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ON THE COMPLETENESS OF FLAT SURFACES IN S^3

BY

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1. Introduction. A surface M is called *flat* if the Gaussian curvature of M is identically equal to zero. In a recent article [2], Sasaki produced many examples of complete flat surfaces isometrically immersed in S^3, the hypersphere of radius 1 in R^4.

Let $a(u^1)$, $b(u^2)$ be a pair of real-valued differentiable functions defined on the whole real line satisfying

$$0 < a(u^1) + b(u^2) < \pi \quad \text{for all } u^1, u^2.$$

Sasaki constructed a flat surface $M \subset S^3$ for which u^1 and u^2 are asymptotic coordinates forming a Tchebyshev net, i.e. the first and second fundamental forms of M in S^3 are given by

$$g_{11} = g_{22} = 1, \quad g_{12} = \cos\big(a(u^1) + b(u^2)\big), \tag{1}$$

$$h_{11} = h_{22} = 0, \quad h_{12} = \sin\big(a(u^1) + b(u^2)\big). \tag{2}$$

Since M is flat, it can be regarded as an isometric immersion of R^2 into S^3, where R^2 has the Riemannian metric g in (1). Sasaki also stated sufficient conditions for the metric g to be complete. As he noted ([2], p. 173), these conditions were communicated to him from the author through K. Nomizu. These conditions are stated in the following theorem which is the main result of this paper:

THEOREM 1. *Let $a(u^1)$, $b(u^2)$ be real-valued differentiable functions defined on the whole real line. Suppose there exist positive constants α, β, A, B such that*

$$0 < \alpha \leqslant a(u^1) + b(u^2) \leqslant \beta < \pi, \tag{3}$$

$$\left|\frac{da}{du^1}\right| \leqslant A, \quad \left|\frac{db}{du^2}\right| \leqslant B \tag{4}$$

for all values of u^1, u^2. Then the metric g on R^2 defined by

$$g_{11} = g_{22} = 1, \quad g_{12} = \cos\big(a(u^1) + b(u^2)\big)$$

is complete.

2. Preliminaries. Let $p = (u_0^1, u_0^2)$ be an arbitrary point in R^2. Let

$$X = X_1 \frac{\partial}{\partial u^1} + X_2 \frac{\partial}{\partial u^2}$$

be a vector which is tangent to R^2 at p such that X has unit length with respect to the metric g. We will show that the geodesic $\gamma(s)$ with initial position p and initial tangent vector X can be extended to arbitrarily large lengths. As is well known (see, for example, [3], p. 64), such a geodesic $\gamma(s) = \big(u^1(s), u^2(s)\big)$ is found by solving the system of differential equations

$$\frac{d^2 u^k}{ds^2} + \sum_{i,j=1}^{2} \frac{du^i}{ds} \frac{du^j}{ds} \Gamma_{ij}^k = 0, \quad k = 1, 2, \tag{5}$$

with initial conditions

$$u^1(0) = u_0^1, \quad u^2(0) = u_0^2, \quad \frac{du^1}{ds}(0) = X_1, \quad \frac{du^2}{ds}(0) = X_2, \tag{6}$$

where Γ_{ij}^k are the Christoffel symbols, derived from the metric g by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{h=1}^{2} g^{hk} \left(\frac{\partial g_{ih}}{\partial u^j} + \frac{\partial g_{jh}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^h} \right).$$

For brevity, we let $\theta = a(u^1) + b(u^2)$. Then, using the explicit expression (1) for g, we can evaluate the Γ_{ij}^k, and the system (5) becomes

$$\begin{aligned} &\frac{d^2 u^1}{ds^2} + (\cot\theta) \frac{da}{du^1} \left(\frac{du^1}{ds} \right)^2 - (\csc\theta) \frac{db}{du^2} \left(\frac{du^2}{ds} \right)^2 = 0, \\ &\frac{d^2 u^2}{ds^2} - (\csc\theta) \frac{da}{du^1} \left(\frac{du^1}{ds} \right)^2 + (\cot\theta) \frac{db}{du^2} \left(\frac{du^2}{ds} \right)^2 = 0. \end{aligned} \tag{7}$$

The system (7) can be reduced to a first-order system by introducing the coordinates

$$y^1 = u^1, \quad y^2 = u^2, \quad y^3 = \frac{du^1}{ds}, \quad y^4 = \frac{du^2}{ds}.$$

In terms of these new coordinates, the system (7) becomes

$$\frac{dy^i}{ds} = f_i(s, y^1, y^2, y^3, y^4), \quad i = 1, \ldots, 4, \tag{8}$$

where

$$\begin{aligned} &f_1(s, y^1, y^2, y^3, y^4) = y^3, \quad f_2(s, y^1, y^2, y^3, y^4) = y^4, \\ &f_3(s, y^1, y^2, y^3, y^4) = -(\cot\theta) \frac{da}{dy^1} (y^3)^2 + (\csc\theta) \frac{db}{dy^2} (y^4)^2, \\ &f_4(s, y^1, y^2, y^3, y^4) = (\csc\theta) \frac{da}{dy^1} (y^3)^2 - (\cot\theta) \frac{db}{dy^2} (y^4)^2. \end{aligned} \tag{9}$$

Initial conditions (6) become

$$(10)\qquad y^1(0) = u_0^1, \quad y^2(0) = u_0^2, \quad y^3(0) = X_1, \quad y^4(0) = X_2.$$

Let $y = (y^1, y^2, y^3, y^4) \in R^4$. We define a norm on R^4 by

$$\|y\| = \sum_{i=1}^{4} |y^i|,$$

and a function f from $R^1 \times R^4$ to R^4 by

$$f(s, y) = \big(f_1(s, y), f_2(s, y), f_3(s, y), f_4(s, y)\big)$$

for $f_i(s, y)$, $i = 1, \ldots, 4$, given by (9). For $y(0)$ as defined by (10), let $S \subset R^1 \times R^4$ be defined by

$$S = \{(s, y)\colon\ |s| \leqslant c,\ \|y - y(0)\| \leqslant d \text{ for some } c, d > 0\}.$$

Since all partial derivatives $\partial f_i/\partial y^k$ are continuous, f satisfies a Lipschitz condition on the compact region S; that is, there exists a constant K such that

$$\|f(s, y) - f(s, z)\| \leqslant K\|y - z\|$$

for any points (s, y), $(s, z) \in S$. Likewise, since each f_i in (9) is a continuous function of s and y, the compactness of S implies the existence of a positive constant M such that $\|f(s, y)\| \leqslant M$ for all $(s, y) \in S$. Thus, by the fundamental existence theorem for solutions of first-order systems (see, for example, [1], p. 251), there exists a unique solution $y(s)$ of (8) subject to initial conditions (10) which is valid for $|s| \leqslant \delta$, where $\delta = \min\{c, d/M\}$.

Equivalently, there exists a solution $\gamma(s) = \big(u^1(s), u^2(s)\big)$ of (7) with initial conditions (6) for values of s satisfying $|s| \leqslant \delta$. The existence of such a local solution is well known. To extend this local solution to arbitrarily large values of s, we need the following simple lemma:

LEMMA. *Let* $\gamma(s) = \big(u^1(s), u^2(s)\big)$ *be a local solution of* (7) *satisfying initial conditions* (6) *and existing for* $|s| \leqslant \delta$ *for some* $\delta > 0$. *Then, for* α, β *as in* (3), *the equation*

$$\left|\frac{du^i}{ds}\right| \leqslant T = \max\{(1 - |\cos\alpha|)^{-1/2},\ (1 - |\cos\beta|)^{-1/2}\}, \qquad i = 1, 2,$$

is valid on the interval $|s| \leqslant \delta$.

Proof. A solution $\gamma(s)$ of (7) with initial conditions (6) is a geodesic in R^2 with the metric g which is parametrized by arc-length parameter s. Hence, the equation

$$(11)\qquad 1 = g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) = \left(\frac{du^1}{ds}\right)^2 + \left(\frac{du^2}{ds}\right)^2 + 2(\cos\theta)\frac{du^1}{ds}\left(\frac{du^2}{ds}\right)$$

holds on the interval $|s| \leqslant \delta$. For any fixed value s_0 in the interval $|s| \leqslant \delta$, let

$$v = \left(\frac{du^1}{ds}\right)(s_0), \qquad w = \left(\frac{du^2}{ds}\right)(s_0).$$

Then, for $s = s_0$, equation (11) becomes

$$1 = v^2 + w^2 + 2(\cos\theta)vw.$$

This is the equation of an ellipse in the vw-plane with major axis of length $(1-|\cos\theta|)^{-1/2}$. Since $\theta(u^1, u^2)$ must satisfy equation (3), the proof is complete.

3. Proof of Theorem 1. By (3) there exist constants P and Q such that

$$(12) \qquad \big|\cot\big(\theta(u^1, u^2)\big)\big| \leqslant P, \qquad \big|\csc\big(\theta(u^1, u^2)\big)\big| \leqslant Q$$

for all values of u^1, u^2. From (4), (9) and (12) we have, for any choice of s and y,

$$(13) \qquad |f_3(s, y)| \leqslant PA(y^3)^2 + QB(y^4)^2, \qquad |f_4(s, y)| \leqslant QA(y^3)^2 + PB(y^4)^2.$$

Thus (9) and (13) yield the following equation which holds for all choices of s and y:

$$(14) \qquad \|f(s, y)\| \leqslant |y^3| + |y^4| + A(P+Q)(y^3)^2 + B(P+Q)(y^4)^2.$$

Equations (6), (10), and the lemma yield

$$(15) \qquad |y^3(0)| \leqslant T, \qquad |y^4(0)| \leqslant T.$$

Let $S \subset R^1 \times R^4$ be the region

$$S = \{(s, y)\colon\ |s| \leqslant c,\ \|y - y(0)\| \leqslant d \text{ for some } c, d > 0\}.$$

Then (15) implies that, for y satisfying $\|y - y(0)\| \leqslant d$, we have

$$(16) \qquad |y^3| \leqslant T + d, \qquad |y^4| \leqslant T + d.$$

From (14) and (16) we see that if $(s, y) \in S$, then

$$(17) \qquad \|f(s, y)\| \leqslant 2(T+d) + (A+B)(P+Q)(T+d)^2.$$

Let $d = 1$; then the fundamental existence theorem yields a solution of (8) with initial conditions (10) which exists for s satisfying

$$|s| \leqslant \delta = 1/\big(2(T+1) + (A+B)(P+Q)(T+1)^2\big).$$

Thus we have a solution $\gamma(s) = \big(u^1(s), u^2(s)\big)$ of (7) with initial conditions (6) for s satisfying $|s| \leqslant \delta$. To extend this solution to larger values of s, we proceed as follows. Let $s_1 = \delta/2$, and let $y(s_1)$ be the solution $y(s)$ of (8) evaluated at $s = s_1$. Consider $S_1 \subset R^1 \times R^4$ defined by

$$S_1 = \{(s, y)\colon\ |s - s_1| \leqslant c,\ \|y - y(s_1)\| \leqslant d \text{ for } c, d > 0\}.$$

We recall that (14) is valid for any choice of s and y. Moreover, the lemma implies

$$(18) \qquad |y^3(s_1)| = \left|\frac{du^1}{ds}(s_1)\right| \leqslant T, \qquad |y^4(s_1)| = \left|\frac{du^2}{ds}(s_1)\right| \leqslant T.$$

From (14) and (18) we see that (17) is valid for all $(s, y) \in S_1$. Thus we can proceed precisely as we did to obtain, for $|s - s_1| \leqslant \delta$, a geodesic $\eta(s)$ with initial position $\gamma(s_1)$ and initial tangent vector $(d\gamma/ds)(s_1)$. By the uniqueness of solutions to systems such as (8) with prescribed initial conditions, we see that $\eta(s)$ extends $\gamma(s)$ to a geodesic with initial conditions (6), existing for $0 \leqslant s \leqslant 3\delta/2$.

By choosing $s_2 = \delta$ and repeating the same process, we obtain, for $0 \leqslant s \leqslant 2\delta$, a geodesic $\gamma(s)$ with initial conditions (6). Obviously, a sufficient number of repetitions of this process yields a geodesic $\gamma(s)$ of arbitrarily large length having the prescribed initial conditions (6). Since the initial choices of p and X were arbitrary, we infer that R^2 with metric g is complete.

REFERENCES

[1] E. A. Coddington, *An introduction to ordinary differential equations*, Englewood Cliffs, N. J., 1961.

[2] S. Sasaki, *On complete surfaces with Gaussian curvature zero in 3-sphere*, Colloquium Mathematicum 26 (1972), p. 165-174.

[3] T. J. Willmore, *An introduction to differential geometry*, London 1959.

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