ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
IX, 2 (1967)

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BREAKDOWN PROCESSES OF SYSTEMS IN SERIES

1. Consider a random sequence of signals $Z_n, n = 0, \pm 1, \pm 2, \ldots$, on a line, arranged in such a way that $Z_{n+1} \geqslant Z_n$. We make the following assumptions about the sequence Z_n:

1° the differences $X_n = Z_{2n+1} - Z_{2n}$ form a sequence of random variables with the common distribution function $F(x) = P(X_n < x)$;

2° the differences $Y_n = Z_{2n} - Z_{2n-1}$ form also a sequence of random variables with the common distribution function $G(y) = P(Y_n < y)$;

3° the random variables X_n and $Y_n, n = 0, \pm 1, \pm 2, \ldots$, are independent en bloc.

In addition we assume the existence of the expected values

$$\bar{x} = \int_0^\infty x dF(x), \quad \bar{y} = \int_0^\infty y dG(y),$$

and that $F(+0) = G(+0) = 0$. For each realisation $z_n, n = 0, \pm 1, \pm 2, \ldots$, the origin of the coordinate system on the line is drawn at random with a uniform distribution on the segment (z_{-1}, z_1).

A stochastic process $a(t)$, defined as follows:

$$(1) \qquad a(t) = \begin{cases} 1 & \text{for} \quad Z_{2n} \leqslant t < Z_{2n+1}, \\ 0 & \text{for} \quad Z_{2n-1} \leqslant t < Z_{2n}, \end{cases} \qquad n = 0, \pm 1, \pm 2, \ldots,$$

is associated with the sequence of signals Z_n. Symbolically, this process will be denoted as

$$(2) \qquad a(t) \equiv \{X, Y\}.$$

An example of a realization of the process $a(t)$ is presented in Fig. 1.

It follows from the above assumptions that the process $a(t)$ is stationary with expected value

$$p = Ea(t) = \frac{\bar{x}}{\bar{x}+\bar{y}}, \tag{3}$$

and with covariance function

$$c(\tau) = Ea(t)a(t+\tau)-p^2. \tag{4}$$

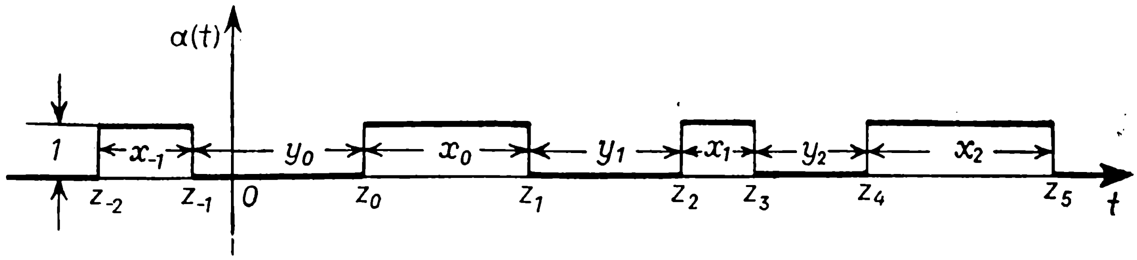

Fig. 1

2. Now consider the sequence of processes $a^{(k)}(t) = \{X^{(k)}, Y^{(k)}\}$, $k = 1, 2, \ldots$ For the process $a^{(k)}(t)$ denote: by $F^{(k)}(x)$ – the distribution function of the random variable $X^{(k)}$, by $G^{(k)}(y)$ – the distribution function of the random variable $Y^{(k)}$, by $p^{(k)}$ – the expected value of the process, and by $c^{(k)}(\tau)$ – its covariance function. The processes $a^{(k)}(t)$, $k = 1, 2, \ldots$, will be called *independent*, if for any set of moments $t_{1,1}, t_{1,2}, \ldots, t_{1,m_1}, t_{2,1}, t_{2,2}, \ldots, t_{2,m_2}, \ldots,$ the random vectors $[a^{(k)}(t_{k,1}), a^{(k)}(t_{k,2}), \ldots, a^{(k)}(t_{k,m_k})]$ are independent en bloc. For independent processes $a^{(k)}(t)$, $k = 1, 2, \ldots, n$, a new process is formed:

$$a_n(t) = \prod_{k=1}^{n} a^{(k)}(t). \tag{5}$$

It is easy to show that $a_n(t)$ is a zero-one valued stationary process with the expected value

$$p_n = Ea_n(t) = \prod_{k=1}^{n} p^{(k)}, \tag{6}$$

and the covariance function

$$c_n(\tau) = \prod_{k=1}^{n} [c^{(k)}(\tau)+(p^{(k)})^2]-p_n^2. \tag{7}$$

3. We shall give now three examples of problems leading to processes $a_n(t)$.

I. *The breakdown process.* We are given n machines working in series, e.g. a sequence of conveyors (see [1]) or a system of television relay stations. Denote by $\ldots X_n^{(k)}, Y_{n+1}^{(k)}, X_{n+1}^{(k)}, \ldots$ the sequence of alternate working and breakdown times of machine k. For the process $a^{(k)}(t)$ which is associated with this sequence of random variables the state $a^{(k)}(t) = 1$ indicates that at moment t the machine k is in working condition while the contrary, i.e. the state $a^{(k)}(t) = 0$, indicates that that machine has a breakdown. If the breakdown of one machine stops the production of the whole system, then the breakdown rate of the system may be described by $a_n(t)$. The process $a_n(t)$ is a zero-one valued process, it has thus the form

$$(8) \qquad a_n(t) = \{X_n, Y_n\}$$

with the exception that the working and breakdown times may be dependent. In section 5 we shall consider the distributions of the random variables X_n and Y_n for some special cases.

It is worth pointing out that in the above considered model the breakdown of anyone machine does not influence the breakdown probabilities for other machines. It is possible to consider other situations, e.g. a model (see [4], p. 149) where the breakdown of anyone machine stops all other machines up to the moment of restouration. From this moment on the system continues to work as if the breakdown time were of length zero.

The interpretation of example I has born the title of the paper and the terminology adopted.

II. *Passing a road crossing.* In stochastic models of road traffic one assumes in the simplest cases (see [2]) that the intervals between vehicles on the road are independent random variables with the same distribution. A vehicle passing a road crossing, occupies it for a random time. For the k-th traffic lane, $k = 1, 2, \ldots, n$, consider the stochastic process $a^{(k)}(t)$ defined as follows: $a^{(k)}(t) = 0$ if crossing is occupied in this lane, $a^{(k)}(t) = 1$ if it is free. A driver in a branch street crossing the main road is interested in knowing whether the road crossing is occupied (denote this state by 0) or free (denote this state by 1). The process of interest is zero-one valued and given by $a_n(t)$, where n is the number of traffic lanes on the main road.

III. *Rectangular impulse processes.* N. M. Sedyakin cites in [5] several papers from radiotechnics and electronics in which processes of type $a_n(t)$ are natural mathematical models of electronic impulses. Processes of type

$$(9) \qquad \gamma_n(t) = \sum_{k=1}^{n} a^{(k)}(t)$$

are considered in [4] and [5]. Those may have the values $0, 1, \ldots, n$. In the present paper while defining the process $a_n(t)$ we joint the states $0, 1, \ldots, n-1$ into one state of breakdown (0), and we call the state n the working state (1).

In parallel systems, e.g. in telecommunication when the question of system availability arises, the system is in failure if and only if all devices are in breakdown. The breakdown process of the system is then defined as

$$(10)\qquad 1-\bar{a}_n(t) = \prod_{k=1}^{n}[1-a^{(k)}(t)].$$

Parallel systems and systems in series are thus dual to each other with respect to interchangeability of the terms "in working condition" and „in breakdown".

4. Before passing to the distributions of the working and breakdown times, X_n and Y_n, in processes $a_n(t) = \{X_n, Y_n\}$ which are of interest to us, we shall find the Laplace transform of the covariance function of a zero-one valued process. We shall prove

LEMMA 1. *The Laplace transform of the covariance function of the process* $a(t) = \{X, Y\}$ *has the form*

$$(11)\qquad c^*(s) = \int_0^\infty e^{-s\tau}c(\tau)\,d\tau = \frac{pq}{s} - \frac{p}{s^2EX}\,\frac{\big(1-f^*(s)\big)\big(1-g^*(s)\big)}{1-f^*(s)g^*(s)},$$

where

$$f^*(s) = \int_0^\infty e^{-sx}dF(x) \quad \textit{and} \quad g^*(s) = \int_0^\infty e^{-sy}dG(y).$$

Proof. The covariance function of a stationary zero-one valued stochastic process with expected value p, calculated after formula (4), is equal to

$$(12)\qquad c(\tau) = pP\big(a(t+\tau) = 1 \mid a(t) = 1\big)-p^2.$$

If at moment t the process $a(t)$ is in the working state, $a(t) = 1$, then the time which will elapse up to the begining of the next breakdown is a continuous random variable X' with the distribution given by

$$P(X' < \tau) = \frac{1}{EX}\int_0^\tau \big(1-F(x)\big)\,dx.$$

We have

$$(13)\qquad P\big(a(t+\tau) = 1 \mid a(t) = 1\big) = P(X_0' > \tau) + \sum_{n=1}^{\infty} P(S_n \leqslant \tau,\ S_n + X_n > \tau)$$

$$= P_0(\tau) + \sum_{n=1}^{\infty} P_n(\tau),$$

where $S_n = X_0' + Y_1 + X_1 + Y_2 + \ldots + X_{n-1} + Y_n$. The density function of the random variable S_n will be denoted by $s_n(x)$. It is easy to show that

$$P_0(\tau) = 1 - P(X' < \tau) = 1 - \frac{1}{EX}\int_0^{\tau} \big(1 - F(x)\big)\,dx,$$

$$P_n(\tau) = \int_0^{\tau} s_n(x)\big(1 - F(\tau - x)\big)\,dx, \qquad n = 1, 2, \ldots,$$

and that the transforms of these probabilities are

$$P_0^*(s) = \frac{1}{s}\left[1 - \frac{1}{sEX}\big(1 - f^*(s)\big)\right],$$

$$P_n^*(s) = s_n^*(s)\frac{1 - f^*(s)}{s} = \frac{\big(1 - f^*(s)\big)^2}{s^2 EX}\big(f^*(s)\big)^{n-1}\big(g^*(s)\big)^n.$$

Due to (12) and (13) we obtain thus

$$c^*(s) = p\left[P_0^*(s) + \sum_{n=1}^{\infty} P_n^*(s)\right] - \frac{p^2}{s}$$

$$= \frac{p}{s}\left[1 - \frac{1}{sEX}\,\frac{\big(1 - f^*(s)\big)\big(1 - g^*(s)\big)}{1 - f^*(s)g^*(s)}\right] - \frac{p^2}{s}.$$

This ends the proof of lemma 1.

COROLLARY 1. *The covariance function $c(\tau)$ does not determine the distributions of random variables X and Y uniquely, however the covariance function $c(\tau)$ and the distribution of one the random variables X or Y do determine the distribution of second random variable.*

COROLLARY 2. *If the working time is a random variable with the exponential distribution*

$$(14)\qquad F(x) = 1 - e^{-\lambda x}, \qquad \lambda > 0,\ x > 0,$$

then

$$c^*(s) = \frac{p}{s + \lambda\big(1 - g^*(s)\big)} - \frac{p^2}{s}.$$

COROLLARY 3. *If the working time X is a random variable with the exponential distribution* (14) *and if the breakdown time Y is a random variable with the exponential distribution*

$$G(y) = 1-e^{-\mu y} \quad \mu > 0,\ y > 0, \tag{15}$$

then

$$c^*(s) = pq/(a+s), \tag{16}$$

$$c(\tau) = pqe^{-a\tau}, \tag{17}$$

where $a = \lambda+\mu$, $p = \mu/a$, $q = 1-p$.

5. We pass now to the investigation of the distributions of the working and breakdown times in the processes $a_n(t) = \prod a^{(k)}(t)$. Generally, we shall assume the mutual independence of the processes $a^{(k)}(t)$, $k = 1, 2, \ldots, n$. We shall mainly deal with the case of exponential working and breakdown time distributions. Where not otherwise stated we shall assume for simplicity that the distribution parameters are the same, i.e. that $\lambda^{(k)} = \lambda$ and $\mu^{(k)} = \mu$; it seems to us that similar results may be obtained (using rather complicated notation) without this restriction. Here are our theorems and corollaries.

THEOREM 1. *If the working times $X^{(k)}$ of the processes $a^{(k)}(t)$ have exponential distributions with parameters $\lambda^{(k)}$, then,*

(a) *the working time X_n of the process $a_n(t)$ is exponentially distributed with parameter*

$$\lambda_n = \sum_{k=1}^{n} \lambda^{(k)}, \tag{18}$$

(b) *if, in addition, the breakdown times $Y^{(k)}$ of the processes $a^{(k)}(t)$ have exponential distributions with parameters $\mu^{(k)}$, then the Laplace transform $g_n^*(s)$ of the distribution of breakdown times Y_n in the process $a_n(t)$ satisfies the equation*

$$\frac{1}{s+\lambda_n\big(1-g_n^*(s)\big)} = \sum \prod_{k=1}^{n} (q^{(k)})^{i_k}(p^{(k)})^{1-i_k} \frac{1}{i_1 a^{(1)}+\ldots+i_n a^{(n)}+s}, \tag{19}$$

where the summation is extended over 2^n components with $i_k = 0, 1$, $k = 1, 2, \ldots, n$, and where $a^{(k)} = \lambda^{(k)}+\mu^{(k)}$, $p^{(k)} = \mu^{(k)}/a^{(k)}$, $q^{(k)} = 1-p^{(k)}$.

(c) *if $\lambda^{(k)} = \lambda$ and if the breakdown times $Y^{(k)}$ have exponential distributions with parameters $\mu^{(k)} = \mu$, then the distribution of the random variable $n\lambda p^n Y_n$ tends for $n \to \infty$ to the exponential distribution with parameter* 1.

Two corollaries follow from part (b) of theorem 1.

COROLLARY 4. *If the working and breakdown times of the processes $a^{(k)}(t)$ are exponentially distributed and if $\lambda^{(k)} = \lambda$ and $\mu^{(k)} = \mu$, then*

$$\frac{1}{s+n\lambda\big(1-g_n^*(s)\big)} = \sum_{k=0}^{n} \binom{n}{k} q^k p^{n-k} \frac{1}{ka+s}, \tag{20}$$

$$EY_n = (p^{-n}-1)/n\lambda, \tag{21}$$

$$EY_n^2 = 2B_n/(n\lambda aqp^{2n}), \tag{22}$$

where

$$B_n = \sum_{k=1}^{n} \binom{n}{k} q^{k+1} p^{n-k} \frac{1}{k}.$$

From this we immediately obtain

$$\lim_{n\to\infty} E(n\lambda p^n Y_n) = 1, \tag{23}$$

$$\lim_{n\to\infty} D^2(n\lambda p^n Y_n) = 1. \tag{24}$$

COROLLARY 5. *If the processes $a^{(k)}(t)$ have exponentially distributed working and breakdown times, then the breakdown time Y_n of the process $a_n(t)$ has a distribution different from the exponential one.*

THEOREM 2. *If the processes $a^{(k)}(t) = \{X^{(k)}, Y^{(k)}\}$ are independent and have working time distributions $F^{(k)}(x)$ which satisfy the condition*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{EX^{(k)}} = \lambda, \tag{25}$$

and for some $\varepsilon > 0$

$$\overline{\lim_{k\to\infty}} \sup_{0\leqslant x\leqslant\varepsilon} f^{(k)}(x) < \infty, \tag{26}$$

where $f^{(k)}(x)$ is the density function of the distribution $F^{(k)}(x)$, then the distribution of the product nX_n, i.e. the working time $a_n(t)$ multiplied by n, tends with $n \to \infty$ to the exponential distribution.

Proof of theorem 1. If at moment t the process $a_n(t) = 1$, then the working time which elapse untill the next breakdown is equal to $X_n' = \min(X^{(1)'}, \ldots, X^{(n)'})$, where $X^{(k)'}$ are exponentially distributed random variables with parameters $\lambda^{(k)}$. It is known (see [4], p. 134) that X_n' and so the full working time X_n have also exponential distributions with the parameter $\lambda_n = \lambda^{(1)}+\lambda^{(2)}+\ldots+\lambda^{(n)}$. This proves part (a) of theorem 1.

It follows from corollary 3 that the covariance function of a process with exponentially distributed working and breakdown times is of the form

$$c^{(k)}(\tau) = p^{(k)}q^{(k)}\exp(-a^{(k)}\tau).$$

From this and equation (7) the covariance function of the process $a_n(t)$ may be found as equal to

$$(27)\quad c_n(\tau) = \prod_{k=1}^{n}[p^{(k)}q^{(k)}\exp(-a^{(k)}\tau)+(p^{(k)})^2]-\Big(\prod_{k=1}^{n}p^{(k)}\Big)^2 = p_n\sum(q^{(1)})^{i_1}\ldots$$
$$\ldots(q^{(n)})^{i_n}(p^{(1)})^{1-i_1}\ldots(p^{(n)})^{1-i_n}\exp\big(-(i_1a^{(1)}+\ldots+i_na^{(n)})\tau\big)-p_n^2,$$

where summation is extendend over all combinations $i_1, \ldots, i_n$, with $i_k = 0, 1\ (k = 1, 2, \ldots, n)$.

The working and breakdown times of the process $a_n(t)$ are independent. Since we have shown earlier that, by our assumptions, the working time X_n is exponentially distributed with parameter λ_n, we obtain from corollary 2

$$(28)\qquad c_n^*(s) = \frac{p_n}{s+\lambda_n\big(1-g_n^*(s)\big)}-\frac{p_n^2}{s},$$

where $g_n^*(s)$ is the Laplace transform of the breakdown time Y_n distribution. A comparison of the Laplace transform of function (27) with (28) leads us to formula (19). This ends the proof of part (b) of theorem 1.

If $\lambda^{(k)} = \lambda$ and $\mu^{(k)} = \mu$, then from (27) we get

$$(29)\qquad c_n^*(s) = p^n\sum_{k=0}^{n}\binom{n}{k}q^kp^{n-k}\frac{1}{ka+s}-\frac{p^{2n}}{s}.$$

Formula (28) takes the form

$$(30)\qquad c_n^*(s) = \frac{p^n}{s+n\lambda\big(1-g_n^*(s)\big)}-\frac{p^{2n}}{s}.$$

From formulas (29) and (30) follows formula (20) in corollary 4.

The Laplace transform of the random variable $n\lambda p^nY_n$ is equal to $\bar{g}_n^*(s) = g_n^*(n\lambda p^n s)$. After a substitution of np^ns in place of s in equation (30) and after a multiplication by $n\lambda p^{-n}$, the right-hand side of this equation tends to

$$(31)\qquad \frac{1}{1-\lim\limits_{n\to\infty}\bar{g}_n^*(s)}-\frac{1}{s}.$$

The right-hand side of (29) may be presented in form

$$n\lambda \sum_{k=1}^{n} \binom{n}{k} q^k p^{n-k} \frac{1}{qp^n s + ka} = n\lambda B_n + o(n\lambda p^n s), \tag{32}$$

where

$$B_n = \sum_{k=1}^{n} \binom{n}{k} q^{k+1} p^{n-k} \frac{1}{k}.$$

Now we shall prove that $\lim_{n\to\infty} nB_n = 1$.

Let $\varepsilon > 0$ and $k > 1/\varepsilon$; then $\frac{1}{k} < \frac{1+\varepsilon}{1+k}$ and thus

$$\sum_{k=1}^{n} \binom{n}{k} q^{n-k} p^{n-k} \frac{1}{k+1} < B_n$$

$$< (1+\varepsilon) \sum_{k=1}^{n} \binom{n}{k} q^{k+1} p^{n-k} \frac{1}{k+1} + \sum_{k=1}^{k_\varepsilon} \binom{n}{k} q^{k+1} p^{n-k} \frac{1}{k},$$

where $k_\varepsilon = [1/\varepsilon]+1$.

We have, however,

$$\sum_{k=1}^{n} \binom{n}{k} q^{k+1} p^{n-k} \frac{1}{k+1} = \frac{1}{n+1}(1-p^{n+1}-(n+1)qp^n),$$

and the sum

$$\sum_{k=1}^{k_\varepsilon} \binom{n}{k} q^{k+1} p^{n-k} \frac{1}{k}$$

is composed of a finite number of terms tending to zero with $n \to \infty$.

Since p^n tends to zero sufficiently fast and since nB_n tends to one, from (31) and (32) we have

$$1/\big(1-\lim_{n\to\infty} \bar{g}_n^*(s)\big) = 1+1/s$$

and

$$\lim_{n\to\infty} \bar{g}_n^*(s) = 1/(1+s).$$

The limit distribution of the random variable $n\lambda p^n Y_n$ is thus an exponential distribution with parameter one, which proves part (c) of theorem 1.

Proof of theorem 2. If at moment t the process $a^{(k)}(t) = 1$, then the time which will elapse till the next breakdown in this process is a random variable $X^{(k)'}$ with distribution function

$$P(X^{(k)'} < T) = \lambda^{(k)} \int_0^T \big(1-F(x)\big)\,dx = \lambda^{(k)}T + f^{(k)}(\theta_k T)T^2/2,$$
$$0 \leqslant \theta_k \leqslant 1, \quad \lambda^{(k)} = 1/EX^{(k)}.$$

From the independence of the processes and from the fact that $X_n' = \min(X^{(1)'}, \ldots, X^{(n)'})$ follows

$$P(nX_n' > T) = \prod_{k=1}^{n} \Big[1-\lambda^{(k)} \int_0^{T/n} \big(1-F^{(k)}(x)\big)\,dx\Big]$$
$$= \prod_{k=1}^{n} \Big(1-\lambda^{(k)}\frac{T}{n}\Big) + o\Big(\frac{1}{n}\Big).$$

From this we obtain

$$\lim_{n\to\infty} \log P(nX_n' > T) = \lambda T.$$

This ends the proof of theorem 2.

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Received on 5. 4. 1966

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PROCESY AWARII SYSTEMÓW O UKŁADZIE SZEREGOWYM

STRESZCZENIE

W pracy rozpatruje się urządzenie opisane przez stochastyczne procesy awarii o wartościach $a(t) = 1$, jeśli urządzenie znajduje się w stanie pracy i $a(t) = 0$, jeśli urządzenie znajduje się w stanie awarii. Zakłada się, że czasy pracy urządzenia są niezależnymi zmiennymi losowymi o jednakowym rozkładzie, czasy awarii urządzenia są niezależnymi zmiennymi losowymi również o jednakowym rozkładzie oraz czasy pracy i czasy awarii są niezależne.

Rozważmy obecnie system złożony z n urządzeń o procesach awarii $a^{(k)}(t)$, $k = 1, 2, \ldots, n$. Mówimy, że system ten ma *układ szeregowy*, jeśli proces awarii systemu jest określony przez iloczyn

$$a_n(t) = \prod_{k=1}^{n} a^{(k)}(t).$$

Dla systemów o niezależnych procesach awarii urządzeń składowych, w których czas pracy i czas awarii są zmiennymi losowymi o rozkładzie wykładniczym z parametrami odpowiednio równymi λ i μ, udowodniono, że czas awarii systemu Y_n ma rozkład, którego transformata Laplace'a $g_n^*(s)$ spełnia równanie

$$\frac{1}{s+n\lambda(1-g_n^*(s))} = \sum_{k=0}^{n} \binom{n}{k} q^k p^{n-k} \frac{1}{(\lambda+\mu)k+s},$$

gdzie $p = \mu/(\lambda+\mu)$, $q = 1-p$. Korzystając z tego równania udowodniono, że rozkład czasu awarii unormowanej $\tilde{Y}_n = Y_n/EY_n$ dąży przy n rosnącym nieograniczenie do rozkładu wykładniczego z parametrem 1.

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ПРОЦЕССЫ АВАРИИ ПОСЛЕДОВАТЕЛЬНО СОЕДИНЁННЫХ СИСТЕМ

РЕЗЮМЕ

В работе рассматривается устройство, описанное стохастическим процессом аварии о значениях $a(t) = 1$, если устройство работает и $a(t) = 0$, если оно находится в состоянии аварии. Предполагается, что 1° промежутки времени работы устройства являются независимыми величинами, имеющими одинаковое распределение; 2° промежутки времени аварии устройства являются независимыми случайными величинами, имеющими тоже одинаковое распределение; 3° промежутки времени работы и аварии независимы.

Рассмотрим сейчас систему из n устройств, которых процессами аварии являются $a^{(k)}(t)$, $k = 1, 2, \ldots, n$, будем говорить, что эта система *последовательно соединена*, если процесс аварии системы определяется произведением

$$a_n(t) = \prod_{k=1}^{n} a^{(k)}(t).$$

Для систем с независимыми процессами аварии для составных устройств, в которых время работы и время аварии являются случайными величинами, имеющими экспоненциальное распределение с параметрами соответственно равными λ и μ доказано, что время аварии системы Y_n имеет распределение, которого преобразование Лапласа $g_n^*(s)$ удовлетворяет уравнению

$$\frac{1}{s+n\lambda(1-g_n^*(s))} = \sum_{k=0}^{n} \binom{n}{k} g^k p^{n-k} \frac{1}{(\lambda+\mu)k+s},$$

где $p = \mu/(\lambda+\mu)$, $q = 1-p$. Пользуясь этим уравнением доказывается, что распределение периода нормированной аварии $\tilde{Y}_n = Y_n/EY_n$ стремится при неограниченно растущим n к экспоненциальному распределению с параметром 1.