < \infty$ *then the distributions* N_a, N_a^+, $a = 1, 0$, *have the form*

$$(52)\qquad N_0^+(x) = k\big(1 + \int_0^x K^*(u, 0)\, du\big),$$

$$(53)\qquad N_0(x) = (v/\mu_1) N_0^+(x),$$

$$(54)\qquad N_1^+(x) = k\big(H_0(R_0(x)) + \int_0^x H_0(R_0(x) - R_0(u)) K^*(u, 0)\, du\big),$$

$$(55)\qquad N_1(x) = (vk/\mu_0)\big(\int_0^x K_0(u, 0)\, du + \int_0^x \int_0^y K_0(y, u) K^*(u, 0)\, du\, dy\big),$$

$x \geqslant 0$.

Proof. Substituting (48) with $a = 1$ into (47) we have the equation

$$n_0^+(x) = K(x, 0) N_0^+(\{0\}) + \int_0^x K(x, u) n_0^+(u) du, \qquad x > 0. \tag{56}$$

Hence, iterating (56) $n-1$ times we obtain

$$n_0^+(x) = N_0^+(\{0\}) \sum_{j=1}^{n} K^{*j}(x, 0) + \int_0^x K^{*n}(x, u) n_0^+(u) du, \qquad x > 0. \tag{57}$$

Using (51) and the bounded convergence, letting $n \to \infty$ in (57) we establish that the unique solution of (56) is $n_0^+(x) = N_0^+(\{0\}) K^*(x, 0)$. It is a density iff $\int_0^\infty K^*(u, 0) du < \infty$. Thus we have (52) and by (48) we have also (53). The formula (54) follows easily from (52) and (6). The distribution N_1 is obtained substituting (52) and (54) into (40) with $a = 1$. In addition we can verify that $N_1(\infty) + N_0(\infty) = 1$.

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