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# *ON A LINEARIZATION OF AN EQUATION OF AN ELASTIC ROD*

**1.** In this paper we consider the system of equations

$$\frac{dX}{ds} = \sin\theta, \quad \frac{dY}{ds} = \cos\theta, \quad IE\frac{d^2\theta}{ds^2} = f\sin\theta,$$

where the unknown functions $X(s)$, $Y(s)$, $\theta(s)$ are defined on the interval $\langle 0, s_0\rangle$ and satisfy the following conditions

$$X(0) = Y(0) = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \frac{d\theta}{ds} = 0 \quad \text{for} \quad s = s_0.$$

The quantities $I$, $E$, $f$ and $s_0$ are given positive constants. The equations under consideration describe the approximate form of the middle line of an elastic rod with a circular cross-section the rod being clamped on one of its ends. The other end is acted upon by a force directed orthogonally to the axis of the rod in the undeformed state ([3], p. 731).

We introduce new variables by the formulas

$$l = \left(\frac{f}{IE}\right)^{1/2} s, \quad z(l) = \frac{\pi}{2} - \theta(s),$$

$$x(l) = \left(\frac{f}{IE}\right)^{1/2} X(s), \quad y(l) = \left(\frac{f}{IE}\right)^{1/2} Y(s).$$

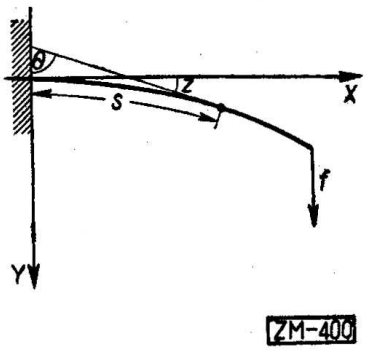


Fig. 1

The preceding equations now take the form

$$z'' + \cos z = 0, \quad z(0) = z'(l_0) = 0 \tag{1.1}$$

and

$$x' = \cos z, \quad y' = \sin z, \quad x(0) = y(0) = 0, \tag{1.2}$$

where $l_0$ denotes $\left(\frac{f}{IE}\right)^{1/2} s_0$ and $' = \frac{d}{dl}$. It is possible to integrate the above system of equations by means of elliptic functions. The final formulas are of a rather complicated form. In this paper comparatively simple

approximating formulas are given for the solutions in question. We get these formulas using a method of linearization ([4], [5]). An estimation of the error of the aproximation is also given.

**2.** Throughout this paper $z(l)$ will denote the solution of the boundary value problem (1.1). It will be approximated by a solution $z_{ap}(l)$ of a linear equation with constant coefficients

$$z''_{ap} + az_{ap} + b = 0 \tag{2.1}$$

with the boundary conditions

$$z_{ap}(0) = 0, \quad z_{ap}(l_0) = z_0, \tag{2.2}$$

where $z_0 = z(l_0)$.

We are now going to estimate the difference $z(l) - z_{ap}(l)$. The result will suggest to us the optimal choice of the constants $a$ and $b$.

In virtue of equations (1.1), (1.2) we have the following identity:

$$(z'' - z''_{ap}, z - z_{ap}) = -(\cos z - az - b, z - z_{ap}) - a(z - z_{ap}, z - z_{ap}).$$

The symbol $(g, h)$ denotes here the scalar product $\int_0^{l_0} gh\,dl$ of the functions $g(l)$ and $h(l)$. We integrate by parts the left-hand member of the last equation. On account of the boundary conditions we get the identity

$$\|z' - z'_{ap}\|^2 = (\cos z - az - b, z - z_{ap}) + a\|z - z_{ap}\|^2. \tag{2.3}$$

Here $\|g\|$ denotes the $L^2$ — norm in $\langle 0, l_0\rangle$, i.e. $\|g\|^2 = (g, g)$. In what follows we limit ourselves to the case where the value of the parameter $a$ is non-positive: $a \leqslant 0$. Then we get the inequality

$$\|z' - z'_{ap}\|^2 \leqslant \|\cos z - az - b\| \cdot \|z - z_{ap}\| \tag{2.4}$$

obtained by omitting the last term in identity (2.3) and applying the Schwarz inequality to the preceding term. On the other hand, the vanishing of $z - z_{ap}$ on both ends of the interval $\langle 0, l_0\rangle$ guarantees that the following two inequalities are valid:

$$\begin{gathered} \|z - z_{ap}\| \leqslant \frac{l_0}{\pi}\|z' - z'_{ap}\|, \\ |z - z_{ap}| \leqslant l_0^{-1/2}\{l(l_0 - l)\}^{1/2} \cdot \|z' - z'_{ap}\|. \end{gathered} \tag{2.5}$$

The first of these inequalities is the well-known Steklov-Wirtinger inequality, whereas the second presents a stronger form of an inequality given in [1], p. 346. From these inequalities and from (2.4) we get two estimates:

$$\|z' - z'_{ap}\| \leqslant \frac{l_0}{\pi}\|\cos z - az - b\| \tag{2.6}$$

and

$$|z-z_{ap}| \leqslant \frac{l_0^{1/2}}{\pi}\{l(l_0-l)\}^{1/2}\|\cos z-az-b\|. \tag{2.7}$$

The last inequality gives the desired estimate of the error of approximation but the right-hand side of the formula depends upon the unknown function $z(l)$. We can overcome this disadvantage as follows. Making use of the first integral

$$z'^2+2\sin z = 2\sin z_0 \tag{2.8}$$

of equation (1.2) we obtain the identity

$$\|\cos z-az-b\|^2 = \int_0^{z_0} \frac{(\cos z-az-b)^2}{\sqrt{2(\sin z_0-\sin z)}}\,dz.$$

Although the right-hand integral does not depend upon the unknown function $z(l)$, it is rather difficult to compute. But the inequality

$$\frac{1}{\sqrt{2(\sin z_0-\sin z)}} \leqslant \left(\frac{z_0}{\cos z_0}\right)^{1/2}\frac{1}{\sqrt{z_0^2-z^2}},$$

valid in the interval $\langle 0, z_0)$, enables us to write down the estimate

$$\|\cos z-az-b\|^2 \leqslant \left(\frac{z_0}{\cos z_0}\right)^{1/2}\int_0^{z_0}\frac{(\cos z-az-b)^2}{\sqrt{z_0^2-z^2}}\,dz, \tag{2.9}$$

where the right-hand integral is now of a comparatively simple form.

Finally, on account of (2.6), (2.7) and (2.9), we get the estimates

$$\begin{aligned} \|z'-z'_{ap}\| &\leqslant \frac{l_0}{\pi}\,\delta(a,b), \\ |z-z_{ap}| &\leqslant \frac{l_0^{3/2}}{\pi}\left\{\frac{l}{l_0}\left(1-\frac{l}{l_0}\right)\right\}^{1/2}\delta(a,b), \end{aligned} \tag{2.10}$$

where $\delta(a,b) = \left(\frac{z_0}{\cos z_0}\right)^{1/4}\Delta(a,b)$ and

$$\Delta^2(a,b) = \int_0^{z_0}\frac{(\cos z-az-b)^2}{\sqrt{z_0^2-z^2}}\,dz = \tag{2.11}$$

$$= \frac{\pi z_0^2}{4}a^2+2z_0ab+\frac{\pi}{2}b^2-2z_0S_0(z_0)a-\pi J_0(z_0)b+\frac{\pi}{4}[1+J_0(2z_0)].$$

Here $J_0$ denotes the Bessel function of the first kind and order 0 and $S_0$ is defined as follows

$$S_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) z^{2n}}{[(2n+1)!!]^2}.$$

In deriving formula (2.11) we used the known integral formulas (cf. [6], p. 191-192).

For the given values of parameters $l_0$, $z_0$, $a \leqslant 0$ and $b$ inequalities (2.10) enable us to calculate the numerical estimates of the errors $|z - z_{ap}|$ and $\|z' - z'_{ap}\|$.

**3.** Now it becomes clear how to choose the parameters $a$ and $b$. We define them by the conditions

$$\Delta^2(a, b) = \text{minimum}, \qquad a \leqslant 0.$$

After a simple calculation we get for the coefficients $a$ and $b$ the formulas

$$\begin{aligned} a &= -r^2 = \frac{4\pi}{\pi^2 - 8} \cdot \frac{S_0(z_0) - J_0(z_0)}{z_0}, \\ b &= J_0(z_0) + \frac{2r^2}{\pi} z_0, \end{aligned} \tag{3.1}$$

and for the minimal value of $\frac{2}{\pi}\Delta^2$ we get

$$\frac{2}{\pi}\Delta^2(a, b) = \frac{1}{2} + \frac{1}{2} J_0(2z_0) - J_0^2(z_0) - \frac{8}{\pi^2 - 8}[S_0(z_0) - J_0(z_0)]^2. \tag{3.2}$$

Before proceeding further let us remark that we can improve estimations (2.10) in the following way.

By applying the Schwarz inequality as well as that given by (2.9) we obtain from formula (2.3) the inequality

$$\alpha^2 + r^2\beta^2 \leqslant \delta\beta,$$

which may be rewritten in the form

$$\frac{\alpha^2}{\left(\frac{\delta}{2r}\right)^2} + \frac{\left(\beta - \frac{\delta}{2r^2}\right)^2}{\left(\frac{\delta}{2r^2}\right)^2} \leqslant 1, \tag{3.3}$$

where $\alpha = \|z' - z'_{ap}\|$, $\beta = \|z - z_{ap}\|$ and $\delta$ has the same meaning as in formula (2.10).

In the geometrical interpretation this inequality determines an ellipse on the plane $\alpha$, $\beta$ with the centre at the point $\alpha = 0$, $\beta = \delta/2r^2$ and with the semi-axes equal to $\delta/2r$ and $\delta/2r^2$ respectively. In view of Steklov's inequality, which may now be written in the form

$$0 \leqslant \beta \leqslant \frac{l_0}{\pi}\alpha, \tag{3.4}$$

the point with the coordinates $\alpha = \|z' - z'_{ap}\|$, $\beta = \|z - z_{ap}\|$ lies in the part of the elliptical region (3.3) where inequality (3.4) holds.

This part of the ellipse is shaded in figure 2 given below, which corresponds to the case where

$$\frac{l_0}{\pi} \leqslant \frac{\delta}{2r^2} : \frac{\delta}{2r} = \frac{1}{r}. \tag{3.5}$$

All the points of the domain determined by (3.3)-(3.5) have the $\alpha$ coordinate not greater than the abscissa of the point $M$. This conclusion leads us to the estimate

$$\|z' - z'_{ap}\| \leqslant \frac{l_0}{\pi} \cdot \frac{\delta}{1 + \left(\frac{rl_0}{\pi}\right)^2}, \tag{3.6}$$

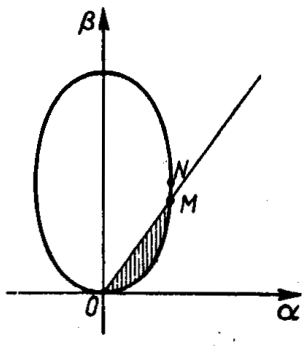


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Fig. 2

which is better than the first of the estimates given by (2.10). In the case where condition (3.5) does not hold, the straight line $OM$ goes above the vertex $N$. In that case the $\alpha$ coordinates of the points satisfying inequalities (3.3) and (3.4) are not greater than the abscissa of the point $N$, which gives the estimate

$$\|z' - z'_{ap}\| \leqslant \frac{\delta}{2r}. \tag{3.7}$$

This estimate is also better than the first of the estimates given by (2.10), for the inequality $1/r < l_0/\pi$ holds in this case.

In virtue of the second inequality of (2.5) we get from (3.6) and (3.7) the estimates

$$|z - z_{ap}| \leqslant \frac{l_0^{3/2}}{\pi} \left\{\frac{l}{l_0}\left(1 - \frac{l}{l_0}\right)\right\}^{1/2} \frac{\delta}{1 + \left(\frac{rl_0}{\pi}\right)^2} \tag{3.8}$$

for $1/r \geqslant l_0/\pi$ and

$$|z - z_{ap}| \leqslant l_0^{1/2} \left\{\frac{l}{l_0}\left(1 - \frac{l}{l_0}\right)\right\}^{1/2} \frac{\delta}{2r} \tag{3.9}$$

for $1/r < l_0/\pi$.

There is a possibility of obtaining a somewhat better estimate, but we will not go into details.

**4.** We shall now discuss the construction of the function $z_{ap}(l)$. On account of equations (2.1), (2.2) this function has the form

$$z_{ap} = z_0 \frac{\operatorname{sh} rl}{\operatorname{sh} rl_0} + \frac{b}{r^2 \operatorname{sh} rl_0} \{\operatorname{sh} rl_0 - \operatorname{sh} rl - \operatorname{sh} r(l_0 - l)\}. \tag{4.1}$$

We have to put into this formula the values of $b$ and $r$ given by (3.1). In the formula obtained in this way, as well as in (3.2) and in estimates (3.8), (3.9) occur the parameters $l_0$ and $z_0$. However, in the boundary value problem (1.1), which has been approximately solved, only one parameter $l_0$ is given. To complete the list of formulas we add to them the known formula

$$l_0 = \int_0^{z_0} \frac{dz}{z'} = \frac{1}{\sqrt{2}} \int_0^{z_0} (\sin z_0 - \sin z)^{-1/2} dz \tag{4.2}$$

establishing the connection between the parameters $l_0$ and $z_0$. This formula follows directly from the first integral (2.8) of equation (1.1). By means of the substitution $\sin z = \sin z_0 - (1 + \sin z_0) v$ we transform equality (4.2) and obtain

$$l_0 = \frac{1}{2} \left( \int_0^1 - \int_{\cos^2\varphi}^1 \right) \frac{dv}{\sqrt{v(1-v)(k'^2 + k^2 v)}},$$

where the parameters $k$, $k'$ and $\varphi$ are defined by the equations

$$k = \sin\left(\frac{\pi}{4} + \frac{z_0}{2}\right), \quad k'^2 + k^2 = 1, \quad \sqrt{2} k \sin\varphi = 1. \tag{4.3}$$

Hence we get the formula for $l_0$, which is expressed in terms of $z_0$,

$$l_0 = K(k) - F(\varphi, k). \tag{4.4}$$

Here $K$ and $F$ denote the normal elliptic integrals of the first kind ([6], p. 302-303). Let us remark that it is sufficient to know only the upper and the lower bounds of $l_0$ in order to obtain from formulas (3.8), (3.9) an estimate of the error $|z - z_{ap}|$. We get these two bounds in virtue of (4.4) by using the tables of elliptic integrals, e.g. [2], [7], without interpolation which is to be carried out with respect to two arguments $\varphi$ and $k$ of the function $F$.

It is easy to give a simple formula for the bounds of $l_0$. For this purpose we use the inequalities

$$\left(\frac{z_0}{2\sin z_0}\right)^{1/2} \frac{1}{\sqrt{z_0 - z}} \leqslant \frac{1}{\sqrt{2(\sin z_0 - \sin z)}} \leqslant \left(\frac{1}{2\cos z_0}\right)^{1/2} \frac{1}{\sqrt{z_0 - z}},$$

which are satisfied in the interval $\langle 0, z_0)$. Hence we get the desired formula by integration:

$$\left(\frac{z_0}{\sin z_0}\right) \leqslant \frac{l_0}{\sqrt{2z_0}} \leqslant \left(\frac{1}{\cos z_0}\right)^{1/2}. \tag{4.5}$$

We conclude this section with a numerical example putting $z_0 = 1$. This corresponds to the bending of a rod when the tangent to its middle line at the free end forms an angle of 1 radian with the axis of the undeformed rod.

From formulas (3.1) we find in this case $b = 1{,}094\cdot 397$, $r = 0{,}719\cdot 100$. Then using tables [2], [7] and in virtue of (4.3), (4.4) we get the following estimates: $1{,}750 \leqslant l_0 \leqslant 1{,}776$. Finally formula (3.8) yields

$$|z - z_{ap}| \leqslant 0{,}015\cdot 44 \leqslant 1{,}55\%\ \text{of}\ z_0.$$

The other examples given in Table 1 have been computed in the same way.

TABLE 1

| $z_0$ | $b$ | $r$ | $\tilde{l}_0$ | $\lvert z - z_{ap}\rvert \leqslant$ | $\frac{1}{z_0}\lvert z - z_{ap}\rvert\cdot 100 \leqslant$ |
|---|---|---|---|---|---|
| 0,25 | 1,006612 | 0,373280 | 0,71 | $2{,}2\cdot 10^{-4}$ | 0,09 |
| 0,50 | 1,025864 | 0,524398 | 1,04 | $1{,}7\cdot 10^{-3}$ | 0,34 |
| 0,75 | 1,056039 | 0,633797 | 1,36 | $6{,}2\cdot 10^{-3}$ | 0,82 |
| 1,00 | 1,094397 | 0,719100 | 1,76 | $1{,}6\cdot 10^{-2}$ | 1,60 |
| 1,25 | 1,135680 | 0,784770 | 2,31 | $4{,}1\cdot 10^{-2}$ | 3,25 |

Remark: $\tilde{l}_0$ in Table 1 denotes the mean value of both bounds of $l_0$ obtained by using tables [2], [7] without interpolation. It is clear that the value of $l_0$ to be put into (4.1) must be computed with greater accuracy.

**5.** In this section we shall give approximate formulas for the functions $x(l)$ and $y(l)$ describing the form of the rod under deformation. On account of equations (1.1), (1.2) we find the relation $y'' = -z'z''$, and hence

$$y(l) = l\sin z_0 - \tfrac{1}{2}\int_0^l z'^2\,dl. \tag{5.1}$$

We replace this exact formula by the approximate one,

$$y_{ap}(l) = l\sin z_0 - \tfrac{1}{2}\int_0^l z_{ap}'^2\,dl, \tag{5.2}$$

where the integration is simple to be carried out (cf. (4.1)).

For the error of the approximation of (5.1) by (5.2) we find

$$|y(l)-y_{ap}(l)| \leqslant \tfrac{1}{2}\int_0^l |z'+z'_{ap}|\cdot|z'-z'_{ap}|\,dl.$$

Then, making use of the Schwarz inequality and of the triangle inequality, we obtain the estimate

$$|y(l)-y_{ap}(l)| \leqslant (\|z'_{ap}\|+\tfrac{1}{2}\|z'-z'_{ap}\|)\,\|z'-z'_{ap}\|. \tag{5.3}$$

The term $\|z'-z'_{ap}\|$ occurring in the last formula has been estimated in section 3. The other member $\|z'_{ap}\|$ may easily be calculated.

We now proceed similarly for the function $x(l)$. From the basic equations (1.1), (1.2) we find the exact formula for $x(l)$

$$x(l) = z'(0)-z'(l), \tag{5.4}$$

which we replace by the approximation formula

$$x_{ap}(l) = z'_{ap}(0)-z'_{ap}(l). \tag{5.5}$$

The estimation of the error will follow from the identity

$$x(l)-x_{ap}(l) = \int_0^l (z''-z''_{ap})\,dl,$$

which yields, by means of Schwarz's inequality,

$$|x(l)-x_{ap}(l)| \leqslant \sqrt{l}\,\|z''-z''_{ap}\|.$$

In virtue of the equations which are satisfied by the functions $z$ and $z_{ap}$ we can replace $z''-z''_{ap}$ by $\cos z-az_{ap}-b$. From this new inequality, by the application of the triangle inequality, we get the final estimate

$$|x(l)-x_{ap}(l)| \leqslant \sqrt{l}\,(\|\cos z-az-b\|+r^2\|z-z_{ap}\|). \tag{5.6}$$

The bound for the first term of the right-hand side is given by formulas (2.9), (2.11) and (3.1), (3.2). The estimate of the second term follows from (2.5) and (3.6), (3.7).

Other approximating formulas for $x(l)$ and $y(l)$ can be given. For example we can use the fact that the function $x(l)$ may be expressed in terms of $z(l)$ as follows:

$$x(l) = (2\sin z_0)^{1/2}\left\{1-\left(1-\frac{\sin z}{\sin z_0}\right)^{1/2}\right\}.$$

If the inequality $z_0 < b/r^2$ holds, then we can prove that $z_{ap}(l) \leqslant z_0$ in the whole interval $\langle 0, l_0\rangle$. This fact enables us to replace the function $z$ in the last formula by $z_{ap}$. The function obtained in this way will present another approximation of $x(l)$.

**6.** If the bending of the rod is sufficiently small, we can hope that the equation

$$z''_{ap} + b = 0 \tag{6.1}$$

will give us a practically useful approximation of $z(l)$. Equation (6.1) is much simpler to deal with than the equation proposed in section 2. Under the same boundary conditions (2.2) the solution of (6.1) is

$$z_{ap}(l) = \frac{b}{2}\, l(l_0 - l) + \frac{z_0}{l_0}\, l. \tag{6.2}$$

To obtain the desired estimates we start from formula (2.7) by putting in it $a = 0$. This procedure gives us

$$|z - z_{ap}| \leqslant \frac{l_0^{1/2}}{\pi} \{l(l_0 - l)\}^{1/2} \|\cos z - b\|. \tag{6.3}$$

To get the best estimate for $|z - z_{ap}|$ we define $b$ by the condition $\|\cos z - b\|$ = minimum. The identity

$$\|\cos z - b\|^2 = \int_0^{z_0} \frac{\cos^2 z}{\sqrt{2(\sin z_0 - \sin z)}}\, dz - 2b(2\sin z_0)^{1/2} + b^2 l_0, \tag{6.4}$$

together with the condition formulated above, yields

$$b = \frac{1}{l_0}(2\sin z_0)^{1/2}. \tag{6.5}$$

To compute the integral occuring in (6.4) we have to use elliptic integrals. To avoid them we maximize this integral in the same way as has been done in section 2. This leads us to the inequality

$$\|\cos z - b\|^2 \leqslant \frac{\pi}{4}\left(\frac{z_0}{\cos z_0}\right)^{1/2} [1 + J_0(2z_0)] - \frac{2\sin z_0}{l_0}, \tag{6.6}$$

which, together with (6.3), yields the estimate of $|z - z_{ap}|$, $z_{ap}$ being defined by (6.2) and (6.5). The parameter $l_0$ is given again by (4.4).

An observation should be made. Let us note that dealing with formulas (6.5), (6.6) may be troublesome if we do not use sufficiently accurate tables of the elliptic integrals. From the point of view of numerical calculations, the situation is now different from that presented in the section 3 (cf. (3.1), (3.2) and (3.8), (3.9)) since, unlike the present case (cf. (6.6)), no difference of the expressions dependent on both $l_0$ and $z_0$ has occurred in the formulas given there.

The appearance of such a difference with a very small numerical value makes it desirable to have the parameters $l_0$ and $z_0$ given with a high

accuracy in order to get the numerical results with a small relative error. We will therefore derive other formulas, where the difficulty discussed above will disappear. For this purpose we choose another value for $b$:

$$b = J_0(z_0) = 1 - \frac{1}{2!^2}\left(\frac{z_0}{2}\right)^2 + \frac{1}{4!^2}\left(\frac{z_0}{2}\right)^4 - \ldots \tag{6.7}$$

It can readily be verified that this value of $b$ provides the function $\Delta(0, b)$ with its minimal value. Formulas (2.9)-(2.11) may now be used by putting there $a = 0$, $b = J_0(z_0)$ to yield

$$|z - z_{ap}| \leqslant \frac{l_0^{3/2}}{\pi}\left\{\frac{l}{l_0}\left(1 - \frac{l}{l_0}\right)\right\}^{1/2}\left(\frac{z_0}{\cos z_0}\right)^{1/4} \Delta\big(0, J_0(z_0)\big) \tag{6.8}$$

with $z_{ap}$ given now by formulas (6.2), (6.7).

As indicated at the beginning of this section, we limit the domain of the application of approximation (6.2) to the range of small values of $z_0$. Therefore it is meaningful to ask for simpler formulas to work with, although the estimates given by them will be a little worse.

To this end, in view of inequality (4.5), we estimate first of all the term $l_0^{3/2}$ occurring in (6.8). Next we develop $\frac{2}{\pi}\Delta^2$ into a power series in $z_0$ ([6], p. 352). We find that this expression has the form of an alternating series

$$\frac{2}{\pi}\Delta^2\big(0, J_0(z_0)\big) = \sum_{n=2}^{\infty}(-1)^n \frac{z_0^{2n}}{2n!^2}\left\{1 - 2\frac{(2n-1)!!}{(2n)!!}\right\}.$$

Taking into consideration only the first term of the series and replacing $\cos z_0$ by its estimate $1 - \frac{z_0^2}{2}$, we finally get, on account of (6.8), the desired simple estimate

$$\frac{|z - z_{ap}|}{z_0} \leqslant \left(\frac{1}{512\pi^2}\right)^{1/4} \frac{\frac{1}{2}z_0^2}{1 - \frac{1}{2}z_0^2}. \tag{6.9}$$

The following Table 2 contains the values both of the last estimate and of the parameter $b$ for a few specific values of $z_0$. There is also added a column (the last one) of the values illustrating the inaccurracy in the determination of $l_0$ when using formula (4.5).

The comparison of the last two lines of Table 2 with Table 1 shows that

the error of the "parabolic" approximation of $z(l)$ considered in this section is only about 5 times greater than the error of the more precise but at the same time more complicated approximation considered before.

TABLE 2

| $z_0$ | $b$ | $\frac{1}{z_0}\lvert z - z_{ap}\rvert \cdot 100 \leqslant$ | $l_0$ |
|---|---|---|---|
| 0,025 | 0,999843 | 0,0037 | $0{,}23363667 \pm 5{,}07 \cdot 10^{-6}$ |
| 0,050 | 0,999375 | 0,015 | $0{,}3163614 \pm 6{,}43 \cdot 10^{-5}$ |
| 0,075 | 0,998594 | 0,034 | $0{,}387667 \pm 1{,}77 \cdot 10^{-4}$ |
| 0,100 | 0,997501 | 0,060 | $0{,}447963 \pm 3{,}72 \cdot 10^{-4}$ |
| 0,250 | 0,984435 | 0,383 | $0{,}71458 \pm 3{,}78 \cdot 10^{-3}$ |
| 0,500 | 0,938470 | 1,695 | $1{,}0444 \pm 2{,}31 \cdot 10^{-2}$ |

## References


[1] Л. В. Канторович, В. И. Крылов, *Приближенные методы высшего анализа*, изд. 3, Москва 1950.

[2] Keiichi Hayashi, *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*, Berlin 1930.

[3] Л. Л. Ландау, Л. М. Лифшиц, *Механика сплошных сред*, изд. 2, Москва 1954.

[4] A. Rybarski, *Über eine gewisse Linearisationsmethode der Differentialgleichungen von Pendeltypus*, Bull. Acad. Pol. Sc. 6 (1958), pp. 175-179.

[5] — *Pewna metoda linearyzacji równań różniczkowych typu równania wahadła*, Zast. Matem. 5 (1960), pp. 247-259.

[6] И. М. Рыжик и И. С. Градштейн, *Таблицы интегралов сумм, рядов и произведений*, изд. 3, Москва 1951.

[7] G. W. and R. M. Spenceley, *Smithsonian elliptic functions tables*, Smithsonian Misc. Coll. (109) 1947.


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## *LINEARYZACJA RÓWNAŃ PRĘTA SPRĘŻYSTEGO*

### STRESZCZENIE

W pracy rozważa się układ trzech nieliniowych równań różniczkowych zwyczajnych, opisujących kształt sprężystego pręta, zamocowanego sztywno na jednym końcu i obciążonego na drugim siłą prostopadłą do osi pręta nieodkształconego. Jak wiadomo, badany układ całkuje się przez kwadratury, za pomocą całek eliptycznych.

W pracy podano pewne przybliżone rozwiązanie równań, zawierające tylko funkcje elementarne i funkcje Bessla. Te przybliżone rozwiązania otrzymano przez zastosowanie metody linearyzacji, pokrewnej metodzie linearyzacji harmonicznej, znanej w teorii drgań nieliniowych. Otrzymano również oszacowanie błędu rozwiązań przybliżonych. Osobno rozpatrzono przypadek małych deformacji pręta. Podano przykłady numeryczne.

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## *ЛИНЕАРИЗАЦИЯ УРАВНЕНИЙ УПРУГОГО ПРУТА*

### РЕЗЮМЕ

В статье исследуется система трех нелинейных дифференциальных уравнений, изображающих форму упругого прута, закрепленного неподвижно одним концом и подвергнутого на другом конце действию силы перпендикулярной к оси недеформированного прута. Как известно, исследуемая система интегрируема при помощи эллиптических интегралов.

В работе приводится некоторые приближенные решения уравнений содержащее элементарные функции и функции Бесселя. Эти приближенные решения получены путем применения метода линеаризации, родственного метода гармонической линеаризации, известной из теории нелинейных колебаний. Получена также оценка погрешности приближенных решений. Отдельно рассматривается случай малых деформаций прута. Приведено числовые примеры.

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