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AN ALGORITHM FOR OPTIMAL PARTITIONING OF A GRAPH

Segmentation of a computer programme, i.e., its division into smaller segments, is a method applied in the cases where the size of the programme is larger than the size of the computer core store which can be used in the realization of the programme.

Segments of a programme are in the auxiliary store of the computer and during the realization they are entered into the core store. This process is connected with the problem of optimal division of a programme with known logical structure, which leads to the minimal number of transmissions of segments between different levels of storage and, therefore, to the time minimization of the processor of the computer used in the realization of the programme.

In many papers, e.g., [1], [3], [5] and [6], the model of a programme is considered as a graph G the n vertices of which have sizes and whose edges have costs. Thus the problem of effective programme segmentation is reduced to finding a partition of G into subsets of limited size so that the sum of costs on the edges joining vertices in different subsets is minimal.

The problem of optimal partitioning of a graph may exactly be solved for very small n by the use of such methods as branch and bound [4] or heuristic methods for considerably greater n (see [6] and [7]).

An algorithm of graph partition in a particular case, where only vertices having consecutive numbers can belong to each of the subsets, is presented in [5].

In this paper we present an algorithm for finding an optimal partition of an n -vertex connected graph using the hypergraph concept.

The use of the topological structure and properties of the graph G makes it possible to simplify the problem and, consequently, to obtain a simple algorithm.

Let a network $S = \langle X, R, f, c \rangle$ be given, where $G = \langle X, R \rangle$ is a finite, symmetric and connected graph without loops; the function f with positive values is defined on the set X of vertices of the graph, and

the function c with real non-negative values is defined on the set U of edges of the graph G . (Typically f is a function with integer values, but this is not necessary.)

By the *admissible level of the network* we mean a number p such that

$$\max_{x \in X} f(x) \leq p < \sum_{x \in X} f(x).$$

The number $\sum_{x \in Y} f(x)$, where $Y \subset X$, is called the *size* of the set Y .

The subset Y of vertices of the graph G for which we have

$$(1) \quad \sum_{x \in Y} f(x) \leq p$$

is called an *admissible subset of the set of vertices of the network* S . Consider p to be fixed for the rest of the paper.

Let $Y \subset X$. We denote by Γ_y the set of those $x \in X$ for which there exists a $y \in Y$ such that xRy . The set $\Gamma_y - Y$ is called a *neighbourhood* of the set Y . If for every x belonging to the neighbourhood of the set $Y \subset X$ the set $Y \cup \{x\}$ is not an admissible subset of the set X , then X is said to be *saturated*. Otherwise, the set Y is called *non-saturated*.

An ordered pair $H = \langle X, \mathcal{E} \rangle$, where \mathcal{E} is the family of all admissible subsets of the vertex set of the graph G , is called a *hypergraph* of the admissible subsets of the network S . A *matching* of the hypergraph H is a division of the set X into non-empty subsets X_1, X_2, \dots, X_m such that

$$X_i \cap X_j = \emptyset \quad \text{for } i \neq j,$$

$$\bigcup_{i=1}^m X_i = X \quad \text{and} \quad X_i \in \mathcal{E} \quad \text{for } i = 1, 2, \dots, m.$$

The edge $[x, y]$ of the graph G is called an *outer edge with respect to the matching* $V = \{X_1, X_2, \dots, X_m\}$ if the vertices x and y belong to different subsets of the matching V . The remaining edges are called *inner edges* of V . The set of all outer edges with respect to the matching V is called the *cut* of V and is denoted by U_V . The number

$$P(V) = \sum_{k \in U_V} c(k)$$

is called *cost of the matching* V .

The matching of the hypergraph H is called *optimal* if its cost is minimal.

The problem of finding an optimal partition of the graph G leads to finding the optimal matching of the hypergraph H .

Obviously, we have

LEMMA 1. *If $k = [x, y]$ is an edge of the graph G and if condition (2) is not satisfied for the set $\{x, y\}$, then $k \in U_V$ for any matching V .*

This implies the following

COROLLARY 1. *Edges k for which Lemma 1 holds can be removed from the graph G and the problem of finding the optimal matching can be solved for the corresponding partial graph.*

Remark 1. The number $\sum_{k \in U_0} c(k)$, where U_0 denotes the set of edges removed from the graph G , is obviously to be added, in view of Corollary 1, after some calculations, to the cost of the optimal matching of the hypergraph H .

LEMMA 2. *Let $G = \langle X_j, R_{X_j} \rangle$, $X_j \in V$, be a non-connected subgraph of the optimal matching $V = \{X_1, X_2, \dots, X_m\}$, and let $G_i = \langle X_j^{(i)}, R_{X_j^{(i)}} \rangle$, $i = 1, 2, \dots, k$, be its connected components. Then the matching*

$$\bar{V} = \{X_1, X_2, \dots, X_{j-1}, X_j^{(i)}, \dots, X_j^{(k)}, X_{j+1}, \dots, X_m\}$$

is also optimal.

Indeed, the matchings V and \bar{V} have identical sets of outer edges and, therefore, their costs are equal.

Lemma 2 implies immediately the following

COROLLARY 2. *The problem of finding the optimal matching of the hypergraph $H = \langle X, \mathcal{E} \rangle$ may be limited to its partial hypergraph $H' = \langle X, \mathcal{E}' \rangle$, where \mathcal{E}' is a subfamily of the family \mathcal{E} and for every $E \in \mathcal{E}'$ the graph $G = \langle E, R_E \rangle$ is a connected subgraph.*

We assume in the sequel that the hypergraph $H = \langle X, \mathcal{E} \rangle$ of admissible subsets satisfies the conditions of Corollary 2; the elements of the family \mathcal{E} are called *admissible subsets* of the set X .

Remark 2. If the graph G of the network S is non-connected, then the problem of finding the optimal matching V can be solved separately for every component of G .

Obviously, we have

LEMMA 3. *If in the network S there exists a vertex x such that the set $\{x\}$ is saturated, then the set $\{x\}$ belongs to every matching V of the hypergraph $H = \langle X, \mathcal{E} \rangle$.*

This implies immediately

COROLLARY 3. *The vertex x for which Lemma 3 holds can be removed from the graph G and the problem of finding the optimal matching may be solved for the corresponding subhypergraph H' .*

Remark 3. The number $\sum_{y \in \Gamma_x} c(x, y)$ is to be added, having performed calculations, to the cost of the optimal matching of the hypergraph H , and the set $\{x\}$ is to be joined to this matching.

An essential restriction for the family of admissible subsets is obtained in virtue of the following

THEOREM 1. *If $X_i \subset X$ is not a saturated set of the network S and if it belongs to an optimal matching V , then for every vertex x belonging to the neighbourhood of the set X_i such that the set $X_i \cup \{x\}$ satisfies condition (1) we have the inequality*

$$(2) \quad \sum_{y \in \Gamma_x \cap X_i} c(x, y) \leq \sum_{z \in \Gamma_x - X_i} c(x, z).$$

Proof. Let $X_i \subset X$ be a non-saturated set belonging to the optimal matching $V = \{X_1, X_2, \dots, X_m\}$ of the hypergraph of admissible subsets of the network S . Adjoining a vertex $x \in X_j$ to the set X_i we get the set $X' = X_i \cup \{x\}$ belonging to the family of admissible subsets. The set $X_j - \{x\} = X'_j$ also belongs to this family. So

$$V' = \{X_1, X_2, \dots, X'_i, \dots, X'_j, \dots, X_m\}$$

is also a matching:

$$P(V') = P(V) - \sum_{y \in \Gamma_x \cap X_i} c(x, y) + \sum_{z \in \Gamma_x - X_i} c(x, z).$$

If we had

$$\sum_{y \in \Gamma_x \cap X_i} c(x, y) > \sum_{z \in \Gamma_x - X_i} c(x, z),$$

then the cost $P(V')$ would be smaller than $P(V)$, in contradiction to our assumption.

Every admissible subset $E \in \mathcal{E}$ is called a *proper set* if E is a saturated set or if it satisfies condition (2).

Hence, obviously,

COROLLARY 4. *Any subset $Y \subset X$ belonging to the optimal matching V of the hypergraph of admissible subsets of the network S is a proper set.*

Let us now formalize the operation of obtaining all matchings of the hypergraph $\bar{H} = \langle \bar{X}, \bar{\mathcal{E}} \rangle$ of proper subsets $\bar{\mathcal{E}} \subset \mathcal{E}$. Let x_1 be any vertex of the partial subgraph $G = \langle \bar{X}, R_{\bar{X}} \rangle$, $|\bar{X}| = n$, obtained from G according to Corollaries 1 and 3. Denote by $E_i^{(1)}$ all proper subsets of the family which contain vertex x_1 . Let us choose an arbitrary vertex $x_2 \in \Gamma_{x_1}$ and denote by $E_j^{(2)}$ all proper subsets containing x_2 and different from $E_i^{(1)}$. Apply this method successively to all vertices belonging to $\Gamma_{x_1}, \Gamma_{x_2}, \dots$ till we exhaust all vertices of the set X . Proper subsets $E_i^{(s)}$ cannot contain those vertices x_l for which $l < s$. Consider the (formal) product of (formal) sums

$$(3) \quad (E_1^{(1)} + E_2^{(1)} + \dots + E_{k_1}^{(1)})(E_1^{(2)} + E_2^{(2)} + \dots + E_{k_2}^{(2)}) \dots \\ \dots (E_1^{(n)} + E_2^{(n)} + \dots + E_{k_n}^{(n)}).$$

Multiply the factors of (3) as polynomials; successive matchings are obtained in the form of the formal product

$$E_{i_1}^{(1)} E_{i_2}^{(2)} \dots E_{i_n}^{(n)}.$$

The sets $E_{i_j}^{(j)}$ do not belong to the matchings if

$$\bigcup_{s < j} E_{i_s}^{(s)} \cap E_{i_j}^{(j)} \neq \emptyset.$$

The remaining sets of the formal product form matchings of the hypergraph \bar{H} if their set-theoretical sum is equal to the set \bar{X} .

We calculate now the cost of the first determined matching and we store it with its matching. Then we determine the next matching, we calculate its cost, we compare it with the cost stored in the memory and we store this one of the matchings whose cost is lower.

Continuing this procedure we determine all the matchings of the hypergraph \bar{H} and find the matching whose cost is minimal.

The formalized process of obtaining the optimal matching is illustrated by the following

Example 1. In Fig. 1 a network with 11 vertices is shown. The numbers at the vertices of the network are values of the function f , and the numbers at the edges are the values of the function c . The vertices of the network are denoted by x_1, x_2, \dots, x_{11} . Let us also assume that the number p which is the admissible level of the network is equal to 9.

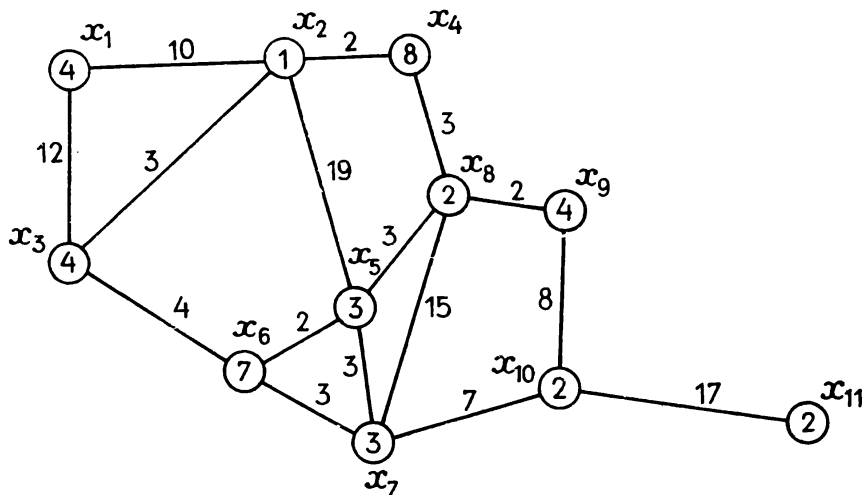


Fig. 1. Network with 11 vertices

By Corollary 3, the vertex x_6 together with edges $[x_3, x_6]$, $[x_5, x_6]$ and $[x_6, x_7]$ are to be removed from the graph G of the given network. According to Corollary 1 we also remove the edge $[x_4, x_8]$. We add the number $4 + 2 + 3 + 3 = 12$ to the cost of the optimal matching of the

hypergraph H . We obtain thus a partial subgraph \bar{G} for which we now find the family of all proper subsets.

Proper sets containing the vertex x_1 :

$$E_1^{(1)} = \{x_1, x_3\}, \quad E_2^{(1)} = \{x_1, x_2, x_3\}, \quad E_3^{(1)} = \{x_1, x_2, x_5\}.$$

Proper sets containing the vertex x_2 :

$$E_1^{(2)} = \{x_2, x_4\}, \quad E_2^{(2)} = \{x_2, x_5\}, \quad E_3^{(2)} = \{x_2, x_3, x_5\}, \\ E_4^{(2)} = \{x_2, x_5, x_7, x_8\}, \quad E_5^{(2)} = \{x_2, x_5, x_7, x_{10}\}.$$

Similarly, we find

$$E_1^{(4)} = \{x_4\}, \quad E_1^{(5)} = \{x_5, x_8, x_9\}, \quad E_1^{(7)} = \{x_7, x_8\}, \\ E_2^{(7)} = \{x_7, x_9, x_{10}\}, \quad E_3^{(7)} = \{x_7, x_8, x_9\}, \quad E_4^{(7)} = \{x_7, x_8, x_{10}, x_{11}\}, \\ E_1^{(8)} = \{x_8, x_9, x_{10}\}, \quad E_1^{(9)} = \{x_9\}, \quad E_2^{(9)} = \{x_9, x_{10}, x_{11}\}.$$

Remark 4. The sets of vertices of the network given in Fig. 1 which are admissible but not proper, since they are not saturated and do not satisfy inequality (2), are the following:

$$\{x_1\}, \{x_1, x_2\}; \quad \{x_2\}, \{x_2, x_3\}, \{x_2, x_5, x_7\}, \{x_2, x_5, x_8\}; \quad \{x_3\}; \\ \{x_5\}, \{x_5, x_7\}, \{x_5, x_8\}, \{x_5, x_7, x_8\}, \{x_5, x_7, x_{10}\}; \\ \{x_7\}, \{x_7, x_{10}\}, \{x_7, x_{10}, x_{11}\}; \quad \{x_8\}, \{x_8, x_9\}; \quad \{x_9, x_{10}\}; \\ \{x_{10}\}, \{x_{10}, x_{11}\}; \quad \{x_{11}\}.$$

We now form expression (3) and find all matchings of the hypergraph $\bar{H} = \langle \bar{X}, \bar{\mathcal{E}} \rangle$:

$$V_1 = \{\{x_1, x_3\}, \{x_2, x_5\}, \{x_4\}, \{x_7, x_8\}, \{x_9, x_{10}, x_{11}\}\}, \quad P(V_1) = 30, \\ V_2 = \{\{x_1, x_3\}, \{x_2, x_5, x_7, x_8\}, \{x_4\}, \{x_9, x_{10}, x_{11}\}\}, \quad P(V_2) = 24.$$

None of the remaining formal products, obtained after multiplying (3), forms matchings of the graph \bar{H} of the given network.

The matching V_2 with the least cost is the optimal matching of \bar{H} .

Taking into account Remark 2 we find the optimal matching V of the hypergraph H of the network given in Fig. 1:

$$V = \{\{x_1, x_3\}, \{x_2, x_5, x_7, x_8\}, \{x_4\}, \{x_8\}, \{x_9, x_{10}, x_{11}\}\}.$$

Then we calculate its cost: $P(V) = 24 + 12 = 36$.

In order to realize practically the method of determining the optimal matching presented in this paper for Odra 1300 computers, a programme was written in the PLAN language. This programme consists in the successive realization of the following three procedures:

1. Determination of the partial subgraph of the graph G according to Corollaries 1 and 3 and to Remark 2.

2. Formation of the family of proper subsets $\bar{\mathcal{E}}$ according to Corollaries 2 and 4.

3. Formation of all matchings of the hypergraph \bar{H} and determination of the optimal matching based on the previously presented formalization.

The detailed block diagram of the programme is presented in [3]. With the use of this programme the calculations have been performed. The time necessary to determine the optimal matching for the network mentioned in Example 1 was about 7 secs.

The times of determining the optimal matchings for a network with 20 vertices and 48 edges and for a network with 30 vertices and 46 edges were 115 and 150 secs., respectively, for fixed values of p (admissible level of the network).

If the graph $G = \langle X, R \rangle$ is a simple path, the method may essentially be simplified in order to avoid using a computer.

Indeed, arrange the vertices x_1, x_2, \dots, x_n of the path according to their order. We say that the subset $Y = \{x_i, x_{i+1}, \dots, x_k\}$ of the set X is a *right-hand saturated set* if the set $\{x_i, x_{i+1}, \dots, x_k, x_{k+1}\}$ is not admissible. Analogously we define a left-hand saturated set.

By Theorem 1 we have

LEMMA 4. *If the edge $[x_q, x_{q+1}]$ of the graph G belongs to the optimal matching V and the function c decreases in the segment of the path containing $[x_q, x_{q+1}]$, i.e., $c(x_{q-1}, x_q) > c(x_q, x_{q+1})$, then the set $X_1 = \{x_p, x_{p+1}, \dots, x_q\}$ is right-hand saturated. If c is an increasing function, then the set $X_{i+1} = \{x_q, x_{q+1}, \dots, x_r\}$ is left-hand saturated.*

Let us illustrate the method of obtaining the optimal matching in case of a simple path, based on this lemma.

Fig. 2 shows a network with admissible level $p = 10$:

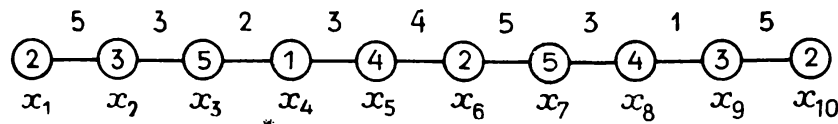


Fig. 2

Lemma 4 implies that the set $\{x_1, x_2, x_3\}$ must belong to the matching of the hypergraph H of admissible sets of the network. The vertex x_4 does not form a set belonging to the optimal matching, since the set $\{x_5, x_6\}$, which would then belong to this matching, does not satisfy Lemma 4.

If the set $\{x_4, x_5\}$ belonged to the admissible matching, then the sets $\{x_6, x_7\}$ and $\{x_8, x_9, x_{10}\}$ would belong to this matching by virtue of Lemma 4. The cost of the matching obtained is $P(V_1) = 9$. If the set $\{x_4, x_5, x_6\}$ belongs to the admissible matching, then the sets $\{x_7, x_8\}$ and $\{x_9, x_{10}\}$ must also belong to this matching and $P(V_2) = 8$. Hence

$$V_2 = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}, \{x_9, x_{10}\}\}$$

is the required optimal matching.

This example of a network, which is a simple path, leads to the following analytical generalization of the problem.

Let $f(x)$ be an arbitrary real-valued function integrable in the segment $[a, b]$ of the number axis and let $c(x)$ be an arbitrary real-valued function with non-negative values defined in this segment. Let

$$\max f(x) \leq p < \int_a^b f(x) dx,$$

where p is a number.

PROBLEM. Find such a division of the segment $[a, b]$ in subsegments $[x_i, x_{i+1}]$ that

$$\int_{x_i}^{x_{i+1}} f(x) dx \leq p \quad \text{for } i = 1, 2, \dots, m-1$$

and that the sum $\sum_{i=1}^{m-1} c(x_i)$ of values of the function c at the points of division is minimal.

A suitable formulation of Lemma 4 for the analytical example gives a simple solution in the case where the function $c(x)$ is monotonic or convex. For $c(x)$ being an arbitrary function we cannot give an analytical method of solving the problem.

The problem can obviously be generalized in a natural manner to the case of the n -dimensional cube. It seems that the analytical solution of this optimization problem is practically impossible. We think that the only reasonable way consists in its discretization (i.e., in reducing practically to the case of the network considered in this paper) in order to obtain an approximate solution.

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ALGORYTM OPTYMALNEGO ROZBICIA ZBIORU WIERZCHOŁKÓW GRAFU

STRESZCZENIE

W pracy przedstawiono metodę optymalnego podziału zbioru wierzchołków sieci $S = \langle X, R, f, c \rangle$, gdzie f i c są odpowiednio funkcjami określonymi na zbiorze wierzchołków i krawędzi grafu symetrycznego $G = \langle X, R \rangle$, na niepuste podzbiory X_1, X_2, \dots, X_m , dla których spełniony jest warunek

$$\sum_{x \in X_i} f(x) \leq p.$$

Przez *podział optymalny* rozumie się podział o minimalnej przepustowości $\sum c(k)$ na krawędziach zewnętrznych podziału.

Przedstawiony algorytm może znaleźć zastosowanie w automatycznej, efektywnej segmentacji programów dla maszyn cyfrowych.