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RELATIVELY FREE LATTICES*

BY

B. JÓNSSON (NASHVILLE)

1. Introduction. A basic question in the study of a variety (equational class) $\mathscr{V}$ of algebraic structures is the decision problem for the equational theory of $\mathscr{V}$. This is the problem of finding an effective procedure for determining whether or not a given identity holds in $\mathscr{V}$, or of showing that no such procedure exists. This is equivalent to the word problem for $\mathscr{V}$-free algebras A, i.e., the problem of deciding whether two terms s and t represent the same element in A when the variables are replaced by generators of A.

If $\mathscr{V}$ is a variety of lattices, then the equality $s = t$ is equivalent to the conjunction of two inclusions, $s \leqslant t$ and $t \leqslant s$, and it therefore suffices to obtain a decision procedure for inclusions. Now s is either a variable, or else it is a lattice sum (join) or product (meet) of simpler terms, and similarly for t. Of the nine cases that arise in this manner, five are trivial and use only criteria that hold in every lattice:

$$a+b \leqslant t \text{ iff } a \leqslant t \text{ and } b \leqslant t,$$

$$s \leqslant cd \text{ iff } s \leqslant c \text{ and } s \leqslant d.$$

Accordingly, if s is a sum or t is a product, then the inclusion $s \leqslant t$ is equivalent to the conjunction of two simpler inclusions. We may therefore assume that s is a variable or a product, and that t is a variable or a sum; i.e., we need only consider the four inclusions

$$x \leqslant y, \quad ab \leqslant y, \quad x \leqslant c+d, \quad ab \leqslant c+d,$$

where x and y are variables, a, b, c and d are arbitrary terms.

For the particular case when $\mathscr{V}$ is the class of all lattices the word problem was solved by Whitman [5] who showed that in a free lattice

(W1) $x \leqslant y$ iff $x = y$,

(W2) $ab \leqslant y$ iff $a \leqslant y$ or $b \leqslant y$,

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(W3) $x \leqslant c+d$ iff $x \leqslant c$ or $x \leqslant d$,

(W4) $ab \leqslant c+d$ iff $a \leqslant c+d$ or $b \leqslant c+d$ or $ab \leqslant c$ or $ab \leqslant d$.

The central result of this note asserts that the first three of Whitman's conditions hold in every relatively free lattice.

2. The principal theorem. In order to state the main theorem in a somewhat more general form, we use in place of the notion of a relatively free algebra a more general concept that was introduced in [1].

Definition 1. Given a class $\mathscr{K}$ of algebras, an algebra A is said to be *$\mathscr{K}$-freely* generated by a set X* if A is generated by X and every map of X into a member B of $\mathscr{K}$ can be extended to a homomorphism of A into B. If in addition, $A \in \mathscr{K}$, then A is said to be *$\mathscr{K}$-freely generated by X.*

LEMMA 2. *Suppose L is a lattice generated by a set X and suppose $a \in L$. If, for every non-empty finite subset Y of X,*

(i) $$\prod Y \leqslant a \text{ implies that } y \leqslant a \text{ for some } y \in Y,$$

then (i) *holds whenever Y is a non-empty finite subset of L.*

Proof. Given a subset S of L we define $P(a, S)$ to mean that every non-empty finite subset Y of S satisfies (i), and we let S^s be the set of all sums of elements of S, and S^p the set of all products of elements of S.

By hypothesis $P(a, X)$ holds, and it is obvious that $P(a, S)$ implies $P(a, S^p)$. We claim that $P(a, S)$ also implies $P(a, S^s)$. In fact, suppose

$$y \nleqslant a \text{ for all } y \in Y,$$

where Y is a non-empty finite subset of S^s. Each element y of Y is a sum $y = \sum Z(y)$, where $Z(y)$ is a non-empty finite subset of S. Since $y \nleqslant a$, there exists $z(y) \in Z(y)$ such that $z(y) \nleqslant a$. Since the elements $z(y)$ belong to S, we infer from the property $P(a, S)$ that their product u is not included in a. Inasmuch as $u \leqslant \prod Y$, this implies that $\prod Y \nleqslant a$. Thus $P(a, S^p)$ holds.

Let $S_0 = X$ and $S_{n+1} = (S_n)^{sp}$ for $n = 0, 1, \ldots$ Then $P(a, S_n)$ holds for each n. Since the sets S_n form a non-decreasing sequence whose union is L, and since the property $P(a, S)$ is of finite character, we conclude that $P(a, L)$ holds, and the proof is complete.

LEMMA 3. *Consider the following three properties of a lattice L and a generating set X for L:*

(i) *For any non-empty finite subsets Y and Z of X, if $\prod Y \leqslant \sum Z$, then $Y \cap Z \neq \emptyset$.*

(ii) *X is multiplicatively and additively irredundant.*

(iii) *For all $x, y \in X$ and $a, b, c, d \in L$,* W1, W2 *and* W3 *hold.*

Properties (ii) *and* (iii) *are equivalent and are implied by* (i).

Proof. Property (ii) means that if $x \in X$ and if Y and Z are non-empty finite subsets of X, then

$$\prod Y \leqslant x \text{ implies } x \in X, x \leqslant \sum Z \text{ implies } x \in Z.$$

Thus (ii) is equivalent to the two special cases of (i) in which either Z or Y consists of just one element.

If (ii) holds, then W1 is obviously satisfied, while W2 follows with the aid of Lemma 2, and W3 can be inferred by duality. Thus (iii) holds.

Finally assume (iii). Suppose $x \in X$, Y is a non-empty finite subset of X, and $\prod Y \leqslant x$. By an obvious generalization of W2 there exists $y \in Y$ with $y \leqslant x$, and it follows by W1 that $x = y \in Y$. Thus X is multiplicatively irredundant. Dually, X is additively irredundant, and (ii) therefore holds.

THEOREM 4. *If the class $\mathscr{K}$ of lattices is non-trivial (i.e. at least one member of $\mathscr{K}$ has more than one element), and if L is a lattice that is $\mathscr{K}$-freely* generated by a set X, then L and X have properties* (i)-(iii) *of Lemma* 3.

Proof. Choose a member A of $\mathscr{K}$ with at least two distinct elements. Then there exist $a, b \in A$ such that $a \nleqslant b$. Given disjoint non-empty finite subsets Y and Z of X, choose a map f of X into A that maps each member of Y onto a and every member of Z onto b. There exists a homomorphism g of L into A that agrees with f on X. Since

$$g(\prod Y) = a \nleqslant b = g(\sum Z),$$

this shows that $\prod Y \nleqslant \sum Z$.

3. Applications. It is well-known that in a free lattice the generating set is uniquely determined. We now extend this result to relatively free lattices.

THEOREM 5. *Suppose $\mathscr{K}$ is a non-trivial class of lattices and L is $\mathscr{K}$-freely* generated by X. Then X is the set of all those elements $x \in L$ that are both multiplicatively and additively irreducible. Hence every set that generates L contains X, and X is the only subset of L that $\mathscr{K}$-freely* generates L.*

Proof. By definition, an element x is said to be *multiplicatively irreducible* if $x = ab$ always implies that $x = a$ or $x = b$, and *additive irreducibility* is defined dually. Since L and X satisfy conditions W2 and W3, it follows at once that every element of X is both additively and multiplicatively irreducible in L. Conversely, each element x of L that is additively and multiplicatively irreducible must belong to every set that generates L, because $L - \{x\}$ is a sublattice of L. The second part of the conclusion is an immediate consequence of the first.

It is interesting to contrast the next theorem with the situation that exists for relatively free groups, cf. Neumann [2].

THEOREM 6. *Suppose $\mathscr{K}$ is a non-trivial class of lattices and L is $\mathscr{K}$-freely* generated by X. If the order of L is greater than four, then L is directly indecomposable.*

Proof. Suppose $L = A \times B$, where A and B are non-trivial lattices. For $u \in L$ let u_0 and u_1 be the A-component of u and the B-component of u, respectively, so that $u = \langle u_0, u_1\rangle$.

Consider any element $x \in X$. For any $a \in A$ and $b \in B$,

$$x \leqslant \langle x_0, b\rangle + \langle a, x_1\rangle$$

and therefore, by W3, $x \leqslant \langle x_0, b\rangle$ or $x \leqslant \langle a, x_1\rangle$, i.e.,

$$x_1 \leqslant b \quad \text{or} \quad x_0 \leqslant a.$$

Since this must hold for all $a \in A$ and $b \in B$, we infer that either A has a zero element 0_A and $x_0 = 0_A$ or B has a zero element 0_B and $x_1 = 0_B$. Dually, either $x_0 = 1_A$ or $x_1 = 1_B$. Now A and B were assumed to be non-trivial, so that $0_A \neq 1_A$ and $0_B \neq 1_B$, and we therefore infer that X can have at most two elements, $\langle 0_A, 1_B\rangle$ and $\langle 1_A, 0_B\rangle$, and from this it follows that the order of L is at most four.

THEOREM 7. *If Y is a non-empty finite set of variables, and if t is any lattice term, then the inclusion $\prod Y \leqslant t$ either holds in every lattice or else holds only in the one-element lattice.*

Proof. It suffices to show that if the given inclusion holds in a non-trivial relatively free lattice L when the variables are replaced by distinct generators, then it holds in every lattice. By Lemma 2 together with property (i) in Lemma 3 we see that $\prod Y \leqslant t_1 + t_2$ holds iff either $\prod Y \leqslant t_1$ or $\prod Y \leqslant t_2$ holds, and of course $\prod Y \leqslant t_1 t_2$ holds iff both $\prod Y \leqslant t_1$ and $\prod Y \leqslant t_2$ hold. Therefore the problem readily reduces to the case in which t is a variable, and in this case we again invoke the property (i) in Lemma 3.

A different proof of the preceding theorem has been given by Professor Stephen D. Comer.

We can use the number of summation signs and multiplication signs that occur in a term as a measure of the complexity of the term; we call this number the *weight* of the term. More precisely, we let $w(x) = 0$ if x is a variable, and if t is the sum or the product of two or more terms $s_0, s_1, \ldots, s_k$, then we let

$$w(t) = 1 + \max\{w(s_i) \mid i = 0, 1, \ldots, k\}.$$

Let us refer to an inclusion $s \leqslant t$ as an (m, n)-inclusion if $w(s) = m$ and $w(t) = n$. It follows from Theorem 7 that if $m \leqslant 1$ or $n \leqslant 1$, then every (m, n)-inclusion is either a consequence of the lattice axioms or else holds only in the one-element lattice. The simplest inclusions for which this is not the case will therefore be $(2, 2)$-inclusions. The distributive law is of this type, but the modular law $x(y+xz) \leqslant xy+xz$ is a $(3,2)$-inclusion. It follows from the next theorem that the modular lattices cannot be characterized by a $(2,2)$-inclusion. A different proof of this fact has been given by Professor Comer, who showed that any $(2,2)$-inclusion that holds in every lattice of dimension two holds in every lattice.

THEOREM 8. *Every $(2,2)$-inclusion that implies the modular law also implies the distributive law.*

Proof. Let the given $(2,2)$-inclusion be

$$s = \prod_{j \in J} \left(\sum Y_j\right) \leqslant \sum_{k \in K} \left(\prod Z_k\right) = t. \tag{1}$$

If this implies the modular law, then it must fail in the pentagon $N = \{0, u, v, w, 1\}$, where $u < w$, $uv = wv = 0$ and $u+v = w+v = 1$. It is therefore possible to assign a value $x' \in N$ to each variable x that occurs in (1) in such a way that $s' \nleqslant t'$, where s' and t' are the values of s and t under this assignment. If we identify two variables whenever they receive the same value under this assignment, we obtain a new inclusion which also fails in N under this assignment, and which is implied by (1). It is therefore sufficient to show that the new inclusion implies the distributive law. We may therefore assume without loss of generality that there are only five variables x_0, x_1, x_u, x_v, x_w that occur in (1), and that they are assigned the values $0, 1, u, v$ and w, respectively.

We may assume that (1) holds in some non-trivial lattice, and therefore holds in every distributive lattice. In this case, if φ is the homomorphism of N onto the distributive sublattice $D = \{0, u, v, 1\}$ that maps w onto u and each member of D onto itself, then $\varphi(s') \leqslant \varphi(t')$, and this clearly can only happen if $s' = w$ and $t' = u$. From this we can infer that each of the sets Y_j must contain either x_w or x_1 or x_u and x_v, and that each Z_k must contain x_u or x_0 or x_w and x_v. Consequently, if we let

$$s_1 = x_w x_1 (x_u + x_v) \qquad \text{and} \qquad t_1 = x_u + x_0 + x_w x_v,$$

then the inclusions $s_1 \leqslant s$ and $t \leqslant t_1$ hold in every lattice, and therefore $s_1 \leqslant t_1$ is a consequence of (1). Finally, $s_1 \leqslant t_1$ implies

$$x_w (x_u + x_v) \leqslant x_u + x_w x_v,$$

and this is well known to imply the distributive law.

A weaker form of this last theorem is stated by Takeuchi in [4], p. 6, but there appears to be a gap in the proof. In fact, the argument is based on a corollary on p. 62 in the earlier paper [3], and the proof of this makes reference to another corollary on p. 60 of the same paper. However, the earlier corollary concerns $-\mu^*$-transformations, but the intended application would seem to require $-\mu$-transformations.

REFERENCES

[1] B. Jónsson and A. Tarski, *On two properties of free algebras,* Mathematica Scandinavica 9 (1961), p. 95-101.

[2] P. M. Neumann, *A note on the direct decomposability of relatively free groups,* Quarterly Journal of Mathematics 18 (1967), p. 67-79.

[3] Kensuke Takeuchi, *On free modular lattices,* Japanese Journal of Mathematics 21 (1951), p. 53-65.

[4] — *On free modular lattices II,* The Tôhoku Mathematical Journal 11 (1959), p. 1-12.

[5] P. M. Whitman, *Free lattices,* Annals of Mathematics 42 (1941), p. 325-330.

VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE

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