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REMARKS ON CONGRUENCE RELATIONS AND WEAK AUTOMORPHISMS IN ABSTRACT ALGEBRAS

BY

K. URBANIK (WROCŁAW)

In this note we adopt the definitions and notations given by E. Marczewski in [2]. In particular, for a given abstract algebra $\mathfrak{A} = (A; \boldsymbol{F})$ with a carrier A and a family of fundamental operations $\boldsymbol{F}$ we denote by $\boldsymbol{A}^{(n)}$ the family of all n-ary algebraic operations. Moreover, for any positive integer p, $\mathfrak{R}_p(\mathfrak{A})$ denotes the p-reduct of $\mathfrak{A}$, i.e. the algebra $(A; \boldsymbol{A}^{(p)})$. In the present note we consider finite algebras only and by a we denote the number of elements of carrier A of $\mathfrak{A}$. An equivalence relation $\sim$ in A is said to be a *congruence relation* in $\mathfrak{A}$ if for every $n \geqslant 1$ and every $f \in \boldsymbol{A}^{(n)}$ the relations $a_i \sim b_i$ $(i = 1, 2, \ldots, n)$ imply the relation $f(a_1, a_2, \ldots, a_n) \sim f(b_1, b_2, \ldots, b_n)$. Each permutation h of the set A induces by the formula

$$f^*(x_1, x_2, \ldots, x_n) = h\big(f(h^{-1}(x_1), h^{-1}(x_2), \ldots, h^{-1}(x_n))\big)$$

a mapping from the family of all finitary operations in $\mathfrak{A}$ onto itself. Following A. Goetz and E. Marczewski (see [1]) we say that a permutation h of A is a *weak automorphism* of the algebra $\mathfrak{A}$ if the induced mapping $f \to f^*$ transforms the family of all algebraic operations in $\mathfrak{A}$ onto itself. If $f = f^*$ for every algebraic operation f, then h is an automorphism of $\mathfrak{A}$.

E. Marczewski remarked that each automorphism of $\mathfrak{R}_a(\mathfrak{A})$ is an automorphism of $\mathfrak{A}$ and asked the following two questions:

1. Is every congruence relation in $\mathfrak{R}_a(\mathfrak{A})$ a congruence relation in $\mathfrak{A}$?

2. Is every weak automorphism of $\mathfrak{R}_a(\mathfrak{A})$ a weak automorphism of $\mathfrak{A}$?

The aim of this note is to give an affirmative answer to the first question and a negative answer to the second one.

Congruence relations. We start with a simple lemma.

LEMMA. *Let $\sim$ be a congruence relation in $\mathfrak{R}_a(\mathfrak{A})$. If $f \in A^{(n)}$, $a_1, a_2, \ldots, a_n \in A$ and $a_1 \sim a_2$, then*

$$f(a_1, a_2, a_3, \ldots, a_n) \sim f(a_1, a_1, a_3, \ldots, a_n).$$

Proof. We shall prove the Lemma by induction with respect to n. For $n \leqslant a$ the Lemma is obvious. Suppose that $n > a$. Further, suppose that the Lemma is true for all $(n-1)$-tuples of elements of A and all operations from $A^{(n-1)}$. Put

$$b_1 = f(a_1, a_2, a_3, \ldots, a_n), \qquad b_2 = f(a_1, a_1, a_3, \ldots, a_n).$$

We have to prove that $b_1 \sim b_2$.

First assume that at least two elements among $a_1, a_3, \ldots, a_n$ are identical, say $a_p = a_q$, where $p \neq q$ and $p, q \in \{1, 3, \ldots, n\}$. We define an auxiliary $(n-1)$-ary algebraic operation g by the formula

$$g(x_1, x_2, \ldots, x_{q-1}, x_{q+1}, \ldots, x_n) = f(x_1, x_2, \ldots, x_{q-1}, x_p, x_{q+1}, \ldots, x_n).$$

It is clear that

$$\begin{aligned} b_1 &= g(a_1, a_2, a_3, \ldots, a_{q-1}, a_{q+1}, \ldots, a_n), \\ b_2 &= g(a_1, a_1, a_3, \ldots, a_{q-1}, a_{q+1}, \ldots, a_n). \end{aligned}$$

Hence, by the inductive assumption, we get the relation $b_1 \sim b_2$.

Now suppose that all elements $a_1, a_3, \ldots, a_n$ are different. In this case, by the inequality $n > a$, we infer that $n = a+1$. Consequently, there exists an index $r \in \{1, 3, \ldots, n\}$ such that $a_r = a_2$. If $r = 1$, then the relation $b_1 \sim b_2$ is obvious. Suppose that $r \geqslant 3$ and put

$$\begin{aligned} h_1(x_1, x_2, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n) &= f(x_1, x_2, \ldots, x_{r-1}, x_2, x_{r+1}, \ldots, x_n), \\ h_2(x_1, x_2, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n) &= f(x_1, x_1, x_3, \ldots, x_{r-1}, x_2, x_{r+1}, \ldots, x_n). \end{aligned}$$

Of course, $h_1, h_2 \in A^{(n-1)}$ and

$$\begin{aligned} b_1 &= h_1(a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n), \\ b_2 &= h_2(a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n), \end{aligned} \tag{1}$$

$$\begin{aligned} &h_1(a_1, a_1, a_3, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n) \\ &\qquad = h_2(a_1, a_1, a_3, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n). \end{aligned} \tag{2}$$

Further, by the inductive assumption, we have the relations

$$h_1(a_1, a_2, a_3, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n) \sim h_1(a_1, a_1, a_3, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n)$$
$$h_2(a_1, a_2, a_3, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n) \sim h_2(a_1, a_1, a_3, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n).$$

Hence and from (1) and (2) the relation $b_1 \sim b_2$ follows, which completes the proof of the Lemma.

THEOREM 1. *Each congruence relation in $\mathfrak{R}_a(\mathfrak{A})$ is a congruence relation in $\mathfrak{A}$.*

Proof. To prove the Theorem it suffices to prove for $n \geqslant 1$, $f \in A^{(n)}$ and $a_1, a_2, \ldots, a_n, b_1 \in A$ that the relation $a_1 \sim b_1$ implies the relation $f(a_1, a_2, \ldots, a_n) \sim f(b_1, a_2, \ldots, a_n)$. We shall prove the last statement by induction with respect to n. For $n \leqslant a$ our statement is obvious. Suppose that $n > a$ and the statement is true for all n-tuples of elements of A and all operations from $A^{(n-1)}$. Put

$$c_1 = f(a_1, a_2, \ldots, a_n), \qquad c_2 = f(b_1, a_2, \ldots, a_n). \tag{3}$$

Since $n > a$, at least two elements among $a_1, a_2, \ldots, a_n$ are identical. Suppose that $a_p = a_q$, where $p < q$ and $p, q \in \{1, 2, \ldots, n\}$. We define an auxiliary $(n-1)$-ary algebraic operation h by means of the formula

$$h(x_1, x_2, \ldots, x_{q-1}, x_{q+1}, \ldots, x_n) = f(x_1, x_2, \ldots, x_{q-1}, x_p, x_{q+1}, \ldots, x_n).$$

Obviously $c_1 = h(a_1, a_2, \ldots, a_{q-1}, a_{q+1}, \ldots, a_n)$ and, by the inductive assumption,

$$c_1 \sim h(b_1, a_2, \ldots, a_{q-1}, a_{q+1}, \ldots, a_n). \tag{4}$$

First consider the case $p = 1$. By Lemma we have the relations

$$\begin{aligned} h(b_1, a_2, \ldots, a_{q-1}, a_{q+1}, \ldots, a_n) &= f(b_1, a_2, \ldots, a_{q-1}, b_1, a_{q+1}, \ldots, a_n) \\ &\sim f(b_1, a_2, \ldots, a_{q-1}, a_1, a_{q+1}, \ldots, a_n) = f(b_1, a_2, \ldots, a_n). \end{aligned}$$

Hence and from (3) and (4) the relation $c_1 \sim c_2$ follows.

In the remaining case $p > 1$ we have the formula

$$c_2 = h(b_1, a_2, \ldots, a_{q-1}, a_{q+1}, \ldots, a_n)$$

which, by formula (4), implies the relation $c_1 \sim c_2$. Theorem 1 is thus proved.

Weak automorphisms. Since the permutation group of the carrier of a finite algebra is finite, we infer that for every finite algebra $\mathfrak{A}$ there exists an index p such that each weak automorphism of $\mathfrak{R}_p(\mathfrak{A})$ is a weak automorphism of $\mathfrak{A}$. The least index p in the last statement will be denoted by $p(\mathfrak{A})$. Obviously, the affirmative answer to the second Marczewski's question is equivalent with the inequality $p(\mathfrak{A}) \leqslant a$.

Taking into account the complete description of all two-element algebras given by E. L. Post in [3], we can prove the inequality $p(\mathfrak{A}) \leqslant 2$ for all two-element algebras $\mathfrak{A}$. But, in general, the inequality $p(\mathfrak{A}) \leqslant a$ is not true. Namely we shall prove the following theorem:

THEOREM 2. *For every integer k there exists a three-element algebra $\mathfrak{A}$ for which the inequality $p(\mathfrak{A}) > k$ holds.*

Proof. Let k be an arbitrary positive integer and $A = \{0, 1, 2\}$. We define n-ary symmetrical operations f_n and g_n on A as follows: $f_n(2, 2, \ldots, 2) = 1$, $g_n(2, 2, \ldots, 2) = 0$ and $f_n(x_1, x_2, \ldots, x_n) = 0$, $g_n(x_1, x_2, \ldots, x_n) = 1$ otherwise. Put $\mathfrak{A} = (A; f_k, g_{k+1})$. By F_n we shall denote the family of n-ary operations on A consisting of the trivial operations $e_1^{(n)}, e_2^{(n)}, \ldots, e_n^{(n)}$, the constant operations 0, 1 and the compositions $f_k(e_{i_1}^{(n)}, e_{i_2}^{(n)}, \ldots, e_{i_k}^{(n)})$, $g_{k+1}(e_{i_1}^{(n)}, e_{i_2}^{(n)}, \ldots, e_{i_{k+1}}^{(n)})$, where $i_1, i_2, \ldots, i_{k+1} \in \{1, 2, \ldots, n\}$. Since

$$f_k\big(g_{k+1}(x_1, x_2, \ldots, x_{k+1}), x_2, \ldots, x_k\big) \equiv 0$$

and

$$g_{k+1}\big(f_k(x_1, x_2, \ldots, x_k), x_2, \ldots, x_{k+1}\big) \equiv 1,$$

we infer that 0 and 1 are algebraic constants in $\mathfrak{A}$. Consequently, all operations from F_n are algebraic in the algebra $\mathfrak{A}$. Let $u_1, u_2, \ldots, u_k \in F_n$ and $v_1, v_2, \ldots, v_{k+1} \in F_n$. If at least one operation among $u_1, u_2, \ldots, u_k$ and among $v_1, v_2, \ldots, v_{k+1}$ is non-trivial, then $f_k(u_1, u_2, \ldots, u_k) \equiv 0$ and $g_{k+1}(v_1, v_2, \ldots, v_{k+1}) \equiv 1$. Thus the family F_n is closed under the composition with the fundamental operations f_k and g_{k+1}. Hence we get the equation $A^{(n)} = F_n$ $(n = 1, 2, \ldots)$. From this equation it follows that g_{k+1} is the only operation from $A^{(k+1)}$ depending on every variable and, consequently, the operation f_{k+1} is not algebraic in $\mathfrak{A}$. Further, taking into account the formula

$$g_{k+1}(x_1, x_2, \ldots, x_{k-1}, x_k, x_k) = g_k(x_1, x_2, \ldots, x_k),$$

we infer that the family $A^{(k)}$ consists of the trivial operations $e_1^{(k)}, e_2^{(k)}, \ldots, e_k^{(k)}$, the constant operations 0, 1 and the compositions $f_k(e_{j_1}^{(k)}, e_{j_2}^{(k)}, \ldots, e_{j_k}^{(k)})$, $g_k(e_{j_1}^{(k)}, e_{j_2}^{(k)}, \ldots, e_{j_k}^{(k)})$, where $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, k\}$. Consequently, $\mathfrak{R}_k(\mathfrak{A}) = (A; f_k, g_k)$.

We define a permutation h of A by the formulas $h(0) = 1$, $h(1) = 0$ and $h(2) = 2$. It is clear that $f_n^* = g_n$, $g_n^* = f_n$ $(n = 1, 2, \ldots)$, where $f \to f^*$ is the mapping induced by h. Hence it follows that this mapping transforms the family of fundamental operations of $\mathfrak{R}_k(\mathfrak{A})$ onto itself. Thus h is a weak automorphism of $\mathfrak{R}_k(\mathfrak{A})$. Further, we know that f_{k+1} is not algebraic in $\mathfrak{A}$. Hence it follows that g_{k+1}^* is not algebraic in $\mathfrak{A}$ and, consequently, h is not a weak automorphism of $\mathfrak{A}$. Thus $p(\mathfrak{A}) > k$, which completes the proof.

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INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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