COLLOQUIUM MATHEMATICUM

VOL. XXII 1970 FASC. 1

# ON THE NUMBER OF POLYNOMIALS OF A UNIVERSAL ALGEBRA, I

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Let $\mathfrak{A}$ be a (universal) algebra [2], and for $n > 1$ let $p_n(\mathfrak{A})$ denote the number of $n$-ary polynomials over $\mathfrak{A}$ depending on all $n$ variables; let $p_1(\mathfrak{A})$ be the number of non-constant unary polynomials excluding $e_0^1$ (the projection, in other words, $x$); let $p_0(\mathfrak{A})$ be the number of constant unary polynomials (which is the same as the number of nullary polynomials if there is a nullary operation).

A sequence $\langle p_0, \ldots, p_n, \ldots\rangle$ is called *representable* if, for all $n, p_n = p_n(\mathfrak{A})$ for some algebra $\mathfrak{A}$.

In this paper some sufficient conditions for representability will be given.

An easy combinatorial argument shows that the problem of representability of sequences is equivalent to Problem 42 of [1], p. 195; thus the results of this paper are partial solutions of Problem 42.

THEOREM. *Each of conditions* (i)-(iv) *listed below is sufficient for the representability of the sequence* $\langle p_0, p_1, \ldots, p_n, \ldots\rangle$:

(i) $p_0 > 0$;

(ii) $p_0 = 0$ *and* $p_n > 0$ *for all* $n > 0$;

(iii) $p_0 = 0$, $2n$ *divides* $p_{2n}$, *and* $p_{2n-1} > 0$ *for* $n > 0$;

(iv) $p_0 = 0$, $p_1 > 0$, *and* $n$ *divides* $p_n$ *for all* $n > 0$.

Observe that an algebra representing a sequence satisfying one of (ii)-(iv) has no constants and is not idempotent ($p_0 = 0, p_1 > 0$). It is likely that (ii)-(iv) comes close to being necessary for non-idempotent algebras without constants. Algebras with constants are taken care of by (i).

Thus the remaining problems are: firstly, to try to strengthen (ii)-(iv) for non-idempotent algebras without constants (**P 691**); secondly, to consider idempotent algebras without constants (**P 692**). The paper of Urbanik [2] could prove to be useful for the latter problem.

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(1) Research supported by the National Research Council of Canada.

(2) Notions and notations of this paper are that of [1].

The proof of any of (i)-(iv) consists of two steps. First, we construct a set $A$, and a set of functions $F$ on $A$. Second, we prove that $F$ contains all projections and $F$ is closed under substitution of functions, hence $F$ is the set of all polynomials of the algebra $\langle A; F\rangle$. Then it becomes obvious that $\langle A; F\rangle$ represents the sequence it is supposed to represent.

The first steps contain the ideas behind the theorem. The second step is completely routine but sometimes long and tedious. For instance, in case (iii) there are three kinds of functions, so there are nine statements to be proved about substitutions, which break down to twelve cases. Therefore we shall present only the constructions and leave the verifications of the second steps to the reader.

Let $\mathfrak{p} = \langle p_0, \ldots, p_n, \ldots\rangle$ be a fixed sequence, and $a$ the smallest ordinal with $p_i < \bar{a}$ for all $i$.

For every $i < a$ we take a countable set $A_i$ such that $A_i$ and $A_j$ are disjoint for $i \neq j$.

Case (i). Let $|K| = p_0+1$, $k_0, k_1 \in K$, $k_0 \neq k_1$; let $K$ be disjoint to the $A_i$. Set $A = K \cup \bigcup (A_i | i < a)$. For every $k \in K$, $k \neq k_0$, we define a nullary operation $f_k$ whose value is $k$. For $n > 0$, $0 \leqslant i < p_n$ we define an $n$-ary operation $f_i^n$ on $A$ as follows:

$$f_i^n(a_0, \ldots, a_{n-1}) = \begin{cases} k_0, & \text{if } a_0, \ldots, a_{n-1} \in A_i,\ |\{a_0, \ldots, a_{n-1}\}| = n, \\ k_1, & \text{otherwise.} \end{cases}$$

Let $F$ consist of all projections (variables) $f_k$ for $k \in K$, $k \neq k_0$, and $f_i^n$ for $0 < n$, $i < p_n$. Then $\langle A; F\rangle$ represents $\mathfrak{p}$.

(Note that an $f_i^n$ substituted into an $f_j^m$ always yields $f_{k_1}$; that is why this construction does not work unless $p_0 > 0$.)

Case (ii). Let $A$ consist of the union of the $A_i$ and three more elements: $t_0, t_1, t_2$.

For every $n$, $0 < n$, we construct an $n$-ary operation $g^n$:

$$g^n(a_0, \ldots, a_{n-1}) = \begin{cases} t_0, & \text{if } a_0 = \ldots = a_{n-1} = t_0, \\ t_2, & \text{otherwise.} \end{cases}$$

For every $n$, $0 < n$, and $i$, $0 < i < p_n$, we define an $n$-ary operation $f_i^n$:

$$f_i^n(a_0, \ldots, a_{n-1}) = \begin{cases} t_0, & \text{if } a_0 = \ldots = a_{n-1} = t_0, \\ t_1, & \text{if } a_0, \ldots, a_{n-1} \in A_i \text{ and } |\{a_0, \ldots, a_{n-1}\}| = n, \\ t_2, & \text{otherwise.} \end{cases}$$

Let $F$ consist of all operations $g^n$ and $f_i^n$ and all the projections (variables). Then $\langle A; F\rangle$ represents $\mathfrak{p}$.

(Note that an $f_i^n$ substituted into an $f_j^m$ gives a $g^k$; hence we must have $p_i > 0$ for all $i > 0$ for this construction to work.)

Case (iii). Take $A$ as in (ii). For odd $n$ we define $g^n$:

$$g^n(a_0, \ldots, a_{n-1}) = \begin{cases} t_0, & \text{if the number of } a_i \text{ equal to } t_0 \text{ is odd,} \\ t_2, & \text{otherwise.} \end{cases}$$

Also, for an odd $n$ and $0 < i < p_n$ we define $f_i^n$:

$$f_i^n(a_0, \ldots, a_{n-1}) = \begin{cases} t_0, & \text{if the number of } a_i \text{ equal to } t_0 \text{ is odd,} \\ t_1, & \text{if } a_0, \ldots, a_{n-1} \in A_i,\ |\{a_0, \ldots, a_{n-1}\}| = n, \\ t_2, & \text{otherwise.} \end{cases}$$

For an even $n$, $i < n$, and $j < p_n/n$ we define $h_{i,j}^n$:

$$h_{i,j}^n(a_0, \ldots, a_{n-1}) = \begin{cases} t_0, & \text{if } a_i = t_0, \\ t_1, & \text{if } \{a_0, \ldots, a_{n-1}\} \in A_j,\ |\{a_0, \ldots, a_{n-1}\}| = n, \\ t_2, & \text{otherwise.} \end{cases}$$

Note that for $n$ odd we defined $g^n$ and $p_n - 1$ operations $f_i^n$; $p_n$ operations altogether; for even $n$ we have $n(p_n/n) = p_n$ operations $h_{i,j}^n$. Thus we take $F$ as the set of all projections, $g^n$ ($n$ odd), $f_i^n$ ($n$ odd), and $h_{i,j}^n$ ($n$ even); then $\langle A; F \rangle$ represents $\mathfrak{p}$.

Case (iv). For every $n > 1$ we take the $h_{i,j}^n$ of case (iii), $i < n$, $j < p_n/n$. For $n = 1$ we take $g^1$ (of case (iii)) and $h_{0,j}^1$ for $j < p_1 - 1$.

This concludes the proof of the Theorem.

In conclusion we remark that even though the proof allows the $p_i$ to be arbitrary cardinals, and sequences arbitrarily long (only the size of the $A_i$ has to be increased) the interesting case is when all $p_i$ are finite and the sequence is an $\omega$-sequence.

For $\omega$-sequences of integers it is reasonable to ask when is it possible to represent such sequences by finite algebras of finite type. Our constructions do not shed light on this problem. Note that any one of the four cases covers $2^{\aleph_0}$ sequences while finite algebras of finite type represent only $\aleph_0$ sequences.

## REFERENCES

[1] G. Grätzer, *Universal algebra*, The University Series in Higher Mathematics, Princeton, N. J. 1968.

[2] K. Urbanik, *On algebraic operations in idempotent algebras*, Colloquium Mathematicum 13 (1965), p. 129-157.

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*Reçu par la Rédaction le 12. 9. 1969*