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A NOTE ON CYCLIC SUBELEMENT THEORY – REDUCIBILITY OF LOCAL CONNECTEDNESS AND LOCAL SIMPLE CONNECTEDNESS

BY

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1. Introduction*. A number of ways have been proposed for refining G. T. Whyburn's very useful theory of cyclic elements. (For a historical survey of the situation, see [5].) One recent such refinement decomposes a locally connected T_1-space X into subsets called subelements of X. By a *subelement* of X is meant the closure of any component C of the interior of the set of all non-local cut points of X. (The set C itself is called a *presubelement*.) The theory of subelements was initiated in [6]. In [7], the authors developed the theory in several degrees of generality, characterizing subelements in various ways, developing analogues of A-sets, H-sets and simple cyclic chains, and giving some applications.

In the Whyburn theory, a topological property is called *cyclic element reducible* [*extensible*] if its possession by the space X [by each cyclic element of X] implies its possession by each cyclic element of X [by the space X].

Analogously, we may say that a property P is *subelement reducible* [*extensible*] provided that whenever X has P, so does each subelement [whenever each subelement has P, so does X].

There are numerous cyclic element reducible and extensible properties (cf. [9]). In particular, when X is a semi-locally connected continuum, the fixed point property is cyclicly reducible and extensible ([9], p. 243). Unicoherence is another such property.

Most local properties are evidently "presubelement reducible" in the sense that if X has the (local) property P, and if C is a presubelement, then C also has the property P. But, strictly subelement reducible pro-

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perties (i. e., such that each $\bar{C}$, not merely each C, has the property), seem to be rather rare.

The principal purpose of this paper [1] is to show that the property of being a plane, locally simply connected, Peano continuum is subelement reducible, but that if we drop the planarity we lose the theorem. Section 2 shows, by an example, that there is a locally simply connected Peano continuum in E^3 with a subelement that is not locally simply connected. For a locally simply connected Peano continuum *in the plane*, however, Section 3 shows that subelements are locally simply connected, and Section 4 — that subelements are locally connected. Because one might wonder whether local simple connectedness of X is needed in Section 4, we add, in Section 5, an example of a plane Peano continuum with a non-locally connected subelement. Finally, in Section 6, we append a proof of the equivalence — in the plane only — of two forms of local simple connectedness. The usual form (which we shall call the *homotopy* or *strong* form) is as follows: X *is* (*homotopically*) *locally simply connected* if and only if for each point p and each neighborhood N of p there is a subneighborhood V of p such that each map of the unit circle into V can be extended to a map of the unit disk into N. Next, the *weak* form which we have used for convenience in our proofs is that X is (*weakly*) *locally simply connected* if and only if for each point p and each neighborhood N of p there is a subneighborhood V of p such that each simple closed curve in V bounds a disk in N. (Until Section 6, local simple connectedness will always be taken in the latter sense.)

2. Local simple connectedness is not generally subelement reducible. We give an example in E^3 of a locally simply connected Peano continuum which has a subelement $\bar{C}$ which is not locally simply connected.

Let A_k $(k = 0, 1, 2, \ldots)$ be the solid torus defined, in cylindrical coordinates, by

$$\left[r - \frac{1}{2^k}\right]^2 + z^2 \leqslant \frac{1}{2^{k+2}}$$

and let $A = \bigcup_k A_k$. Let B be a closed solid right circular cone with axis along the x-axis, vertex at the origin, vertex angle 1°, and base in the plane $x = 2$. In our example, the set $C = A \cup B$ will prove to be the subelement in question. Note that it is clearly not locally simply connected at the origin in either of the two senses referred to in Section 1. To complete the example, we go through two steps. In the first step, we add enough extra points to C to restore local simple connectedness and, in the second step, we add still more points in such a way that the points added in the first step are removed when we settle down to locating the unique subelement C of the resulting space. In the first step, for k

$= 0, 1, 2, \ldots$, we let D_k be a horizontal closed disk with center at $(0, 0, 2^{-k-2})$ and radius 2^{-k}, and form $D = C \cup \big(\bigcup_k D_k\big)$. All we need do for the second step is make sure that every point of $\bigcup_k D_k$ is a limit point of the set of local cut points of our final space. To this end, we choose on each D_k a countable dense set $p_1, p_2, p_3, \ldots$ and, for $m = 1, 2, 3, \ldots$, let $s_{m,k}$ be a segment of length 2^{-k-m-3} perpendicular to D_k at P_m and extending in the positive z-direction. Now, let $E = E \cup \big(\bigcup_{m,k} S_{m,k}\big)$ and E is the required example.

3. Reducibility of local simple connectedness. The following theorem shows that, for plane Peano continua, local simple connectedness is subelement reducible.

THEOREM 1. *If X is a plane Peano continuum locally simply connected at a point p and if p belongs to a subelement $\bar{C}$ of X, then $\bar{C}$ is locally simply connected at p.*

Proof. The conclusion is immediate if $p \in C$, so we take $p \in \bar{C} - C$. Suppose that $\bar{C}$ is not locally simply connected at p. Then some neighborhood U of p has the property that in each open subneighborhood N of p there is a simple closed curve J_N in $N \cap \bar{C}$ that bounds no disk in $N \cap \bar{C}$. However, since X is locally simply connected at p, we may so choose N that every simple closed curve J in N bounds a disk D_J in U. But, clearly, D_J and C cannot meet so that $J \subseteq \bar{C} - C$. This implies that J_N is a subset of the boundary of $N \cap C$. By an accessibility argument (e.g., using Theorem 3.23, p. 83, of [9]), there exist two arcs xp and xq in $N \cap C$, except that p and q lie in J_N, such that $xp \cap xq = x$. But, then either subarc pq of J_N forms with xp and xq a simple closed curve K in $N \cap \bar{C}$. And, since $pq \subseteq \bar{C} - C$, K contains either local cut points of X or limit points of the set of local cut points of X. But, then there are points of the plane that are not in X inside K, so that K bounds no disk in U. This is a contradiction.

4. Reducibility of local connectedness. In E^2, local connectednees is subelement reducible for Peano continua.

THEOREM 2. *If X is a locally simply connected plane Peano continuum and if $\bar{C}$ is a subelement of X, then $\bar{C}$ is locally connected.*

Proof. Suppose the contrary. Then there is a circular open neighborhood R of p and a sequence $\{N_i\}$ of distinct components of $\bar{R} \cap \bar{C}$ that converges to a limit continuum N that includes p, but meets no N_i. Of course $N \subseteq \bar{C} \cap \bar{R}$. Since X is locally simply connected, there is an open neighborhood U of p contained in R and such that every simple closed curve in U bounds a disk in R. Also, since X is a Peano continuum, there is an open neighborhood V of p contained in U that is arcwise connected.

If K denotes the component of $N \cap V$ that includes p, if z is another point of K and if x belongs to some one of the sets $V \cap N_i$, then there exist arcs xp and xz in V. Letting the last point of xp on xz be denoted by w, we obtain subarcs wp and wz whose union is also an arc pwz. Since X lies in the plane, we may assume, without loss of generality, that there are infinitely many values of i such that N_i meets both wp and wz. Let k be one of these values. There is a positive number ε samll enough that the ε-neighborhood S of N misses N_k. Then the component Q of $S \cap V$ that contains K is a generalized continuum and is arcwise connected. Thus, it contains an arc pz.

Now, let wa be the subarc of wp from w to the first point of pz reachable along wp after passing through a point of N_k. Let wb be the subarc of wz from w to the first point of pz reachable along wp after passing through a point of N_k. And let ab be the subarc of pz from a to b. Then $wa \cup ab \cup wb$ forms a simple closed curve contained in U, and hence bounds a disk D in R. Using planarity again, some arc α running from w through the interior of D to an interior point of the arc ab must meet N_k. But then D meets N_k at an interior point of D, and hence by the definition of subelement $D \subseteq N_k$. By our choice of ε, this is impossible.

From Theorems 1 and 2 we at once deduce

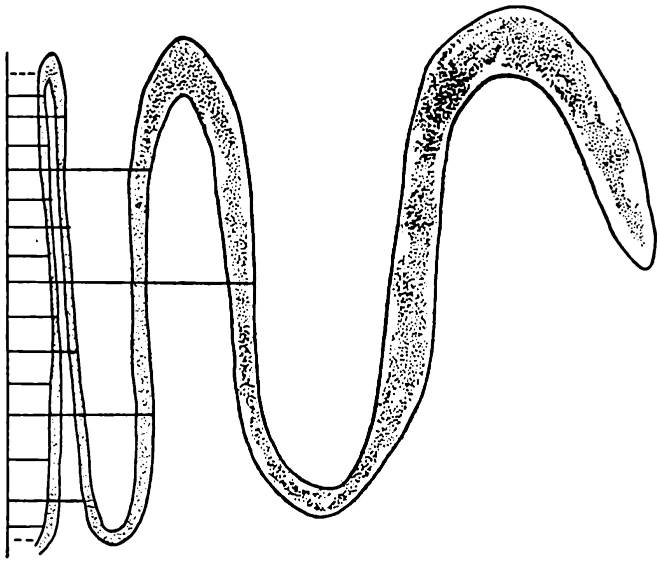
THEOREM 3. *The property of being a plane, locally simply connected, Peano continuum is subelement reducible.*

5. Local connectedness (alone) is not subelement reducible, even for plane continua. Below, we provide an example of a Peano continuum in E^2 which contains a subelement which is not locally connected.

Let A be the set of all points in the xy-plane, for which $0 < x \leqslant \pi$ and $\cos \pi/x - x/3 \leqslant y \leqslant \cos \pi/x + x/3$, and let B be the segment from $(0, 1)$ to $(0, -1)$. For $i = 1, 2, 3, \ldots$, let A_i be the portion of A for which $1/(i+1) \leqslant x \leqslant 1/i$ and let C_i be the union for $-1 \leqslant j/2^i < 1$ of the line segments parallel to the x-axis reaching from $(0, j/2^i)$ to a point of A_i. Let $C = \bigcup_i C_i$.

It is easy to see that $X = A \cup B \cup C$ is locally connected. However, the only subelement is $A \cup B$, which is not locally connected. See the figure.

6. Equivalence in the plane of the concepts of weak and strong (homotopy) local simple connectedness. It is well known that the usual homotopy form of local simple connectedness is not equivalent to the weak notion used in our proof of Theorem 1. However, for plane Peano continua, the two concepts are equivalent.



THEOREM 4. *Let P be a Peano continuum in the plane E^2 such that each simple closed curve in P bounds a disk (topological 2-cell) in P. Then each mapping of the boundary B of a disk D into P can be extended to a mapping of D into P (i. e., P is LC^1). Indeed, P is LC^n for each positive integer n.*

Proof. Since P is weakly simply connected, each cyclic element C of P is also weakly simply connected and C fails to separate E^2. Hence, P does not separate E^2. Borsuk [1], p. 132, has observed that a plane Peano continuum which does not separate the plane is an absolute retract. In particular, then, P is an absolute retract, and hence is a retract of E^2. Now, by [3], p. 26, every retract of a locally contractible space is locally contractible. Since the plane is locally contractible, so is P. By a theorem of Dugundji (quoted by Hu [3], p. 175), P is LC^n for each n.

THEOREM 5. *Suppose that P is a Peano continuum in E^2 such that P is weakly locally simply connected. Then P is simply connected in the homotopy sense (LC^1). Indeed, P is LC^n.*

Proof. By a theorem of Borsuk [1], p. 138, a subset X of E^2 is an *ANR*-set iff X is a locally connected compactum such that $E^2 - X$ has at most a finite number of components. Since P is a Peano continuum which is locally simply connected, $E^2 - P$ has at most a finite number of components. A component of $E^2 - P$ has an outer boundary which is a simple closed curve [8]. If $E^2 - P$ had infinitely many components, then these components would form a null sequence. Thus, for $\varepsilon > 0$, there would exist a simple closed curve J in P such that diameter $J < \varepsilon$ and J would bound a disk not contained in P — a contradiction. Hence, P is an *ANR*. By a theorem [3], p. 118, P is locally contractible. And, by another theorem [3], p. 168, P is LC^n.

THEOREM 6. *If P is a Peano continuum in E^2 which is LC^1, then P is (weakly) simply connected.*

A proof is clear from the definitions involved.

7. Remarks. By using a result of Johnson [4], p. 144, we can prove easily the following theorem.

THEOREM 7. *Suppose that P is a continuum which is lc^n. Furthermore, if each pre-subelement is ulc^n, then each subelement is lc^n.*

QUESTIONS. Suppose that P is a Peano continuum which is an AR. Then each cyclic element is an AR. Under what reasonable conditions is the property of being an AR (ANR) subelement reducible? (**P 772**).

Suppose that P is a locally simply connected Peano continuum. Under what conditions are subelements of P simply connected? (**P 773**). Suppose that the set L of all local separating points of P are not dense in any open set (nowhere dense) or L is not dense in any locally connected subcontinuum X of P. Is each subelement of P locally simply connected? (**P 774**).

These questions have not been pursued by us. They are, however, questions to consider for further progress in this area.

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