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CONCRETE METHODS FOR SUMMING ALMOST PERIODIC FUNCTIONS AND THEIR RELATION TO UNIFORM DISTRIBUTION OF SEMIGROUP ACTIONS

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1. Introduction. Let G be a second contable locally compact group, $AP(G)$ the bounded, continuous Bohr almost periodic functions on G, and M the unique invariant mean on AP. If $\delta\colon G \to \Omega$ is the canonical homomorphism into the Bohr compactification, we obtain an isometric isomorphism $\delta^*\colon C(\Omega) \to AP(G)$ by $\delta^* f(g) = f(\delta(g))$ which identifies M as normalized Haar measure m_Ω on Ω,

$$\langle M, \delta^* f\rangle = \int_\Omega f(w)\, dm_\Omega(w).$$

Since G is second countable, there exist *summing sequences* $\{U_n\}$ of sets in G such that averages over these sets converge in the sup norm,

$$\lim_{n\to\infty} \frac{1}{|U_n|} \int_{U_n} R_g f dg = M(f)\cdot 1 \quad \text{for all } f \in AP(G),$$

where $R_g f(x) = f(xg)$, and $|U|$ is right Haar measure. The existence of such sequences has long been known if G is amenable; then sequences of Følner sets (see [12], Section 3.6) provide obvious candidates for the summing sequences. For general G, Davis [7] has shown that there is always a net of sets in G that sums the almost periodic functions, and one can actually show that there is a sequence if G is second countable.

The problem of constructing such summing sequences still exists. For amenable G some constructions for Følner sequences have been given in [13]. Davis [7] gave a construction for discrete countable G; starting with the class of all finite subsets in G, he shows that one can systematically find a sequence of finite sets which sum almost periodic functions. However, this construction totally ignores the algebraic structure of G;

one has the feeling that there are much more natural choices of a summing sequence which take the structural features of G into account.

We will show how to find explicit, natural summing sequences in $G = F_2$, the discrete free group on two generators a and b. What we say extends to n generators. One might expect the sets of elements in F_2 with length at most n,

$$S_n = \{x_1^{i_1}\cdot \ldots \cdot x_k^{i_k}:\ 0 \leqslant k \leqslant n,\ i_j = \pm 1,\ x_j = a \text{ or } b\},$$

to form a summing sequence in F_2. This is not so. As we will see, the almost periodic functions on F_2 can be summed by taking slightly more complicated sets $U_n = S_n \cup aS_n$ or $S_n \cup bS_n$ $(n = 1, 2, \ldots)$. The free groups make an interesting test case, since they are decidedly non-amenable, and since they yield information about general finitely generated discrete groups.

In an elegant note [1], Arnold and Krylov deal with the uniform distribution of orbits in the 2-sphere S^2 under the action of free subgroups of SO(3). They studied the action

$$\mathrm{SO}(3) \times L^2(S^2) \to L^2(S^2)$$

using certain special properties of spherical harmonics such as the lack of non-trivial 1-dimensional representations in the irreducible decomposition of $L^2(S^2)$. The main point of the present note is to show that their methods can be generalized to deal with the action

$$\Omega \times L^2(\Omega) \to L^2(\Omega),$$

Ω being the Bohr compactification of F_2, and to solve the summability problem for F_2. This done, we observe that uniform distribution questions concerning the action $F_2 \times \Omega \to \Omega$ can readily be translated into summability questions for $AP(F_2) \cong C(\Omega)$. Furthermore, this action is "universal" in a sense: all uniform distribution questions concerning any "almost periodic" action of F_2 on a compact space X can be read out as corollaries. In particular, we quickly obtain the original uniform distribution results of [1], where $X = S^2$, or later results by Avez [4], [5], where X is a compact connected Riemannian manifold on which F_2 acts by isometries. We also get results concerning actions on homogeneous spaces. Almost periodic actions correspond to equicontinuous actions on compact uniform spaces, as explained in [3]. We also note that the methods of Arnold and Krylov have been exploited in other directions in the work by Guivarc'h (see [14] and [15], Section 5).

Finally, in spite of the fact that our analysis of the universal action $F_2 \times \Omega \to \Omega$ relies heavily on group theoretic methods, we are able to solve the uniform distribution problem for almost periodic actions $F_2^+ \times X \to X$ of a free semigroup F_2^+ on two generators (see Section 3).

2. Explicit choices of summing sequences in $F = F_2$ (the free group). Similar arguments work for F_n being the free group on n generators. Introduce the following sets:

$$E_n = \{x_1 \cdot \ldots \cdot x_n : x_i = a \text{ or } b\},$$

$$F_n = \{x_1 \cdot \ldots \cdot x_k : 1 \leqslant k \leqslant n, x_i = a \text{ or } b\} = \bigcup_{j=1}^{n} E_j,$$

$$R_n = \{x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} : x_j = a \text{ or } b, i_j = \pm 1\}, \quad R_0 = \{e\},$$

$$S_n = \{x_1^{i_1} \cdot \ldots \cdot x_k^{i_k} : 0 \leqslant k \leqslant n, x_j = a \text{ or } b, i_j = \pm 1\} = \bigcup_{j=0}^{n} R_j.$$

If $x = x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$ ($i_j = \pm 1$, $x_i = a$ or b), its *signature*

$$\theta(x) = (-1)^{i_1 + \cdots + i_n}$$

is a well-defined one-dimensional representation of G. If x is a reduced word, its *length* is just $|x| = n$; set $|e| = 0$. Thus, S_n denotes all x such that $|x| \leqslant n$. For $g \in G$ we write $R_g f(x) = f(xg)$.

THEOREM. *The averages*

$$A_n f = \frac{1}{|U_n|} \sum_{g \in U_n} R_g f \tag{1}$$

($|U_n|$ is the cardinality of U_n) converge to the constant function $M(f) \cdot 1$ in the sup norm for all $f \in AP(F_2)$ in the following situations:

(i) $A_n f \to M(f) \cdot 1$ *in the Cesàro $(C, 1)$ sense if we take $U_n = E_n, F_n, R_n$, or S_n.*

(ii) $A_n f \to M(f) \cdot 1$ *if we take*

$$U_n = E_n \cup aE_n \cup \ldots \cup a^{n-1} E_n \quad \text{or} \quad U_n = E_n \cup bE_n \cup \ldots \cup b^{n-1} E_n.$$

(iii) $A_n f \to M(f) \cdot 1$ *if we take $U_n = S_n \cup aS_n$ or $S_n \cup bS_n$.*

None of the choices $U_n = E_n, F_n, R_n$, or S_n gives $A_n f \to M(f) \cdot 1$ for all $f \in AP(F_2)$.

Proof. Let $R_g f(x) = f(xg)$ be the right regular representation of the almost periodic compactification $\Omega = \Omega(F_2)$ on $L^2(\Omega) = \bigoplus \{H_i : i \in I\}$ which decomposes into finite-dimensional irreducible subspaces: $R | H_i = \pi_i$. Here π_i runs over all the irreducible unitary representations $\hat{\Omega}$, each $\pi \in \hat{\Omega}$ occurring $\dim(\pi)$ times. Let $i = 0$ correspond to the identity representation $\pi_0 = 1$, so $H_0 = C$ are constant functions. Let $i = 1$ correspond to $\pi_1 = \theta$, so $H_1 = C\theta$. Now $H_i \subseteq C(\Omega)$ for all i; if $\{\varphi_j\}$ is an approximate identity in $L^1(\Omega) \cap C(\Omega)$, then

$$\varphi_j * f \in C(\Omega) \cap H_i \text{ for } f \in H_i, \quad \text{and} \quad \|\varphi_j * f - f\| \to 0.$$

Since the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent on the finite-dimensional spaces H_i, f is continuous. The algebraic linear span $\sum\{H_i : i \in I\}$ is a $\|\cdot\|_\infty$-norm dense subalgebra of $C(\Omega)$ by Stone-Weierstrass. Let $A = R_a$, $B = R_b$, and $L = \frac{1}{2}(A+B)$. Then $|E_n| = 2^n$ and

$$\frac{1}{|E_n|}\sum_{g\in E_n} R_g f = L^n(f).$$

By examining $L^n f$ for $f \in H_i$ we will show that averages (1) converge to $M(f)\cdot 1$ in the sup norm as in (i)-(iii). Since (1) decreases the norm, convergence holds for all $f \in C(\Omega)$. Since $\dim(H_i) < \infty$, we may replace the norm $\|\cdot\|_\infty$ by $\|\cdot\|_2$. The result is obvious if $i = 0$, otherwise $M(f) = 0$ and we want to show that

$$\left\|\frac{1}{|U_n|}\sum_{g\in U_n} R_g f\right\|_2 \to 0 \quad \text{for all } f \in H_i \ (i \neq 0).$$

This is not quite true if $U_n = E_n$, but let us see what does happen.

Let $\mu_1, \ldots, \mu_r$ be the eigenvalues of $L|H_i$. Now $|\mu_i| \leqslant 1$, and if $|\mu_i| < 1$ for all i, then $\|L^n\| \to 0$ and we are done. Otherwise, there are eigenvalues with $|\mu| = 1$; since H_i is complex, there is an eigenvector with $\|f\|_2 = 1$. Now $L^n(f) = \mu^n f$ for $n = 0, 1, 2, \ldots$ The $\|\cdot\|_2$-norm unit ball in H_i is uniformly convex, and

$$1 = \|f\|_2 = \|Lf\|_2 = \left\|\frac{1}{2}Af + \frac{1}{2}Bf\right\|_2 \quad \text{with } \|Af\|_2 = \|Bf\|_2 = 1,$$

so we must have $Af = Bf = \mu f$. Since H_i is irreducible, it must be one-dimensional, $H_i = Cf$, and $L^n v = A^n v = B^n v = \mu^n v$ for all $v \in H_i$, $n \in Z$. The corresponding representation π_i on H_i satisfies $\pi_i(a) = \pi_i(b) = \mu I$; thus π_i is obtained from a character a_i on Z and factoring through

$$F_2 \xrightarrow{t} Z \xrightarrow{a_i} \mathrm{Hom}(H_i),$$

where t is the homomorphism determined by setting $t(a) = t(b) = 1$. Now $\ker(t)$ denotes all words with signature $\theta(w) = +1$, so we have $\ker(t) \supseteq [F_2, F_2]$. Since $i \neq 0$, we have $\mu \neq 1$; hence

$$T_{i_1} \ldots T_{i_n}(f) = \mu^n f \quad \text{if } T_{i_j} = A \text{ or } B,$$

so $L^n f = \mu^n f$ and the Cesàro means converge:

$$\frac{1}{n}(L + \ldots + L^n)f = \frac{1}{n}(\mu + \ldots + \mu^n)f \to 0 \quad \text{for all } f \in H_i.$$

Actually, convergence holds without taking Cesàro means on all H_i except those for which π_i is a one-dimensional representation such that $\pi_i(a) = \pi_i(b) = \mu \neq 1$, for example $\pi_i = \theta$.

Averages formed with $U_n = F_n$ are at least as well behaved as those for E_n, but simple computations show that they also assign divergent sums to the troublesome characters:

$$\frac{1}{|F_n|}\sum_{g\in F_n} R_g f = \frac{2\mu+\ldots+(2\mu)^n}{2+\ldots+2^n} = \frac{(2\mu)^n-1}{2^n-1}\,\frac{\mu}{2\mu-1}.$$

By taking summing sets $U_n = E_n \cup aE_n \cup \ldots \cup a^{n-1}E_n$ we get convergence without Cesàro means for all H_i. Let I' be the indices corresponding to the troublesome characters, $I'' = I' \cup \{0\}$. If $i \notin I''$, then

$$\Big\| \sum_{x\in a^k E_n} R_x f\Big\|_2 = \Big\|A^k \sum_{x\in E_n} R_x f\Big\|_2 \leqslant \Big\| \sum_{x\in E_n} R_x f\Big\|_2.$$

Since $|U_n| = n|E_n|$, averages (1) converge to zero for $f\in H_i$. But if $i\in I'$, we have $R_a f = R_b f = \mu f$ ($\mu \neq 1$) and the average over U_n has the norm

$$\Big\| \frac{1}{n}\mu^n(1+\mu+\ldots+\mu^{n-1})f\Big\|_2 \to 0.$$

Note. Averages formed with E_n, F_n, or U_n as above involve only positive powers of the generators, hence they can be used to study actions of free semigroups with two generators (see Section 3). Results concerning R_n and S_n apply only to group actions.

Convergence of the averages for R_n and S_n does not follow directly from what we know about E_n and F_n. They are handled using a basic idea from [1]. Write

$$T = \frac{1}{4}(A+B+A^{-1}+B^{-1}).$$

Let

$$(2)\quad T_0 = I,\ T_1 = T,\ T_2 = \frac{4}{3}T_1^2 - \frac{1}{3}I,\ \ldots,\ T_{n+1} = \frac{4}{3}T_1T_n - \frac{1}{3}T_{n-1},\ \ldots$$

These operators are self-adjoint since T is, and commute pairwise; moreover, $\|T_n\| \leqslant 1$ on L^2 for $n = 0, 1, 2, \ldots$, so $|\mu| \leqslant 1$ for any eigenvalue of any T_n. These operators are defined so that

$$T_n f = \frac{1}{|R_n|}\sum_{x\in R_n} R_x f \qquad (n = 1, 2, \ldots)$$

as one can check by induction. Formulas (2) force certain restraints on the eigenvalues $\mu_1, \ldots, \mu_r$ of T on a space H_i ($i \neq 0$). In fact, the eigenvalues μ_j' of T_2 on H_i satisfy

$$|\mu_j'| = \left|\frac{4}{3}\mu_j^2 - \frac{1}{3}\right| \leqslant 1 \quad \text{with } |\mu_j| \leqslant 1.$$

This forces $|\mu_j| < 1$ unless we happen to have $\mu_j = \pm 1$. As before, if $|\mu_j| < 1$ for all j, we get $\|T_n f\|_2 \to 0$ for all $f \in H_i$ and are done. Suppose that $\mu = +1$ occurs as an eigenvalue. Take an eigenfunction $\|f\|_2 = 1$. By uniform convexity the formula

$$f = T_1 f = \frac{1}{4}\,(Af + Bf + A^{-1}f + B^{-1}f)$$

insures that $Af = Bf = A^{-1}f = B^{-1}f = f$. Hence $A^n f = B^n f = f$ for all $n \in Z$ and $T_{i_1} \ldots T_{i_n} f = f$ if we take $T_{i_j} \in \{A, B, A^{-1}, B^{-1}\}$. Thus, $R_g f = f$ for all $g \in F_2$, hence for all $g \in \Omega$, so $f = \text{const}$ and $H_i = H_0 = C \cdot 1$, a case we have already dealt with. The other possibility is that $\mu = -1$ occurs, $T_1 f = -f$. Again, by uniform convexity we have $A^n f = B^n f = (-1)^n f$ for all $n \in Z$, so $H_i = Cf$ and on H_i we have $\pi_i(a^{\pm 1}) = -I = \pi_i(b^{\pm 1})$. Thus $i = 1$, $\pi_i = \theta$. Now direct calculations show that the averages over R_n and S_n do not converge for this character, though they converge to $M(f)\cdot 1$ on all H_i orthogonal to H_1. In fact,

$$\frac{1}{|R_n|}\sum_{x \in R_n} R_x f = (-1)^n f \qquad \text{for } f \in H_1.$$

Furthermore, $S_n = R_n \cup R_{n-1}$ for $n = 2, 3, \ldots$ and $R_n \cap R_{n-1} = \emptyset$ since words in R_n after reduction can only have lengths $n, n-2, \ldots$, while words in R_{n-1} have lengths of opposite parity. After some calculation we see that

$$|S_n| = 4\cdot 3^{n-1} + 4\cdot 3^{n-2} + \ldots + 4 + 1 = 2\cdot 3^n - 1$$

and

$$\frac{|R_n|}{|S_n|} \to \frac{3}{4}, \qquad \frac{|R_{n-1}|}{|S_n|} \to \frac{1}{4},$$

so that

$$\frac{1}{|S_n|}\sum_{x \in S_n} R_x = (-1)^n \frac{|R_n| - |R_{n-1}|}{|S_n|} \approx \frac{(-1)^n}{2}$$

on H_1. To get convergence without taking Cesàro means, it suffices to take slightly different summing sets $U_n = R_n \cup aR_n$ or $S_n \cup aS_n$; note that $R_n \cap aR_n = \emptyset$. Thus the proof is complete.

3. Application to uniform distribution problems. Suppose that S is a semigroup with two generators consisting of homeomorphisms on a compact space X, such that

(i) there is an invariant measure ν on X, so

$$\langle \nu, R_s f \rangle = \langle \nu, f \rangle \quad \text{for all } s \in S, f \in C(X);$$

(ii) there is a point $p \in X$ with dense orbit;

(iii) the action is *almost periodic*: for all $f \in C(X)$ the orbit $\{R_s f : s \in S\}$ is relatively compact in the norm $\|\cdot\|_\infty$.

Note. Other authors [3] consider the notion of *almost periodic orbits* $q\cdot S$ in a transformation group: the liftback $j_q^* f(s) = f(q\cdot s) = R_s f(q)$ is almost periodic on S in the sense that right translates $\{R_s(j^*f) : s \in S\}$ form a relatively compact set in $(l^\infty(S), \|\cdot\|_\infty)$ for any $f \in C(X)$. However, the existence of one dense almost periodic orbit implies (iii) since the liftback $j^*: C(X) \to l^\infty(S)$ is an isometry commuting with action of S. On the other hand, given (ii) and (iii) we conclude that all orbits are dense and almost periodic. Indeed, only density needs a proof. If $Y = (q\cdot S)^- \neq X$, then there is an $f \in C(X)$ such that $f|Y = 0$, $f(p) = 1$. There is also a net such that $p\cdot s_i \to q$; taking a subnet we get $\|R_{s_i} f - h\|_\infty \to 0$ for some h. Obviously,

$$h(p) = \lim f(p\cdot s_i) = f(q) = 0.$$

Since $Y\cdot S \subseteq Y$, we get $h(p\cdot s) = f(q\cdot s) = 0$, so $h|p\cdot S = 0$ and $h = 0$, which is impossible since $\|R_s f\|_\infty = 1$ for all s.

Let $\{U_n\}$ be any sequence which sums the almost periodic functions on F_2 such that

$$U_n \subseteq F_2^+ = \{x_1\cdot \ldots \cdot x_k : k \geqslant 0, x_i = a \text{ or } b\}.$$

There is a natural surjective homomorphism $j: F_2^+ \to S$; form the associated weighted averages of translates in $C(X)$:

$$B_n f = \frac{1}{|U_n|} \sum_{s \in j(U_n)} \operatorname{card}\{x \in U_n : j(x) = s\} \cdot R_s f. \tag{3}$$

Clearly, the weights are determined only by the relations among the generators in S, independent of any concrete action of S on some space X. For any action $S \times X \to X$ satisfying (i)-(iii) we will show that all orbits are uniformly distributed in the sense that

$$B_n f \to \Big(\int_X f\,d\nu\Big)\cdot 1_X \text{ in the sup norm} \tag{4}$$

for all $f \in C(X)$. If S is a free semigroup, the weights are trivial and we get the usual averages

$$B_n f = \frac{1}{|U_n|} \sum_{s \in j(U_n)} R_s f.$$

Once (4) is known for continuous functions, there are many standard extensions to larger classes of functions by monotone limit arguments.

For example, if W is a Borel set with $\nu(\text{bdry}\, W) = 0$, and if S acts freely, then (4) applies to the characteristic function $f = \chi_W$ to give

$$\nu(W) = \lim_{n\to\infty} \frac{\text{card}\{s \in j(U_n): q\cdot s \in W\}}{\text{card}\{j(U_n)\}}.$$

This holds for any point $q \in X$ since all orbits are dense, as noted above.

Condition (iii) is not unreasonably strong: it holds for all equicontinuous actions on a compact uniform space. These include the following examples mentioned in the literature:

(i) X compact metric, S a semigroup of isometries preserving some finite measure ν.

(ii) X a compact Riemannian manifold, S a semigroup of isometries, ν the usual measure.

(iii) X a homogeneous space: $X = H\backslash G$, where G is a compact group in which F_2^+ is homomorphically imbedded so that $H\cdot S$ is dense in G, ν the unique G-invariant measure, $p = He$.

The results of Avez [4] are covered by (i) and (ii), and those of Arnold and Krylov [1] by (iii). In (ii) we place no conditions on the Betti numbers of X as in Avez' corrected notes [5] and [4], since we use a somewhat different family of sets to form our averages, the sets U_n defined in Theorem 1 (ii). His notes concentrate exclusively on the following choice of summing sets (for freely acting S):

$$j(U_n) = \{s_1\cdot \ldots \cdot s_n\colon\ s_i = a \text{ or } b\}.$$

These sets, as we have seen, cannot be expected to sum all functions. Cohomological conditions on X essentially prevent the occurrence of the troublesome functions (the presence of troublesome characters) in the liftback $j^*\big(C(X)\big) \subseteq AP(F_2)$.

Proof of the uniform distribution (4). Since $S \subseteq \text{Hom}(X)$ is the group of homeomorphisms, we may discuss the group S' generated by S within $\text{Hom}(X)$.

Note. S' might not be free even if S is; the affine group of the real line is solvable, but contains free semigroups on two generators.

We first remark that the semigroup orbit $\{R_s f\colon\ s \in S\}$ is relatively compact in $C(X)$ if and only if the full orbit $\{R_s f\colon\ s \in S'\}$ is. We prove this as a lemma at the end. The map $j\colon F_2^+ \to S$ extends to a surjective homomorphism $j\colon F_2 \to S'$. Using the dense orbit $p\cdot S$ we lift functions back to F_2 by $j^*f(g) = f\big(p\cdot j(g)\big)$, an isometry between norms

$\|\cdot\|_\infty$ which commutes with actions as follows:

$$\begin{array}{ccc} C(X) & \xrightarrow{j^*} & AP(F_2) \\ {\scriptstyle R_{j(g)}}\downarrow & & \downarrow{\scriptstyle R_g} \\ C(X) & \xrightarrow[j^*]{} & AP(F_2) \end{array}$$

If $U_n \subseteq F_2^+$ is any summing sequence for $AP(F_2)$, form the averages

$$(5) \qquad A_n f = \frac{1}{|U_n|} \sum_{x \in U_n} R_x f \quad \text{for } f \in AP(F_2);$$

then $A_n f \to M(f)\cdot 1$ in sup norm. But

$$A_n\big(j^*(f)\big) = j^*\big(B_n(f)\big) \quad \text{for all } f \in C(X),$$

where $B_n f$ are averages (3). In particular, 1 lies in the closed subspace $j^*\big(C(X)\big)$ and $B_n f$ converges in norm to a function h such that $j^*(h) = \text{const.}$ Obviously, $h = c\cdot 1_X$ due to density of the orbit, but, in fact,

$$c = \int_X f\,d\nu,$$

since

$$\int B_n f\,d\nu = \int f\,d\nu \quad \text{for all } n.$$

Hence we have (4).

It remains to prove that relative compactness for the orbit $f\cdot S$ implies that of the full orbit $f\cdot S'$. We follow now some standard ideas of Glicksberg and de Leeuw [10], [12] concerning almost periodic semigroups of transformations in a Banach space (see also [6]). Let $B = C(X)$ and $\mathscr{I}(B)$ be the (not necessarily invertible) isometries. Multiplication of operators is jointly continuous on bounded sets in the strong operator topology. Now $f\cdot S$ is relatively compact in $C(X)$ for all f if and only if the strong operator closure of $\{R_s\colon s \in S\}$ is compact in $\big(\mathscr{I}(B), (\mathrm{so})\big)$, and similarly for the full orbits $f\cdot S'$. Let M be the (so)-closure of $\{R_s\colon s \in S\}$, an (so) compact semigroup. We want to show that M' being the (so)-closure of $\{R_s\colon s \in S'\}$ is compact; we do it by showing that $M' \subseteq M$. For fixed $s \in S$ it suffices to show that $R_{s^{-1}} = R_s^{-1}$ lies in H = (so) closure of the iterates $\{R_s^k\colon k = 1, 2, \ldots\} \subseteq M$. Write $R = R_s$. Note that the R^k have an accumulation point $T \in H$; thus if $f_1, \ldots, f_m \in B$, $\varepsilon > 0$ and an integer N are given, then there exist k and $l > k+N$ such that

$$\|R^l f_i - Tf_i\| < \varepsilon \quad \text{and} \quad \|R^k f_i - Tf_i\| < \varepsilon,$$

which implies that

$$\|f_i - R^{l-k} f_i\| = \|R^l f_i - R^k f_i\| < 2\varepsilon \quad \text{for all } i.$$

Thus the identity I is also an accumulation point of $\{R^k\}$. Take any subnet $\{R^{k(\delta)}\}$ such that $R^{k(\delta)} \to I$. Now $R^{k(\delta)-1}$ lies in the compact set H, so by taking a subnet we may insure that

$$R^{k(\zeta)} \to I \quad \text{and} \quad R^{k(\zeta)-1} \to A$$

(A being some element in H). Since the R^k commute, joint continuity of multiplication gives

$$R \cdot A = A \cdot R = I = \lim \{R^{k(\zeta)}\colon \zeta\},$$

i.e. $A = R_s^{-1} \in H$ as desired.

As it is well known, any action satisfying (ii) and (iii) is uniquely ergodic: formula (4), valid for any invariant measure ν on X, leads to the conclusion that $\nu(f) = M(j^*f)$ for all $f \in C(X)$.

4. Remarks on weakly almost periodic actions. Ergodic properties of weakly almost periodic functions $W(G)$, the continuous bounded functions f on G such that $\{R_g f\colon g \in G\}$ is weakly relatively compact, were first studied by Eberlein [9]. Although there is a unique (two-sided) invariant mean M on $W(G)$ for all G, as explained in [12], Section 3.1, virtually nothing is known about the nature of summing sequences $\{U_n\} \subseteq G$ such that

$$\frac{1}{|U_n|} \int_{U_n} f(g)\,dg \to M(f) \quad \text{for all } f \in W(G), \tag{6}$$

except when G is amenable and the usual Følner sequences do the job. Davis [8] proved the existence of a net of relatively compact open sets for which this is true, but it seems to be unknown whether there is always a sequence of summing sets if G is second countable (or even finitely generated, discrete). Indeed, nothing seems to be known about $G = F_2$.

Information about summing sequences would lead directly to uniform distribution theorems for dense, weakly almost periodic orbits under actions of second countable G on compact metric spaces X: dense orbits $p \cdot G$ such that all functions $f \in C(X)$ lift back to weakly almost periodic functions on G. This property, discussed in [3], is connected with distality of the action. Much as in Section 3, we would be able to conclude that if ν is any normalized invariant measure on X, then $p \cdot G$ is uniformly distributed in the sense that

$$\frac{1}{|U_n|} \int_{U_n} f(p \cdot g)\,dg \to \int_X f\,d\nu = M(j^*f) \quad \text{for all } f \in C(X). \tag{7}$$

In particular, the action $G \times X \to X$ is uniquely ergodic if there exists one dense weakly almost periodic orbit. To prove (7) we first note that the liftback $j^*: C(X) \to W(G)$ maps $C(X)$ isometrically into a closed subspace Y, and the adjoint j^{**} maps $W(G)^*$ onto $M(X) = C(X)^*$. Thus it is easy to see that the weak orbit closure $O_f = \{R_g f: g \in G\}^-$ is weakly compact in $C(X)$, as is its weak (i.e. norm) closed convex hull C_f. The averages

$$A_n f = \frac{1}{|U_n|} \int_{U_n} R_g f dg \quad \text{for } f \in C(X)$$

lie in C_f since they can be written as norm limits of convex "Riemann sums" of translates. Now C_f is weak sequentially compact by the Eberlein-Šmulian theorem. If h is any weak sequential accumulation point, some subsequence $A_{n(k)}f$ converges weakly, hence pointwise, to h. But h must be constant on X, since at points in the orbit we have

$$\begin{aligned} h(p \cdot g) &= \lim_k A_{n(k)} f(p \cdot g) = \lim_k \frac{1}{|U_{n(k)}|} \int_{U_{n(k)}} (j^* f)(gs) ds \\ &= \lim_n \frac{1}{|U_n|} \int_{U_n} L_g(j^* f) ds = M\big(L_g(j^* f)\big) = M(j^* f). \end{aligned}$$

The value of the constant c must actually be $\int_X f d\nu$ by the dominated convergence theorem:

$$\begin{aligned} c &= \int_X h d\nu = \lim_k \int_X A_{n(k)} f(x) d\nu \\ &= \lim_k \frac{1}{|U_{n(k)}|} \int_{U_{n(k)}} \left[\int_X f(x \cdot g) d\nu\right] dg = \int_X f d\nu. \end{aligned}$$

This holds for every weak sequential limit point h in C_f: the limit point is unique and

$$A_n f \to h = \Big(\int_X f d\nu\Big) \cdot 1$$

weakly, hence pointwise. At $x = p$ we get (7).

It is interesting to compare these uniform distribution theorems with the results on actions of nilpotent groups presented in [2]. The Auslander and Brezin results require entirely new methods, since the actions they study, where $X = \Gamma \backslash N$ for a discrete co-compact subgroup in a nilpotent Lie group N, are not even weakly almost periodic. Perhaps the future will reveal a theory of uniform distribution of group actions powerful enough to subsum all of these results. For a recent survey of diverse results and definitions of uniform distribution see [17].

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