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Some spectral properties of operator-valued representations of function algebras

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In the present paper our interest centres about representations which are not necessarily contractive. For such representations we discuss the existence of the spectral measure and its properties. Our theorems contain in particular the well-known results of Sz.-Nagy and Foiaş concerning properties of the spectrum of unitary dilations of contractions. We also present some examples (see section 3).

1. Throughout the present paper B stands for a (complex) reflexive Banach space with the norm $|f|$ $(f \in B)$. The algebra of all bounded linear operators in B is denoted by $L(B)$. I is the identity operator in B. For $V \in L(B)$, $|V|$ is the norm of V, $V|_{B_1}$ is the restriction of V to a subspace B_1 which is invariant under V.

Let X be a compact Hausdorff space and let $C(X)$ be the Banach algebra of all complex continuous functions on X with the norm $\|u\| = \sup |u|$. A function algebra on X is by definition a subalgebra of $C(X)$ containing constants. In what follows A stands for a function algebra. Let $u|_\sigma$ be the restriction of $u \in C(X)$ to the set $\sigma \subset X$. $A|_\sigma = \{u|_\sigma : u \in A\}$. The closed set $\sigma \subset X$ is called a *peak set* of A if there exists a $u \in A$ such that $\|u\| = 1$ and, $\sigma = \{x : |u(x)| = 1\}$.

We consider in the present paper merely Borel measures on X. Let μ be a complex measure on X. We write $\mu \perp A$ if $\int_X u d\mu = 0$ for all $u \in A$. If σ is a Borel subset of X, then μ_σ is a measure defined by $\mu_\sigma(\varrho) = \mu(\varrho \cap \sigma)$ for all Borel sets ϱ. $\|\mu\|$ denotes the total variation of μ over X.

The bounded algebra homomorphism $T: A \to L(B)$ is called a *representation* of A on B if $T(1) = I$. It is clear that if T is a representation, then $\|T\| \geqslant 1$. A representation is said to be *contractive* if $\|T\| = 1$.

Let B_1 be a subspace of B. We say that B_1 is invariant under the representations T if is invariant under each $T(u)$ $(u \in A)$. The restriction $T|_{B_1}$ of a representation T to B_1 the representation is given by the formula $(T|_{B_1})(u) = T(u)|_{B_1}$ $(u \in A)$. We assume in this definition that B_1 is invariant under T.

Suppose we are given the representation T of A. It is a consequence of the Hahn–Banach theorem and the Riesz representation theorem that there are measures $\mu(\varphi, f)$ $(\varphi \in B^*, f \in B)$ such that $\|\mu(\varphi, f)\| \leqslant \|T\| |\varphi| |f|$, $\varphi(T(u)f) = \int_X u d\mu(\varphi, f)$ for $\varphi \in B^*$ and $f \in B$. We then say that $\mu(\varphi, f)$ are *elementary measures* of the representation T.

The representation T is called *σ-supported* (σ is a Borel subset of X) if it has a system of elementary measures vanishing outside of σ.

The representation T is called *absolutely continuous* with respect to a positive scalar measure m if it has a system of elementary measures absolutely continuous with respect to m. We write $T \ll m$ if T is absolutely continuous with respect to m.

We say that a mapping F defined on the totality of Borel subsets of X, with values in B is a *spectral measure* if: (i) $F(X) = I$, (ii) for every $\varphi \in B^*$ and $f \in B$ the mapping $\sigma \to \varphi(F(\sigma)f)$ is a regular scalar measure, (iii) $F(\varrho \cap \sigma) = F(\varrho)F(\sigma)$, (iv) $|F(\sigma)| \leqslant M$ for all Borel sets σ and some finite M independent of σ.

For a given spectral measure F and fixed $f \in B$ we denote by Ff the mapping $\sigma \to F(\sigma)f$. It is known that, for every $f \in B$, Ff is a vector measure (in the sense of [2]).

For $V \in L(B)$ and the bounded Borel function u we write

$$V = \int_X u dF$$

if for all $f \in B$

$$Vf = \int_X u dFf.$$

Furthermore, let $\|Ff\|$ denote the semivariation of the vector measure Ff over X. Define the semivariation $\|F\|$ of the spectral measure F as $\sup\{\|Ff\| : |f| \leqslant 1\}$. An easy calculation shows that F satisfying (ii) satisfies (iv) if and only if $\|F\| < +\infty$.

The following result will be needed for our purposes.

THEOREM S. *If T is a representation of $C(X)$ on a reflexive Banach space B, then there exists a unique spectral measure F such that*

$$T(u) = \int_X u dF, \quad u \in C(X). \tag{1}$$

Conversely, if F is a spectral measure, then T given by (1) *is a representation of $C(X)$ on B. Moreover, if T and F are related as in* (1), *then $\|T\| = \|F\|$.*

Proof. Fix $f \in B$ and set $T_f(u) = T(u)f$. Since B is reflexive (cf. [2], cor. 3, p. 482), we may apply Theorem 3 of [2], p. 498, to the operator

$T_f: u \to T_f(u)$. This theorem implies that there exists a unique vector measure μ_f with values in B such that:

$$T_f(u) = T(u)f = \int_X u d\mu_f, \quad u \in C(X), \tag{2}$$

$$\varphi \circ \mu_f \text{ is a regular measure for every } \varphi \in B^*, \tag{3}$$

$$\|T_f\| = \|\mu_f\|. \tag{4}$$

For every Borel set σ we define a mapping $F(\sigma)$ from B to B by the formula $F(\sigma)f = \mu_f(\sigma)$, $f \in B$. We want to prove that each $F(\sigma)$ belongs to $L(B)$. One verifies directly that the uniqueness of μ_f corresponding to T_f implies the linearity of the mapping $f \to \mu_f$ and, consequently, the linearity of $F(\sigma)$. Since

$$\|\mu_f(\sigma)\| \leqslant \|\mu_f\| = \|T_f\| \leqslant \|T\| |f|,$$

each $F(\sigma)$ is bounded.

We claim that F has all the properties of a spectral measure. We have

$$F(X)f = \mu_f(X) = \int_X d\mu_f = T(1)f = f.$$

This gives (i). Condition (ii) follows, by definition of F, from (3). To prove (iii), let us define $p(\sigma) = \int_\sigma ud(\varphi \circ Ff)$ for a fixed but arbitrary $u \in C(X)$. Then, for $v \in C(X)$,

$$\int_X v dp = \int_X uvd(\varphi \circ Ff) = \varphi\big(T(u)T(v)f\big) = \big(T(u)^*\varphi\big)\big(T(v)f\big)$$
$$= \int_X vd\big(\big(T(u)^*\varphi\big)\circ Ff\big).$$

Since v is arbitrary, we get

$$\int_\sigma ud(\varphi \circ Ff) = \big(T(u)^*\varphi\big)\big(F(\sigma)f\big)$$

for every σ. But

$$\big(T(u)^*\varphi\big)\big(F(\sigma)f\big) = \int_X ud\big(\varphi \circ FF(\sigma)f\big),$$

which implies that

$$\int_X \varphi_\sigma ud(\varphi \circ Ff) = \int_X ud\big(\varphi \circ FF(\sigma)f\big).$$

Since u is arbitrary,

$$\varphi\big(F(\varrho \wedge \sigma)f\big) = \int_\varrho \chi_\sigma d(\xi \circ Ff) = \varphi\big(F(\varrho)F(\sigma)f\big),$$

which completes the proof of assertion (iii). Condition (iv) follows from the relation $\|T\| = \|F\|$, by the remark preceding this theorem. Thus F has the required properties.

It is easily seen that if F is a spectral measure, then T given by (1) is linear, bounded and $T(1) = I$. We only need to show that T is multiplicative, i.e. that $T(u)T(v) = T(uv)$ for all $u, v \in A$. Observe that, by (iii), this equality holds true if u and v are simple Borel functions and, consequently, for all bounded Borel functions.

It remains to prove that $\|T\| = \|F\|$. To this end it is enough to note that

$$\|T\| = \sup_{\|u\|\leqslant 1}\{\sup_{|f|\leqslant 1}|T(u)f|\} = \sup_{|f|\leqslant 1}\|T_f\| = \sup_{|f|\leqslant 1}\|\mu_f\| = \sup_{|f|\leqslant 1}\|Ff\| = \|F\|,$$

Q.E.D.

We say that the representation T of A is an *X-representation* if it has an extension on the whole $C(X)$, which is a representation of $C(X)$ with the same norm as that of T. By Theorem S, the following is a trivial observation:

PROPOSITION 1. *The representation T of A is an X-representation if and only if there exists a spectral measure F such that* (1) *holds true and* $\|T\| = \|F\|$.

We say that the representation T is an *X-pure representation* if no restriction of T to any invariant subspace $B_1 \neq \{0\}$ of B is an X-representation.

In the case where B is a Hilbert space we have the following two definitions(1):

The representation T is a *$*$-X-representation* if it has an extention on $C(X)$ which is a $*$-representation (i.e. an involution preserving representation) of $C(X)$. The representation T is called a *$*$-X-pure representation* if no restriction of T to any invariant subspace B_1 of B is a $*$-X-representation.

The interplay between our notions and the typical Hilbert space notions is given by the following almost trivial statements.

PROPOSITION 2. *Suppose T is a representation of A on the Hilbert space B. Then:*

(α) *T is a $*$-X-representation if and only if it an X-representation and is contractive.*

(β) *If T is an X-pure representation, then it is a $*$-X-pure representation.*

(1) Those notions are known (cf. [3]) just as an X-representation and an X-pure representation (for contractive representations only).

(γ) *Suppose T is contractive. If T is a $*$-X-pure representation, then it is a X-pure representation.*

A further discussion of those notions is given in section 3.

2. We begin our main considerations with the observation that any σ-supported representation T satisfies the inequality

$$(5)\qquad |T(u)| \leqslant \|T\|\,\|u\|_\sigma,$$

where $\|u\|_\sigma = \sup\limits_\sigma |u|$.

Indeed, for every $\varphi \in B^*$ and $f \in B$ we have

$$\varphi(T(u)f) = \int\limits_X u d\mu(\varphi, f) = \int\limits_\sigma u d\mu(\varphi, f).$$

Hence

$$(6)\qquad |\varphi(T(u)f)| \leqslant \|T\|\,\|u\|_\sigma |\varphi|\,|f|.$$

It is a consequence of the Hahn-Banach theorem that for every $f \in B$ there exists a $\varphi_f \in B^*$ such that $|\varphi_f| = 1$ and $|\varphi_f(T(u)f)| = |T(u)f|$. Hence, by (6), we have the required inequality (5).

THEOREM 1. *Let T be a σ-supported representation of A. Suppose σ is a closed subset of X such that*

$$(*)\qquad A|_\sigma \text{ is dense in } C(\sigma).$$

Then T is an X-representation.

Proof. Let $u \in C(X)$ be fixed. Suppose $u_n|_\sigma \to u|_\sigma$ (uniformly on σ), $u_n \in A$. By (5), we have $|T(u_n) - T(u_m)| \leqslant \|T\|\,\|u_n - u_m\|_\sigma$. This implies that $T(u_n)$ tends (in the uniform operator topology), say to V. Moreover, by (5), we have

$$(7)\qquad |V| \leqslant \|T\|\,\|u\|.$$

Let v_n be some other sequence such that $v_n|_\sigma \to u|_\sigma$, $v_n \in A$ and $T(v_n) \to V$. Then we have $|V - T(v_n)| \leqslant |V - T(u_n)| + |T(u_n) - T(v_n)| \to 0$. Thus V do not depend on $\{u_n\}$. Set $T_0(u) = V$. We have, in particular, $T_0(u) = T(u)$ for $u \in A$. By direct calculation we verify that the mapping T_0: $u \to T_0(u)$ is an algebra homomorphism. We conclude by (7) that T_0 is a representation of $C(X)$ (with the same norm as T) which restricted to A gives T. This completes the proof.

We are now dealing with some consequences of Theorem 1. These consequences are going into two directions.

(a) Let σ be a closed subset of X. We shall assume that σ satisfies $(*)$ and

$$(8)\qquad \sigma \text{ is an intersection of peak sets of } A.$$

Such a set σ will be called an *interpolation peak set.* Without any difficulty, one may show that for every interpolation peak set σ the following condition is satisfied:

$$\mu_\sigma = 0 \quad \text{for all } \mu \perp A. \tag{9}$$

For algebras separating the points of X, Gliksberg [4], th. 4.8 showed that the converse statement holds true, that is if σ satisfies (9), then it is an interpolation peak set of A (more precisely: then $A|_\sigma = C(\sigma)$). Let us add that, as was shown in [9], condition (8) determines a decomposition of T in the direct sum $T = T_\sigma + T_{\sigma'}$, where, among other things, T is σ-supported.

Heaving dealt with these preliminaries, we are now in a position to apply Theorem 1. So we get

THEOREM 2. *Suppose σ is an interpolation peak set of A. Then every representation T of A on B is a unique direct sum $T = T_\sigma + T_{\sigma'}$, where T_σ ($T_{\sigma'}$) is σ-supported (($X\setminus\sigma$)-supported) representation of A on B_σ ($B_{\sigma'}$), $B = B_\sigma + B_{\sigma'}$ (direct sum). Moreover, T_σ is an X-representation.*

In order to illustrate Theorem 2 we consider the algebra $A(\Gamma)$ = the uniform closure of polynomials on the unit circle Γ. Let σ be a closed subset of Γ such that $m(\sigma) = 0$ (m stands for the linear Lebesgue measure on Γ). It is a consequence of the Fatou-Rudin theorem (see [5], p. 81) that σ is an interpolation peak set of A. Suppose we are given the Γ-pure representation T. Then, according to Theorem 2, the space $B_\sigma = \{0\}$ and the projection P_σ on B_σ is equal to zero. Thus for every $\varphi \in B^*$ and $f \in B$ we have

$$0 = \varphi(P_\sigma f) = \int_X \chi_\sigma d\mu(\varphi, f) = \mu(\varphi, f)(\sigma).$$

Hence, by the regularity of both $\mu(\varphi, f)$ and m we have proved in fact that $m(\sigma) = 0$ implies $\mu(\varphi, f)(\sigma) = 0$ for every Borel set σ.

So we come to the following

COROLLARY 1. *Every Γ-pure representation of the algebra $A(\Gamma)$ must necessarily be absolutely continuous with respect to the linear Lebesgue measure on Γ.*

An interesting special case of this corollary arises when B is a Hilbert space and T is a (contractive) representation of $A(\Gamma)$ determining by a given completely non-unitary contraction V (i.e. such a T that $T(u_i) = V$ for $u_1(z) = z$). Then it is known that T is contractive and Γ-pure. Hence, by Proposition 2, (γ), T is a Γ-pure representation. Corollary 1 gives us $T \ll m$. This yields that the spectral measure of a unitary dilation of V is absolutely continuous with respect to m. So we have arrived at the well-known theorem of Sz.-Nagy and Foiaş (see th. 2 of [8, IV] and also [7], p. 78).

(b) Suppose now that the algebra A satisfies (*) for every proper closed subset σ of X. Such an algebra is said to be *pervasive*. For pervasive algebras the converse version of Theorem 2 is sufficiently interesting to be stated separately.

COROLLARY 2. *If T is not an X-pure representation of a pervasive algebra A, then the representation T is not σ-supported for any proper closed subset σ of X.*

There is a theorem of Sz.-Nagy and Foiaş (cor. 2.2 of [8, III] and also [7]) which says that the spectrum of the unitary dilation of non-unitary contraction convers the whole unit circle. This theorem may be deduced from Corollary 2 in the same manner as the preceding theorem of the above mentioned authors. In addition, note that Corollary 2 generalizes a result of Khanh ([6], Prop. 1b) concerning dilatable representations.

3. Suppose T is a X-representation on the Hilbert space B. Let F be a spectral measure of T. From a theorem of Sz.-Nagy–Dixmier [1] if follows that there exists an invertible operator S such that the measure $\sigma \to S^{-1}F(\sigma)S$ is a self adjoint spectral measure. Consequently, T becomes similar to a $*$-X-representation. We may formulate the following.

PROPOSITION 3. *Every X-representation is similar to a $*$-X-representation.*

But we now show by an example that the representation which is similar to a $*$-X-representation need not be an X-representation.

EXAMPLE 1. *A contractive X-pure representation similar to a $*$-X-representation.*

Let B be a two-sided sequential Hilbert space l^2 and let $f = \{f_h\}$ and $g = \{g_u\}$ belong to l^2. Next, let U be a bilateral shift that is $Uf = g$, where $g_n = f_{n+1}$. Define S by $Sf = g$, where $g_n = \frac{1}{2}f_n$ if $n > 0$ and $g_n = f_n$ if $n \leqslant 0$. Let T_U be a contractive representation of $A(\Gamma)$ such that $T_U(u_1) = U$ $(u_1(z) = z)$. T_U is a $*$-Γ-representation of course. Set $T_S(u) = S^{-1}T_U(u)S$. We show that the representation T_S: $u \to T_S(u)$ of $A(\Gamma)$ is contractive but Γ-pure. To this end, by Proposition 2(γ), it is enough to prove that $V = T_S(u_1) = S^{-1}US$ is a completely non-unitary contraction. Write V explicitly. We have $Vf = g$, where $g_n = f_{n+1}$ for $n \neq 0$, $g_0 = \frac{1}{2}f_1$. One sees that V is a contraction. Suppose f satisfies the following condition:

$$|V^{*k}f| = |V^kf| = |f| \quad \text{for every integer } k. \tag{10}$$

An easy calculation shows that equations (10) may be explicitly written as $\ldots = |f_{-1}|^2 + |f_0|^2 = |f_0|^2 = |f_0|^2 + |f_1|^2 = \quad = 0$. Hence $f = 0$. Note that, as was proved in [7], p. 9, condition (10) characterizes contractions which are completely non-unitary.

EXAMPLE 2. *A representation which is an X-representation and *-X-pure.*

Let B be a 2-dimensional space C^2 and let A be an algebra $A(\Gamma)$. Define the representation T of $A(\Gamma)$ on C^2 as follows:

$$T(u) = \begin{pmatrix} u(1) & u(1)-u(i) \\ 0 & u(i) \end{pmatrix}.$$

The extention of T on $C(\Gamma)$ is given by the same formula as T.

It is clear that $|T(u)| = f\big(u(1), u(i)\big)$, where f is a continuous function (which may be computed explicitly). Let z_1, z_2 be the points at which f attains its maximum on the set $\{(t, s): |t| \leqslant 1, |s| \leqslant 1\}$. Denote by $\tilde{u}$ a polynomial such that $\tilde{u}(1) = z_1$, $\tilde{u}(i) = z_2$. Then the supremum of $|T(u)|$ taken over all functions u from $C(\Gamma)$ with $\|u\| = 1$ is equal to $f(z_1, z_2)$ and is attained for $\tilde{u}$. But $\tilde{u}$ belongs to $A(\Gamma)$. Thus T is a Γ-representation. A direct calculation shows that T has no reducing subspaces. This implies that T is a $*$-Γ-pure representation.

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