ANNALES
POLONICI MATHEMATICI
LII (1990)

On the relations between abstract and geometrical equivalence of abstract objects

by Mieczysław Kucharzewski, Brunon Szociński (Katowice) and Marek Żabka (Gliwice)

Abstract. Let (a) (M, G, F) and (b) (M_1, G, F_1) be two abstract objects with the same group G. We call the objects (a) and (b) abstractly equivalent when there is a pair (h, φ) of mappings, h a bijection of M onto M_1 and φ an automorphism of the group G, which satisfy the condition of equivariance

$$F_1(h(x), \varphi(g)) = h(F(x, g)) \quad \text{for all } x \in M \text{ and } g \in G.$$

We call the objects (a) and (b) geometrically equivalent when there is a bijection $h\colon M \to M_1$ which satisfies the condition

$$F_1(h(x), g) = h(F(x, g)) \quad \text{for all } x \in M \text{ and } g \in G.$$

It is obvious that geometrically equivalent objects are also abstractly equivalent. The example given in this paper (Section 3) shows that the inverse implication is not true. Some conditions under which abstractly equivalent objects are also geometrically equivalent are given. The definition and some properties of pseudoinner automorphisms of a group are given in Section 4.

Introduction. The aim of the paper is to establish relations between abstract and geometric equivalence of abstract objects, which were defined in [1], part I, p. 21 (see also Section 1).

Firstly (Section 2), we observe that if the abstract equivalence is given by a pair (h, φ), where φ is an inner group automorphism, then the objects are geometrically equivalent. However, the example in Section 3 shows that, in general, abstractly equivalent objects need not be geometrically equivalent. In Section 5 we give certain conditions which ensure that abstractly equivalent objects are also geometrically equivalent. The definition and some properties of a pseudoinner automorphism are given in Section 4.

Some general remarks are added at the end of the paper.

1. Abstract and geometric equivalence of abstract objects. Since the literature quoted here is hardly accessible and written in Polish, it seems worthwhile to recall basic notions and their properties, which will be of use in the sequel.

By an abstract object we mean a triple

$$(M, G, F), \tag{1.1}$$

where M is a non-empty set, G is a group and F is the action of the group G on the set M, i.e., a mapping

$$F\colon M \times G \to M$$

which satisfies the translation equation

$$F(F(x, g_1), g_2) = F(x, g_2 \cdot g_1) \quad \text{for } x \in M \text{ and } g_1, g_2 \in G$$

and the identity condition

$$F(x, e) = x \quad \text{for } x \in M$$

([1], part I, p. 12).

If the set M in (1.1) is a differentiable manifold, G is a Lie group and the operation F is differentiable, then (1.1) is a Lie group of transformations of M onto M. This means that abstract objects yield a generalization of the notion of a Lie transformation group, therefore all considerations presented here concern in particular Lie groups.

Let us consider another abstract object

$$(M_1, G_1, F_1). \tag{1.2}$$

A pair of transformations (h, φ), where h is a map from M into M_1 and φ is a homomorphism of the group G into G_1, is called an *equivariant map* of the object (1.1) into object (1.2) if it satisfies the condition

$$F_1(h(x), \varphi(g)) = h(F(x, g)) \quad \text{for all } x \in M \text{ and } g \in G. \tag{1.3}$$

One can check that the class of abstract objects with equivariant mappings as morphisms and with composition as morphism operation forms a category. This category is called the *category of abstract objects* and is denoted by OA ([1], part I, p. 22).

A morphism (h, φ) in this category is an isomorphism if and only if the mapping h is a bijection and φ is a group isomorphism. If this is the case, we say that objects (1.1) and (1.2) are *abstractly equivalent*.

Let OG denote the subcategory of OA consisting of all objects having the same group G and morphisms of the form (h, id_G). The subcategory OG is called a *G-geometry*, or a *geometry of the group* G ([1], part I, p. 20).

We call objects (1.1) and (1.2) *geometrically equivalent* whenever $G = G_1$ and there exists a bijection $h\colon M \to M_1$ such that the pair (h, id_G) is an isomorphism in the category OG, i.e., it satisfies the condition of equivariance, which now has the form

$$F_1(h(x), g) = h(F(x, g)) \quad \text{for } x \in M \text{ and } g \in G. \tag{1.4}$$

It is obvious that geometrically equivalent objects (1.1) and (1.2) are abstractly equivalent. However, the question arises whether the condition $G = G_1$ is sufficient for abstractly equivalent objects (1.1) and (1.2) to be geometrically equivalent. The example in Section 3 gives the negative answer to this question.

2. A sufficient condition for geometric equivalence. Suppose that objects (1.1) and

$$(M_1, G, F_1). \tag{2.1}$$

are abstractly equivalent. We show that if there is an isomorphism (h, φ) of the object (1.1) onto the object (2.1) such that φ is an inner automorphism of the group G, then objects (1.1) and (2.1) are geometrically equivalent. Indeed, if φ is an inner automorphism, then there is $a \in G$ such that

$$\varphi(g) = a^{-1} g a \quad \text{for } g \in G. \tag{2.2}$$

In this case, the condition of equivariance (1.3) is of the form

$$F_1(h(x), a^{-1} g a) = h(F(x, g)) \quad \text{for } x \in M \text{ and } g \in G. \tag{2.3}$$

Using the translation equation, we get

$$F_1(h(x), a^{-1} g a) = F_1\big(F_1(F_1(h(x), a), g), a^{-1}\big). \tag{2.4}$$

One can easily check that for any $a \in G$ the mapping $F_1(\cdot, a)$ is a bijection of M_1 onto M_1 having the property

$$F_1^{-1}(\cdot, a) = F_1(\cdot, a^{-1})$$

([2], p. 68, Lemma 1). So, the mapping

$$\bar{h}(x) = F_1(h(x), a) \quad \text{for } x \in M \tag{2.5}$$

is a bijection M onto M_1, as a composition of bijections.

Using (2.3), (2.4) and (2.5) we get

$$F_1\big(F_1(\bar{h}(x), g), a^{-1}\big) = h(F(x, g)),$$

and therefore

$$F_1(\bar{h}(x), g) = F_1\big(h(F(x, g)), a\big) = \bar{h}(F(x, g)).$$

The last relation means that (1.4) is valid, i.e., the objects are geometrically equivalent.

3. An example. Let us consider a contravariant vector and a covariant vector in the n-dimensional space, where $n \geqslant 2$. We can denote them by triples:

$$(R^n, GL(n, R), F), \quad F(v, A) = A \cdot v, \tag{3.1}$$

$$(R^n, GL(n, R), F_1), \quad F_1(u, A) = u \cdot A^{-1}, \tag{3.2}$$

where the character v stands for a one-column matrix (v^i) and u denotes a one-row matrix (u_i). Symbols $A\cdot v$ and $u\cdot A^{-1}$ denote the ordinary multiplication of matrices.

Let $h\colon R^n \to R^n$ be a bijection of the form $h(v) = v^T$ and let $\varphi\colon GL(n,R) \to GL(n,R)$ be an automorphism of $GL(n,R)$ of the form $\varphi(A) = (A^{-1})^T$. We have

$$F_1(h(v), \varphi(A)) = F_1(v^T, (A^{-1})^T) = v^T \cdot A^T = (A\cdot v)^T = h(F(v, A))$$

for $v \in R^n$ and $A \in GL(n,R)$. So, the pair (h, φ) satisfies the condition of equivariance (1.3) and therefore objects (3.1) and (3.2) are abstractly equivalent.

However, they are not geometrically equivalent. We give the proof for $n = 2$. The proof for higher dimensions is quite analogous.

Suppose that objects (3.1) and (3.2), where $n = 2$, are geometrically equivalent and let $(\bar{h}, \mathrm{id}_G)$ be an isomorphism of the object (3.1) onto the object (3.2). Let us consider the vector $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and the covector $u^0 = \bar{h}(e_1)$. As we know ([3], p. 97, Theorem 15.5), the group of isotropy of the vector e_1 in the object (3.1) must be equal to the group of isotropy of the covector u^0 in the object (3.2). The isotropy group of the vector e_1 in the object (3.1) is of the form

$$G(e_1) = \left\{ \begin{pmatrix}1 & a_2^1\\ 0 & a_2^2\end{pmatrix} \colon a_2^1,\ a_2^2 \in R,\ a_2^2 \neq 0 \right\}.$$

Since the isotropy group of the vector $(0,0)$ in the object (3.2) is equal to the whole group $GL(n,R)$, we have

$$u^0 \neq (0,0). \tag{3.3}$$

On the other hand, we have by the equivariance condition

$$u^0 \cdot A^{-1} = \bar{h}(e_1)\cdot A^{-1} = \bar{h}(A\cdot e_1) \quad \text{for each matrix } A \in GL(2,R). \tag{3.4}$$

In the particular case of

$$A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix},$$

we get respectively

$$(u_1^0, u_2^0)\cdot\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} = (u_1^0, u_2^0) \quad \text{and} \quad (u_1^0, u_2^0)\cdot\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} = (u_1^0, u_2^0),$$

Therefore

$$-u_1^0 + u_2^0 = u_2^0 \quad \text{and} \quad 2u_2^0 = u_2^0.$$

Hence, $u^0 = (0,0)$, which contradicts (3.3).

We can explain the geometrical meaning of this example in tne following way. Contravariant and covariant vectors are equivalent as Klein spaces but they are not equivalent as geometrical objects of the geometry of the group $GL(n,R)$.

4. Pseudoinner automorphisms of a group. We propose the following definition.

DEFINITION 4.1. We will call an automorphism φ: $G \to G$ *pseudoinner relative to an abstract object* (M, G, F) if there exists a bijection $h: M \to M$ such that the pair (h, φ) satisfies the equivariance condition (1.3), i.e., the pair (h, φ) is an automorphism in category OA. (By an automorphism of an object in a category we mean an isomorphism of the object onto itself.) We will call an automorphism φ: $G \to G$ *pseudoinner* if it is pseudoinner relative to every abstract object (M, G, F).

Let us observe that the set of all pseudoinner automorphisms of a group G relative to an object (M, G, F) and the set of all pseudoinner automorphisms are subgroups of the group of all automorphisms of G.

Remark 4.1. Let G be a subgroup of the group S_M of all bijections of a set M and let F be a mapping of the form

$$F(x, g) = g(x) \quad \text{for } x \in M \text{ and } g \in G.$$

Then an automorphism φ of the group G is pseudoinner relative to the object (M, G, F) iff φ is the restriction of an inner automorphism of the group S_M, i.e., there is $h \in S_M$ such that $\varphi(g) = hgh^{-1}$ for $g \in G$.

Proof. For $g \in G$, we have:

$$h(F(x, g)) = F(h(x), \varphi(g)), \qquad x \in M,$$

i.e.,

$$h(g(x)) = \varphi(g)h(x), \qquad x \in M,$$

or, equivalently, $hg = \varphi(g)h$ which gives the equality $\varphi(g) = hgh^{-1}$ and Remark 4.1 follows.

DEFINITION 4.2. Let us consider an object (1.1) and an automorphism φ of the group G. Then

$$(M, G, F_1), \quad F_1(x, g) = F(x, \varphi(g)) \tag{4.1}$$

is called the *object generated by the object* (1.1) *and the automorphism* φ.

Since the pair (id_M, φ) satisfies the condition of equivariance, objects (1.1) and (4.1) are abstractly equivalent.

LEMMA 4.1. *If objects* (1.1) *and* (4.1) *are geometrically equivalent, then the automorphism* φ *is pseudoinner relative to the object* (1.1).

Proof. If objects (1.1) and (4.1) are geometrically equivalent, then there is an isomorphism (h, id_G) of object (1.1) onto (4.1). We have

$$(h, \varphi) = (\mathrm{id}_M, \varphi)\ (h, \mathrm{id}_G)$$

and therefore (h, φ) is an automorphism of the object (1.1). Hence, φ is pseudoinner relative to (1.1).

THEOREM 4.1. *Every inner automorphism of a group G is pseudoinner.*

Proof. Let us consider an object (1.1) and let φ be an inner automorphism of the group G. As we have noticed, (id_M, φ) is an isomorphism of the object (4.1), generated by the object (1.1) and the automorphism φ, onto the object (1.1). Since the automorphism φ is inner, we conclude from Section 2 that objects (1.1) and (4.1) are geometrically equivalent. Hence, the automorphism φ is pseudoinner relative to the object (1.1) (see Lemma 4.1). Since there is no restriction imposed upon (1.1), the automorphism φ is pseudoinner.

Now, we are facing the problem of recognizing when an automorphism is pseudoinner relative to the object (1.1).

THEOREM 4.2. *An automorphism φ of a group G is pseudoinner relative to the object (1.1) iff there exists a bijection $h\colon M \to M$ which satisfies the following conditions:*

(a) *the image $h(M_t)$ of each transitive fibre M_t of the object (1.1) is a transitive fibre,*

(b) *for any transitive fibre M_t of the object (1.1) there is an element $x_t \in M_t$ such that the image $\varphi(G(x_t))$ of the isotropy group $G(x_t)$ of the point x_t is equal to the isotropy group $G(h(x_t))$ of the point $h(x_t)$, i.e.,*

$$\varphi(G(x_t)) = G(h(x_t)).$$

Proof. Let an automorphism φ of a group G be pseudoinner relative to the object (1.1). Then there is a bijection $h\colon M \to M$ such that the pair (h, φ) is an automorphism of the object (1.1), i.e., (h, φ) is an isomorphism of the object (1.1) onto itself. It follows from general theorems on isomorphisms of abstracts (see [4], pp. 239–241) that the bijection h satisfies conditions (a) and (b).

Now, let us consider the abstract object (1.1), an automorphism φ of the group G and a bijection $h\colon M \to M$, which satisfies conditions (a) and (b). Let $h_1\colon M \to M$ be a mapping of the form

$$h_1(x) = F(h(x_t), \varphi(g_x)) \quad \text{for } x \in M_t,$$

where g_x in an element of the group G such that $F(x_t, g_x) = x$. Of course, we must show that the definition of h_1 does not depend on the choice of g_x. Indeed, let us notice that $F(x_t, \tilde{g}_x) = F(x_t, g_x)$ iff there is $g_0 \in G(x_t)$ such that $\tilde{g}_x = g_x \cdot g_0$. So, using condition (b), we get

$$\begin{aligned} F(h(x_t), \varphi(\tilde{g}_x)) &= F(h(x_t), \varphi(g_x \cdot g_0)) \\ &= F(h(x_t), \varphi(g_x) \cdot \varphi(g_0)) = F(h(x_t), \varphi(g_x)). \end{aligned}$$

Observe that the mapping $h_1|_{M_t}\colon M_t \to h(M_t)$ is a bijection. Hence, using hypothesis (a), we conclude that $h_1\colon M \to M$ is also a bijection. To complete the proof it is enough to verify the condition of equivariance for the pair (h_1, φ):

$$F(h_1(x), \varphi(g)) = F(F(x_i, \varphi(g_x)), \varphi(g))$$
$$= F(h(x_i), \varphi(g \cdot g_x)) = h_1(F(x, g)).$$

Hence, the automorphism φ is pseudoinner relative to the object (1.1) and the theorem is thus proved.

Note that for a transitive object (1.1) the existence of a bijection h which satisfies conditions (a) and (b) is equivalent to the condition:

(c) *there exist two points* $x_1, x_2 \in M$ *such that*

$$\varphi(G(x_1)) = G(x_2).$$

So, we have proved the following corollary.

COROLLARY 4.1. *An automorphism φ of a group G is pseudoinner relative to a transitive object* (1.1) *iff condition* (c) *is satisfied.*

It is known (see [1], p. 26) that the isotropy groups of two points of the same transitive fibre of an object (1.1) are conjugate, i.e.,

$$x_2 = F(x_1, a) \Rightarrow G(x_2) = a \cdot G(x_1) \cdot a^{-1}.$$

Thus we have

COROLLARY 4.2. *An automorphism φ of an abelian group G is pseudoinner relative to a transitive object* (1.1) *iff there is a point $x_1 \in M$ such that $\varphi(G(x_1)) = G(x_1)$.*

5. Necessary and sufficient conditions for geometrical equivalence of objects. Now we prove the main result.

THEOREM 5.1. *Let (h, φ) be an isomorphism of object* (1.1) *onto object* (2.1). *Then objects* (1.1) *and* (2.1) *are geometrically equivalent iff the automorphism φ is pseudoinner relative to the object* (1.1).

Proof. The *only if* part. Let (h, φ) and (h_1, id_G) be isomorphisms of (1.1) onto (2.1). Then

$$(h_1^{-1} h, \varphi) = (h_1, \mathrm{id}_G)^{-1} (h, \varphi)$$

is an automorphism of the object (1.1) and therefore the automorphism φ is pseudoinner relative to the object (1.1).

The *if* part. Let (h, φ) be an isomorphism of (1.1) onto (2.1). If the automorphism φ is pseudoinner relative to the object (1.1), then there is an bijection $h_1\colon M \to M$ such that the pair (h_1, φ) is an automorphism. Hence the pair $(hh_1^{-1}, \mathrm{id}_G) = (h, \varphi)(h_1, \varphi)^{-1}$ is an isomorphism of the object (1.1) onto (2.1) and therefore the two objects are geometrically equivalent.

Theorems 4.2 and 5.1 together with Corollary 4.1 imply

COROLLARY 5.1. *Let $(\bar{h}, \varphi)$ be an isomorphism of object* (1.1) *onto object* (2.1). *Then the objects* (1.1) *and* (2.1) *are geometrically equivalent iff there exists a bijection $h: M \to M$ which satisfies conditions* (a) *and* (b).

COROLLARY 5.2. *Let $(\bar{h}, \varphi)$ be an isomorphism of a transitive object* (1.1) *onto an object* (2.1). *Then the objects* (1.1) *and* (2.1) *are geometrically equivalent iff condition* (c) *is satisfied.*

Lemma 4.1 and Theorem 5.1 imply the following two corollaries.

COROLLARY 5.3. *Every object (M_1, G, F_1) abstractly equivalent to a fixed object* (1.1) *is geometrically equivalent to it iff each automorphism φ of the group G is pseudoinner relative to the object* (1.1).

COROLLARY 5.4. *Any two abstractly equivalent objects with the same group G are geometrically equivalent iff each automorphism φ of the group G is pseudoinner.*

6. Some remarks. As we see from Corollary 5.4, it may be interesting to investigate the class K of all groups in which every automorphism is pseudoinner. Of course, if each automorphism of a group G is inner then it is also pseudoinner, but not vice versa, as the example of cyclic groups shows.

The example given in Section 3 and Theorem 5.1 imply that the automorphism φ of the group $GL(n, R)$ (for $n \geqslant 2$) given by $\varphi(A) = (A^{-1})^T$ is not pseudoinner relative to the object (3.1). So, an automorphism of a group G need not be pseudoinner relative to the object (1.1). Besides, an automorphism which is pseudoinner relative to (1.1) need not be inner. Indeed, consider an abelian group G such that $\operatorname{Aut} G \neq \{\mathrm{id}_G\}$ and the object

$$(G, G, L), \quad \text{where } L(x, g) = g \cdot x. \tag{6.1}$$

The pair (φ, φ) is an automorphism of the object (6.1) for each automorphism φ of the group G, since we have for any $x, g \in G$

$$L(\varphi(x), \varphi(g)) = \varphi(g) \cdot \varphi(x) = \varphi(g \cdot x) = \varphi(L(x, g)).$$

Hence, each automorphism φ of the group G is pseudoinner relative to the object (6.1), yet is not inner since the group G is abelian.

Concluding, let us notice the relations between the subgroups of the group $\operatorname{Aut} G$ which we have defined:

$$\operatorname{Int} G \leqslant \operatorname{P Int} G \leqslant (\operatorname{P Int} G)_F \leqslant \operatorname{Aut} G,$$

where $\operatorname{P Int} G$ denotes the group of pseudoinner automorphisms of G and $(\operatorname{P Int} G)_F$ is the group of automorphisms of G which are pseudoinner relative to the object (M, G, F).

References

[1] M. Kucharzewski, *Własności przestrzeni Kleina*, cz. I, Pol. Śląska, Gliwice 1985, skrypt 1209, cz. II Pol. Śląska, Gliwice 1986, skrypt 1295.

[2] —, *Pojęcie przestrzeni Kleina i jej geometrii*, ZN Pol. Śl. 747, Mat.-Fiz. 42 (1983), 65–73.

[3] —, *Elementy teorii obiektów geometrycznych*, Uniwersytet Śląski, Katowice 1969, skrypt.

[4] E. Zaporowski, *Podstawowe własności kategorii obiektów abstrakcyjnych*, ZN Pol. Śl. 560, Mat.-Fiz. 30 (1979), 235–242.

Reçu par la Rédaction le 10.02.1988