ANNALES
POLONICI MATHEMATICI
XXX (1975)

On the dilution of series

by VLADIMIR DROBOT (Santa Clara)

Abstract. If Σa_n is a series of real numbers, the insertion of zeros between the terms has no effects on the ordinary convergence. It may, however, affect the summability of the series by other methods. Thus, for instance, the series $1-1+1-\ldots$ and $1-1+0+1-1-\ldots$ are $(C, 1)$-summable to $1/2$ and $1/3$ respectively. We investigate this phenomena, called the *dilution of the series*, restricting the attention to the $(C, 1)$ method. Let $\{s_n\}$ be the sequence of partial sums of Σa_n.

THEOREM. *If* $|s_n - s| > \varepsilon > 0$, $s_j > s+\varepsilon$ *for* $n_{2k-1} < j < n_{2k}$, $s_j < s-\varepsilon$ *for* $n_{2k} < j < n_{2k+1}$ *and* $\liminf n_{p+1} n_p^{-1} > 1+\lambda > 1$, *then the series cannot be diluted to become* $(C, 1)$*-summable to* s.

THEOREM. *If* $|s_n - s| > \varepsilon > 0$, $s_j > s+\varepsilon$ *for* $n_{2k-1} < j < n_{2k}$, $s_j < s-\varepsilon$ *for* $n_{2k} < j < n_{2k+1}$ *and*

$$(n_{p+1}-n_n)\max\{|s_1|^2, |s_2|^2, \ldots, |s_{n_{p+1}}|^2\} = O(1),$$

then the series can be diluted to become $(C, 1)$*-summable to* s.

Other results of similar type are also proved.

Let Σa_n be a series of real numbers. Inserting zeros between the terms has no effect on the convergence or divergence of the series, it may, however, affect the summability by other methods. Thus, for instance, the series $1-1+1-\ldots$ and $1-1+0+1-1+0+\ldots$ are $(C, 1)$ – summable to $1/2$ and $1/3$ respectively. This phenomena was first studied by Chapman ([1], p. 59), who have it the name "dilution of series". We shall investigate it here in more detail, restricting our attention to the $(C, 1)$ method. For convenience we shall be working with sequences rather than series.

DEFINITION 1. Let $\{s_n\}$ be a sequence. We say that $\{t_n\}$ is a *dilution* of $\{s_n\}$ if there exists a sequence of integers $1 = k_1 < k_2 < \ldots$ such that $t_j = s_n$ for $k_n \leqslant j < k_{n+1}$. We say that $\{t_n\}$ is a *dilution* of $\{s_n\}$ obtained by repeating s_n $(k_{n+1}-k_n)$ times. If the series Σa_n is diluted by inserting ε_n zeros after a_n, the sequence of partial sums $\{s_n\}$ gets diluted by repeating each term ε_n+1 times. We say that a sequence $\{s_n\}$ $(C, 1)$*-converges to* s if $\lim n^{-1}(s_1+\ldots+s_n) = s$.

The following theorem is fairly evident and we state it without proof.

THEOREM 1. *Let* $\{s_n\}$ *be a sequence of real numbers and let* s *be a cluster point of* $\{s_n\}$. *Then* $\{s_n\}$ *can be diluted to become* $(C, 1)$*-convergent to* s.

The idea is, that if s_{n_k} converges to s, we repeat each s_{n_k} sufficiently many times to take care of the deviations introduced into the Cesaro means by the terms s_j for $n_k < j < n_{k+1}$. Given a sequence $\{s_n\}$, it clearly cannot be diluted to become $(C, 1)$-convergent to a number s outside the interval $[\liminf s_n, \limsup s_n]$. There are, however, more subtle limitations on $(C, 1)$ convergence to a preassigned number, as the next theorem shows.

DEFINITION 2. We say that a sequence of real numbers *oscillates slowly around* s, if there exists $\varepsilon > 0$ and a sequence of integers $n_1 < n_2 < \ldots$ such that

1° $s_j \geqslant s+\varepsilon$ for $n_{2k-1} \leqslant j < n_{2k}$,

2° $s_j \leqslant s-\varepsilon$ for $n_{2k} \leqslant j < n_{2k+1}$,

3° $\liminf n_{p+1}/n_p \geqslant 1+\lambda > 1$.

THEOREM 2. *If $\{s_n\}$ is a sequence of real numbers oscillating slowly around s, then it cannot be diluted to become $(C, 1)$-convergent to s.*

(Compare this with the theorems on $(C, 1)$-convergence of slowly oscillating sequences in [1] and [2].)

Proof. Set $k_p = n_{p+1}-n_p$, then $k_p \geqslant c(1+\lambda)^p$ for some constant c. (The integers n_p are the ones given by the definition of slow oscillation around s.) Let $\{t_n\}$ be the dilution of $\{s_n\}$ which $(C, 1)$-converges to s. In the sequence $\{t_n\}$ there are ν_{2p-1} terms $\geqslant s+\varepsilon$ followed by ν_{2p} terms $\leqslant s-\varepsilon$, followed by ν_{2p+1} terms $\geqslant s+\varepsilon$, $p = 1, 2, \ldots$ Of course $\nu_p \geqslant n_{q+1}-n_q = k_q$. Put $V_q = \nu_1+\nu_2+\ldots+\nu_q$, and let σ_j be the j^{th} Cesaro mean of $\{t_n\}$, so that $\lim \sigma_j = s$. We may assume without loss of generality that $n_1 = 1$. Then we have

$$\begin{aligned}(1)\qquad \sigma_{V_{2q}} &\leqslant [V_{2q-1}\sigma_{V_{2q-1}}+\nu_{2q}(s-\varepsilon)](V_{2q-1}+\nu_{2q})^{-1}\\ &= (1+\nu_{2q}V_{2q-1}^{-1})^{-1}[\sigma_{V_{2q-1}}+\nu_{2q}V_{2q-1}^{-1}(s-\varepsilon)].\end{aligned}$$

This shows that $\lim_q \nu_{2q}V_{2q-1}^{-1} = 0$. Indeed, if $0 < \limsup \nu_{2q}V_{2q-1}^{-1} = t < \infty$, letting $q\to\infty$ in (1) along an appropriate subsequence we would obtain

$$s \leqslant (1+t)^{-1}\big(s+t(s-\varepsilon)\big) = s-t\varepsilon(1+t)^{-1} < s.$$

Similarly if $t = \infty$ in the above argument, (1) would give $s \leqslant s-\varepsilon$. From the inequality

$$(2)\qquad \sigma_{V_{2q+1}} \geqslant [V_{2q}\sigma_{V_{2q}}+\nu_{2q+1}(s+\varepsilon)](V_{2q}+\nu_{2q+1})^{-1}$$

we similarly establish that $\lim \nu_{2q+1}V_{2q}^{-1} = 0$. Hence

$$(3)\qquad \lim_q \nu_q V_{q-1}^{-1} = 0.$$

This shows that for sufficiently large q_0

$$(4)\qquad \nu_{q_0+r+1} \leqslant \tfrac{1}{4}\lambda(V_{q_0}+\nu_{q_0+1}+\ldots+\nu_{q_0+r}), \qquad r = 0, 1, 2, \ldots$$

We can now establish by induction that for all r

$$(5)\qquad \nu_{q_0+r} \leqslant \tfrac{1}{4}\lambda V_{q_0}(1+\tfrac{1}{4}\lambda)^{r-1} = d(1+\tfrac{1}{4}\lambda)^{q_0+r}, \qquad d = \text{const}.$$

This will give the desired contradiction, since $\nu_q \geqslant k_q \geqslant c(1+\lambda)^q$ as was remarked at the beginning of the proof. For $r = 1$ equation (5) reduces to (4) (with corresponding $r = 0$). Assuming that (5) is true for $r = 1, 2, \ldots, n$ we have:

$$\begin{aligned} \nu_{q_0+n+1} &\leqslant \tfrac{1}{4}\lambda(V_{q_0}+\nu_{q_0+1}+\ldots+\nu_{q_0+n}) \\ &\leqslant \tfrac{1}{4}\lambda[V_{q_0}+\tfrac{1}{4}\lambda V_{q_0}+\tfrac{1}{4}\lambda V_{q_0}(1+\tfrac{1}{4}\lambda)+\ldots+\tfrac{1}{4}\lambda V_{q_0}(1+\tfrac{1}{4}\lambda)^{n-1}] \\ &= \tfrac{1}{4}\lambda V_{q_0}\{1+\tfrac{1}{4}\lambda[1+(1+\tfrac{1}{4}\lambda)+\ldots+(1+\tfrac{1}{4}\lambda)^{n-1}]\} \\ &= \tfrac{1}{4}\lambda V_{q_0}(1+\tfrac{1}{4}\lambda)^n. \end{aligned}$$

The next theorem gives some information as to which values can be $(C,1)$-limits of appriopriately diluted sequences.

THEOREM 3. *Let $\{s_n\}$ be a sequence of real numbers. Let* $\liminf s_n < s < \limsup s_n$ *be not a cluster point of $\{s_n\}$. Let $\varepsilon > 0$ be such that $|s_n - s| \geqslant \varepsilon > 0$ for large n. Let*

$$0 \leqslant m_0 \leqslant n_0 < m_1 \leqslant n_1 < m_2 \leqslant n_2 < \ldots$$

be such that for $k = 1, 2, 3, \ldots$ we have:

A. *If $n_{2k-2} < p \leqslant n_{2k-1}$, then $s_{m_{2k-1}} \leqslant s_p \leqslant s-\varepsilon$.*

B. *If $n_{2k-1} < p \leqslant n_{2k}$, then $s_{m_{2k}} \geqslant s_p \geqslant s+\varepsilon$.*

Set $\nu_k = n_k - n_{k-1}$, $\mu_k = m_k - m_{k-1}$, $\gamma_k = \max\{|s_1|, |s_2|, \ldots, |s_k|\}$.
If

$$\nu_k \gamma_{n_k}^2 = o(n_{k-1}) \quad \text{as } k \to \infty,$$

then the sequence $\{s_n\}$ can be diluted to become $(C,1)$-convergent to s.

We precede the theorem by a fairly evident lemma which can be verified by a direct computation.

LEMMA 1. *If $\xi_1, \xi_2, \ldots, \xi_q$ are any q numbers and $\xi = \max|\xi_j|$, then for any $1 \leqslant i \leqslant q$*

$$\left|\frac{1}{q}(\xi_1+\ldots+\xi_q) - \frac{1}{q+1}(\xi_1+\ldots+\xi_q+\xi_i)\right| \leqslant \frac{2\xi}{q}.$$

Proof of Theorem 3. To simplify the subscripts we will denote the sequence $\{s_n\}$ by $\{s(n)\}$ and the numbers γ_k by $\gamma(k)$. Since the change of a finite number of terms in the sequence does not affect the $(C,1)$-convergence, we may assume that $0 = m_0 = n_0$, $m_1 = 1$ and $|s(n)-s| \geqslant \varepsilon$ for all n. We define the dilution of $\{s(n)\}$ inductively, by repeating $s(m_k)$ Δ_k times as follows.

Suppose $\Delta_1, \Delta_2, \ldots, \Delta_{2l-1}, \Delta_{2l}$ are already chosen and let the resulting sequence be $\{t(1), t(2), \ldots, t(M_{2l})\}$, where

$$(6) \qquad t(M_{2l}) = s(m_{2l}), \qquad M_{2l} = m_{2l} + (\Delta_1 - 1) + \ldots + (\Delta_{2l} - 1).$$

Write

$$A_{l+1} = \frac{t(1) + \ldots + t(M_{2l}) + s(m_{2l}+1) + \ldots + s(m_{2l+1}) + \ldots + s(m_{2l+2})}{M_{2l} + \mu_{2l+1} + \mu_{2l+2}}.$$

If $A_{l+1} = s$, we set $\Delta_{2l+1} = 1 = \Delta_{2l+2}$, so that the terms $s(m_{2l+1})$ and $s(m_{2l+2})$ do not get repeated.

If $A_{l+1} < s$, we set $\Delta_{2l+1} = 1$ and Δ_{2l+2} to be the smallest integer such that

$$\frac{t(1) + \ldots + t(M_{2l}) + s(m_{2l}+1) + \ldots + s(m_{2l+1}) + \ldots + \Delta_{2l+2} s(m_{2l+2})}{M_{2l} + \mu_{2l+1} + \mu_{2l+2} + \Delta_{2l+2} - 1} \geqslant s.$$

Such Δ_{2l+2} can be found because $s(m_{2l+2}) \geqslant s + \varepsilon$.

If $A_{l+1} > s$, we set $\Delta_{2l+2} = 1$ and Δ_{2l+1} to be the smallest integer such that

$$\frac{t(1) + \ldots + t(M_{2l}) + s(m_{2l}+1) + \ldots + \Delta_{2l+1} s(m_{2l+1}) + \ldots + s(m_{2l+2})}{M_{2l} + \mu_{2l+1} + \mu_{2l+2} + \Delta_{2l+1} - 1} \leqslant s.$$

This is again possible because $s(m_{2l+1}) \leqslant s - \varepsilon$. Notice that Δ_1 and Δ_2 are defined by this method, since the above can be carried out in the case when no t's are present. This defines the dilution of $\{s(n)\}$ into $\{t(n)\}$. If M_{2l} is defined by (6) and $\tau(j)$ denotes the j-th Ceasaro mean of $\{t(n)\}$, then s always lies between $\tau(M_{2l}-1)$ and $\tau(M_{2l})$, hence using Lemma 1 we have

$$(7) \quad |\tau(M_{2l}) - s| \leqslant |\tau(M_{2l}-1) - \tau(M_{2l})| \leqslant \frac{2\gamma(m_{2l})}{M_{2l} - 1} \leqslant \frac{2\gamma(n_{2l})}{n_{2l-1}} \to 0 \qquad (l \to \infty).$$

We desire to obtain an upper bound on Δ_{2l+1} and Δ_{2l+2}, in fact we claim that for some constant c (dependent only on ε) it is true that

$$(8) \qquad \Delta_{2l+1,2} \leqslant c\gamma(n_{2l+2})[\nu_{2l} + \nu_{2l+1} + \nu_{2l+2}].$$

If $A_{l+1} = s$, equation (8) clearly holds since in this case the left-hand side is equal to 1. If $A_{l+1} < s$, then $\Delta_{2l+1} = 1$ and Δ_{2l+2} must satisfy

$$(9) \quad \frac{t(1) + \ldots + t(M_{2l}) + s(m_{2l}+1) + \ldots + s(m_{2l+1}) + \ldots + (\Delta_{2l+2}-1) s(m_{2l+2})}{M_{2l} + \mu_{2l+1} + \mu_{2l+2} + \Delta_{2l+2} - 2} \leqslant s.$$

Since

$$s(m_{2l}+1)+\ldots+s(m_{2l+2}-1) \geqslant -(\mu_{2l+1}+\mu_{2l+2})\gamma(m_{2l+2})$$

and $s(m_{2l+2}) \geqslant s+\varepsilon$, the left-hand side of (9) is larger than

$$X = \frac{t(1)+\ldots+t(M_{2l})-(\mu_{2l+1}+\mu_{2l+2})\gamma(m_{2l+2})+(\Delta_{2l+2}-1)(s+\varepsilon)}{M_{2l}+\mu_{2l+1}+\mu_{2l+2}+\Delta_{2l+2}},$$

so $X \leqslant s$. Multiplying out by the denominator and collecting the terms we get

(10) $$\varepsilon\Delta_{2l+2} \leqslant -\{t(1)+\ldots+t(M_{2l})-sM_{2l}\}+(\mu_{2l+1}+\mu_{2l+2})[\gamma(m_{2l+2})+s]+\varepsilon-s.$$

The term in the parentheses on the right-hand side of (10) is less than $2\gamma(m_{2l})\,M_{2l}(M_{2l}-1)^{-1}$ (a consequence of (7)), and $\mu_{2l+1}+\mu_{2l+2} \leqslant \nu_{2l}+\nu_{2l+1}+\nu_{2l+2}$.

Also $\gamma(\cdot)$, is an increasing function and $m_{2l} < m_{2l+2} \leqslant n_{2l+2}$. So

(11) $$\varepsilon\Delta_{2l+2} \leqslant \gamma(m_{2l+2})[2M_{2l}(M_{2l}-1)^{-1}+\nu_{2l}+\nu_{2l+1}+\nu_{2l+2}+s]-\varepsilon-s.$$

Since $M_{2l}\to\infty$ as $l\to\infty$, it is clear that a constant c can be found such that (8) holds for those l for which $A_{l+1} < s$. If $A_{l+1} > s$, then $\Delta_{2l+2} = 1$ and Δ_{2+1} must satisfy

(12) $$\frac{t(1)+\ldots+t(M_{2l})+s(m_{2l}+1)+\ldots+(\Delta_{2l+1}-1)s(m_{2l+1})+\ldots+s(m_{2l+2})}{M_{2l}+\mu_{2l+1}+\mu_{2l+2}+\Delta_{2l+1}-2} \geqslant s.$$

As in the previous case,

$$s(m_{2l}+1)+\ldots+s(m_{2l+2}) \leqslant (\mu_{2l+1}+\mu_{2l+2})\gamma(m_{2l+2})$$

and $s(m_{2l+1}) \leqslant s-\varepsilon$. Substitution of these inequalities into (12) (and the fact that $|t(1)+\ldots+t(M_{2l})-M_{2l}\cdot s| \leqslant 2\gamma(m_{2l})\,M_{2l}(M_{2l}-1)^{-1}$, which follows from (7)), leads to

(13) $$\varepsilon\Delta_{2l+1} \leqslant 2\gamma(m_{2l})\,M_{2l}(M_{2l}-1)^{-1}+(\mu_{2l+1}+\mu_{2l+2})[\gamma(m_{2l+2})-s]+s+\varepsilon.$$

Arguing as before, we conclude from (13) that a constant satisfying (8) can be found also in this case.

Suppose now that $M_{2l} < p < M_{2l+2}$. By the hypothesis of the theorem we have for any fixed $j = 0, 1, 2,$

(14) $$\frac{\nu_{k+j}\gamma^2(n_{k+j})}{n_k} = \frac{\nu_{k+j}\gamma^2(n_{k+j})}{n_{k+j-1}}\cdot\frac{n_{k+j-1}}{n_{k+j-2}}\cdots\frac{n_{k+1}}{n_k}.$$

The first quotient on the right-hand side is $o(1)$. For each fixed r

$$\frac{n_{k+j-r}}{n_{k+j-r-1}} = 1+\frac{\nu_{k+j-r}}{n_{k+j-r-1}} = 1+o(1).$$

So it follows that the right-hand side of (14) is $o(1)$, i.e.

$$\nu_{k+j}\gamma^2(n_{k+j})n_k^{-1} = o(1). \tag{15}$$

Now

$$\tau_p = \frac{t(1)+\ldots+t(M_{2l})+s(m_{2l}+1)+\ldots+s(q_p)}{M_{2l}+(p-M_{2l})}, \quad q_p \leqslant m_{2l+2}$$

with possibly some of the $s(j)$'s repeated ($j = m_{2l}+1, \ldots, q_p$). So

$$\tau_p \leqslant \frac{t(1)+\ldots+t(M_{2l})+(\nu_{2l}+\nu_{2l+1}+\nu_{2l+2})\gamma(n_{2l+2})+\Delta_{2l+1}(s-\varepsilon)+\Delta_{2l+2}\gamma(n_{2l+2})}{M_{2l}},$$

or, using inequality (8),

$$\tau_p \leqslant \tau_{M_{2l}}+(1+c)(\nu_{2l}+\nu_{2l+1}+\nu_{2l+2})\gamma(n_{2l+2})n_{2l-1}^{-1}+ \\ +(\nu_{2l}+\ldots+\nu_{2l+2})\gamma^2(n_{2l+2})n_{2l-1}^{-1}.$$

We conclude now from (15) that $\tau_p \leqslant \tau_{M_{2l}}+o(1)$. By a completely analogous argument we establish that $\tau_p \geqslant \tau_{M_{2l}}+o(1)$ and this concludes the proof.

If $\{s_n\}$ is bounded, then a little bit more can be said about the possible $(C, 1)$-limits of its dilutions.

THEOREM 4. *Suppose $\{s_n\}$ is a bounded sequence and let s be not a cluster point of $\{s_n\}$. Let $\varepsilon > 0$ be such that $|s-s_n| \geqslant \varepsilon$ for large n. If $\{s_n\}$ can be diluted to become $(C, 1)$-convergent to s, then it can be diluted to become $(C, 1)$-convergent to any number t satisfying $s-\varepsilon < t < s+\varepsilon$.*

Proof. As in the proof of Theorem 2 we construct a sequence of integers

$$0 \leqslant m_0 \leqslant n_0 < m_1 \leqslant n_1 < m_2 \leqslant n_2 < \ldots$$

such that for $k = 1, 2, 3, \ldots$ the following is true:

A. If $n_{2k-2} < p \leqslant n_{2k-1}$, then $s(m_{2k-1}) \leqslant s_p \leqslant s-\varepsilon$.

B. If $n_{2k-1} < p \leqslant n_{2k}$, then $s(m_{2k}) \geqslant s_p \geqslant s+\varepsilon$.

Without loss of generality we may assume that $\{s(n)\}$ $(C, 1)$-converges to s to begin with, and that $m_0 = n_0 = 0$ and $|s(n)| \leqslant 1$. Let $s-\varepsilon < t < s+\varepsilon$ be given. We dilute $\{s(n)\}$ by repeating $s(m_k)$ Δ_k times according to the following rules. Suppose $\Delta_1, \Delta_2, \ldots, \Delta_{2l}$ are already chosen and $s(m_k)$ repeated Δ_k times ($k = 1, 2, \ldots, 2l$), the resulting sequence being $\{t(1), t(2), \ldots, t(M_{2l})\}$, where

$$t(M_{2l}) = s(m_{2l}), \quad M_{2l} = m_{l2}+(\Delta_1-1)+\ldots+(\Delta_{2l}-1). \tag{16}$$

Write

$$X_l = \frac{t(1)+\ldots+t(M_{2l})+s(m_{2l}+1)+\ldots+s(m_{2l+1})+\ldots+s(m_{2l+2})}{M_{2l}+m_{2l+2}-m_{2l}}.$$

If $X_l = t$, we set $\Delta_{2l+1} = \Delta_{2l+2} = 1$. If $X_l < t$, we set $\Delta_{2l+1} = 1$ and Δ_{2l+2} to be the smallest integer such that

$$\frac{t(1)+\ldots+t(M_{2l})+s(m_{2l}+1)+\ldots+s(m_{2l+1})+\ldots+\Delta_{2l+2}s(m_{2l+2})}{M_{2l}+m_{2l+2}-m_{2l}+(\Delta_{2l+2}-1)} \geqslant t.$$

Such Δ_{2l+2} exists because $s(m_{2l+2}) \geqslant s+\varepsilon > t$. If $X_l > t$, we set $\Delta_{2l+2} = 1$ and Δ_{2l+1} to be the smallest integer such that

$$\frac{t(1)+\ldots+t(M_{2l})+s(m_{2l}+1)+\ldots+\Delta_{2l+1}s(m_{2l+1})+\ldots+s(m_{2l+2})}{M_{2l}+m_{2l+2}-m_{2l}+\Delta_{2l+1}-1} \leqslant t.$$

This is again possible because $s(m_{2l+1}) \leqslant s-\varepsilon < t$. Thus the dilution of $\{s(n)\}$ is defined, the resulting sequence being $\{t(n)\}$. If $\{\tau(n)\}$ denotes the sequence of Ceasaro means of $\{t(n)\}$, then t lies between $\tau(M_{2l})$ and $\tau(M_{2l}-1)$, where M_{2l} is defined by (16). Hence it follows from Lemma 1 that

$$(17) \qquad |\tau(M_{2l})-t| \leqslant |\tau(M_{2l})-\tau(M_{2l}-1)| \leqslant \frac{2}{M_{2l}-1} \leqslant \frac{2}{m_{2l}-1} \to 0 \qquad (l\to\infty).$$

Again as in the proof of Theorem 3 we need estimates on Δ's, in fact we claim that for $i = 1, 2$ and some constant c

$$(18) \qquad \Delta_{2l+i} \leqslant c\{M_{2l}(M_{2l}-1)^{-1}+(m_{2l+2}-m_{2l})(1+t)\}.$$

If l is such that $X_l < t$, then $\Delta_{2l+1} = 1$ and Δ_{2l+2} satisfies

$$t(1)+\ldots+t(M_{2l})+s(m_{2l}+1)+\ldots+s(m_{2l+1})+\ldots+(\Delta_{2l+1}-1)s(m_{2l+2}) \leqslant t(M_{2l}+m_{2l+2}-m_{2l}+\Delta_{2l+2}-2).$$

The left-hand side of the above is larger than

$$t(1)+\ldots+t(M_{2l})-(m_{2l+2}-m_{2l})+(\Delta_{2l+2}-1)(s+\varepsilon).$$

Combining these two inequalities we get

$$\Delta_{2l+2}(s+\varepsilon-t) \leqslant |t(1)+\ldots+t(M_{2l})-tM_{2l}|+(m_{2l+2}-m_{2l})(1+t)+O(1).$$

But it follows from (17) that $|t(1)+\ldots+t(M_{2l})-tM_{2l}| \leqslant 2M_{2l}(M_{2l}-1)^{-1}$ so that (18) follows. The case $X_l > t$ is treated in an analogous way. Let now $M_{2l} < p < M_{2l+2}$. Then if $\tau(n)$ denotes the n-th mean of $\{t(j)\}$, we have:

$$\tau(p) = \frac{t(1)+\ldots+t(M_{2l})+s(m_{2l}+1)+\ldots+s(q_p)}{M_{2l}+(p-M_{2l})}, \qquad q_p \leqslant m_{2l+2},$$

with possibly some of the $s(j)$'s repeated ($j = m_{2l}+1, \ldots, q_p$). This gives

$$\tau(p) \leqslant \tau(M_{2l}) + \frac{1}{m_{2l}}\{m_{2l+2} - m_{2l} + \Delta_{2l+1} + \Delta_{2l+2}\} \tag{19}$$

and

$$\tau(p) \geqslant \tau(M_{2l})\frac{M_{2l}}{M_{2l+2}} - \frac{1}{M_{2l+2}}\{m_{2l+2} - m_{2l} + \Delta_{2l+1} + \Delta_{2l+2}\}. \tag{20}$$

Let $0 = m_0 = n_0 < m_1 \leqslant n_1 < m_2 \leqslant n_2 < \ldots$ be the sequence of integers as defined at the beginning of the proof, and let $\sigma(j)$ denote the j-th mean of $\{s(n)\}$. Then

$$\begin{aligned}\sigma(n_{2l+1}) &\leqslant [n_{2l} + (n_{2l+1} - n_{2l})]^{-1}\{n_{2l}\sigma(n_{2l}) + (n_{2l+1} - n_{2l})(s-\varepsilon)\}\\ &= \left(1 + \frac{n_{2l+1} - n_{2l}}{n_{2l}}\right)\left\{\sigma(n_{2l}) + \frac{n_{2l+1} - n_{2l}}{n_{2l}}(s-\varepsilon)\right\}.\end{aligned}$$

Since $\sigma(n_{2l}) \to s$, we conclude that $\dfrac{n_{2l+1} - n_{2l}}{n_{2l}} \to 0$.

Similarly from the relation

$$\sigma(n_{2l+2}) \geqslant \left(1 + \frac{n_{2l+2} - n_{2l+1}}{n_{2l+1}}\right)^{-1}\left\{\sigma(n_{2l+1}) + \frac{n_{2l+2} - n_{2l+1}}{n_{2l+1}}\right\}$$

we obtain $\dfrac{n_{2l+2} - n_{2l+1}}{n_{2l+1}} \to 0$, so that

$$\lim_l \frac{n_{l+1} - n_l}{n_l} = 0. \tag{21}$$

But

$$\frac{n_{l+2} - n_l}{n_l} = \frac{n_{l+2} - n_{l+1}}{n_{l+1}} \cdot \frac{n_l + (n_{l+1} - n_l)}{n_l},$$

$$\frac{n_{l+3} - n_l}{n_l} = \frac{n_{l+3} - n_{l+1}}{n_{l+1}} \cdot \frac{n_l + (n_{l+1} - n_l)}{n_l},$$

e.t.c., so for any fixed r

$$\lim_l \frac{n_{l+r} - n_l}{n_l} = 0. \tag{22}$$

But then

$$\frac{m_{2l+2} - m_{2l}}{m_{2l}} \leqslant \frac{n_{2l+2} - n_{2l-1}}{n_{2l-1}} \to 0, \tag{23}$$

so by (18)

$$\frac{\Delta_{2l+1}}{m_{2l}} \to 0, \quad \frac{\Delta_{2l+2}}{m_{2l}} \to 0 \tag{24}$$

and

$$1 \geqslant \frac{M_{2l}}{M_{2l+2}} \geqslant M_{2l}\{M_{2l}+\Delta_{2l+1}+\Delta_{2l+2}+(m_{2l+2}-m_{2l})\}^{-1}. \tag{25}$$

Since $M_{2l} \geqslant m_{2l}$, (25) (23) and (24) give

$$\lim \frac{M_{2l}}{M_{2l+2}} = 1. \tag{26}$$

But then (23), (24), (26), together with (19) and (20) give

$$\lim \tau(p) = t \qquad (p \to \infty).$$

References

[1] G. H. Hardy, *Divergent series*, Oxford 1959.

[2] A. Peyerimhoff, *Lectures on summability*, Lective Notes in Mathematics 107; Springer Verlag, New York 1969.

Reçu par la Rédaction le 18. 1. 1973