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On the radius of convergence of series solutions of a functional equation

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Introduction. In the present paper we are studying the functional equation

$$\varphi(z^p) - \varepsilon\varphi(z^q) = h(z), \tag{1}$$

where $1 \leqslant p < q$ are integers, ε is a complex constant with

$$|\varepsilon| = 1, \tag{2}$$

and the function $h(z)$ is given by the power series

$$h(z) = \sum_{n=0}^{\infty} a(n) z^n. \tag{3}$$

We shall seek the solution $\varphi(z)$ of equation (1) in the form of a power series

$$\varphi(z) = \sum_{n=0}^{\infty} c(n) z^n. \tag{4}$$

Equation (1) is closely related to the equation

$$\varphi(z^q) = g(z)\varphi(z^p), \tag{5}$$

which generalizes the equation

$$\varphi(z^2) = g(z)\varphi(z) \tag{6}$$

studied by Ganapathy Iyer [1] and [2] (cf. also [3], p. 186-187). Equation (5) will be the subject of another publication.

In the case $p = 1$ the theory given in [5] (cf. also [3], Theorem 8.7) applies to equation (1). The result may be shortly summarized that every formal solution of (1) is actual, i.e. it has a positive radius of convergence. In the present paper we shall show that this is also true in the general case of arbitrary positive integers p, q. However, whereas in the case $p = 1$ the verification of the existence of a formal solution of (1) involves at most one condition, in the case $p > 1$ the existence of a formal solution depends on infinitely many conditions.

The considerations of Section 1 have a purely formal character. We establish necessary and sufficient conditions for the existence of a formal solution of equation (1). In Section 2 we evaluate the radius of convergence r of series (4) satisfying equation (1) in terms of the radius of convergence R of series (3). In Section 3 we show that in the case where $r = 1 < R$, the circle $|z| = 1$ is the natural boundary of the function $\varphi(z)$.

We may confine ourselves to the case where the integers p, q are relatively prime. In fact, if $p = dp'$, $q = dq'$, $(p', q') = 1$, then writing $\Phi(z) = \varphi(z^d)$ we reduce (1) to the equation

$$\Phi(z^{p'}) - \varepsilon\Phi(z^{q'}) = h(z),$$

in which p', q' are relatively prime.

1. Formal solution. In the present section we do not assume the convergence of the series considered; so they are regarded as formal power series. We assume that $1 \leqslant p < q$, $(p, q) = 1$.

Inserting (3) and (4) into (1) we get

$$\sum_{\mu=0}^{\infty} c(\mu) z^{p\mu} - \sum_{\nu=0}^{\infty} \varepsilon c(\nu) z^{q\nu} = \sum_{n=0}^{\infty} a(n) z^n. \tag{7}$$

We must compare the coefficients of z^n in both members of relation (7). For $n = 0$ we have $c(0)(1-\varepsilon) = a(0)$, whence

$$c(0) = \begin{cases} (1-\varepsilon)^{-1} a(0) & \text{if } \varepsilon \neq 1, \\ \text{arbitrary} & \text{if } \varepsilon = 1. \end{cases} \tag{8}$$

Moreover, we get the condition

$$a(0) = 0, \quad \text{whenever } \varepsilon = 1. \tag{9}$$

For $n > 0$ we must take into account the divisibility of n by p and q. If $p \mid n$ and $q \mid n$, say $n = pqk$, then z^n appears in both series on the left-hand side of (7) and consequently

$$c(qk) - \varepsilon c(pk) = a(pqk). \tag{10}$$

If $p \mid n$, but $q \nmid n$, say $n = pqk - ip$, $0 < i < q$, then z^n appears only in the first series on the left-hand side of (7); consequently

$$c(qk-i) = a(pqk-ip), \quad 0 < i < q. \tag{11}$$

Similarly, if $q \mid n$, but $p \nmid n$, say $n = pqk - jq$, $0 < j < p$, then

$$-\varepsilon c(pk-j) = a(pqk-jq), \quad 0 < j < p. \tag{12}$$

Finally, if $p \nmid n$ and $q \nmid n$, then z^n does not appear on the left-hand side of (7) at all. Thus we must have

$$a(n) = 0 \quad \text{whenever } p \nmid n \text{ and } q \nmid n. \tag{13}$$

In order to solve equation (7) we must solve the system of equations (10), (11), (12) ($k = 1, 2, 3, \ldots$). Let us take an $n > 0$. It may be uniquely written in the form

$$n = q^s m, \quad \text{where } q \nmid m. \tag{14}$$

For $s = 0$ we have in view of (11)

$$c(n) = a(pn).$$

Now (10) implies for arbitrary s

$$c(n) = c(q^s m) = \sum_{i=1}^{s+1} \varepsilon^{i-1} a(p^i q^{s+1-i} m). \tag{15}$$

In fact, assuming (15) true for an s, we have by (10) for $n = q^{s+1} m$

$$\begin{aligned} c(n) &= c(q^{s+1} m) = a(pq^{s+1} m) + \varepsilon c(pq^s m) \\ &= a(pq^{s+1} m) + \varepsilon \sum_{i=1}^{s+1} \varepsilon^{i-1} a(p^i q^{s+1-i} pm) \\ &= a(pq^{s+1} m) + \sum_{i=2}^{s+2} \varepsilon^{i-1} a(p^i q^{s+2-i} m) \\ &= \sum_{i=1}^{s+2} \varepsilon^{i-1} a(p^i q^{s+2-i} m). \end{aligned}$$

Thus (15) is generally valid.

The $c(n)$ given by (15) must agree with relation (12) whenever the latter applies. This is the case when m in (14) is not divisible by p. Then also $p \nmid n$ (since $(p, q) = 1$), say $n = pk - j$, $0 < j < p$, and by (12)

$$c(n) = -\varepsilon^{-1} a(qn) = -\varepsilon^{-1} a(q^{s+1} m). \tag{16}$$

Relation (16) compared with (15) yields

$$-\varepsilon^{-1} a(q^{s+1} m) = \sum_{i=1}^{s+1} \varepsilon^{i-1} a(p^i q^{s+1-i} m),$$

or, writing s instead of $s+1$,

$$\sum_{i=0}^{s} \varepsilon^{i-1} a(p^i q^{s-i} m) = 0. \tag{17}$$

Relations (9), (13) and (17) are necessary for the existence of a solution of equation (7). As we shall see, they are also sufficient. Let us note also that (13) may be incorporated into (17) for $s = 0$. Thus (17) is postulated for all positive integers m such that $p \nmid m$ and $q \nmid m$, and for $s = 0, 1, 2, \ldots$

THEOREM 1. *Let $1 \leqslant p < q$ be relatively prime integers, let $\varepsilon \neq 0$ be a complex constant, and let the coefficients $a(n)$ of the formal power series* (3) *fulfil conditions* (9) *and* (17) *for all integers m not divisible by p, q and for $s = 0, 1, 2, \ldots$ Then and only then equation* (1) *has a formal solution* (4). *This solution is unique in the case $\varepsilon \neq 1$ and contains a parameter if $\varepsilon = 1$, and the coefficients $c(n)$ are given by formulae* (8) *and* (15).

Proof. Suppose that conditions (9) and (17) are fulfilled and that the sequence $c(n)$ is given by (8) and (15). In order to prove that series (4) satisfies equation (1) (or, equivalently, equation (7)) we must check relations (10), (11) and (12). Let $k = q^s p^t m$, $s \geqslant 0$, $t \geqslant 0$, $p \nmid m$, $q \nmid m$. Then by (15)

$$c(qk) = c(q^{s+1}p^t m) = \sum_{i=1}^{s+2} \varepsilon^{i-1} a(p^{t+i}q^{s+2-i}m),$$

$$c(pk) = c(q^s p^{t+1} m) = \sum_{i=1}^{s+1} \varepsilon^{i-1} a(p^{t+1+i}q^{s+1-i}m),$$

and

$$c(qk) - \varepsilon c(pk) = \sum_{i=1}^{s+2} \varepsilon^{i-1} a(p^{t+i}q^{s+2-i}m) - \sum_{i=2}^{s+2} \varepsilon^{i-1} a(p^{t+i}q^{s+2-i}m)$$
$$= a(p^{t+1}q^{s+1}m) = a(pqk).$$

The verification of (11) is straightforward, since evidently $q \nmid qk - i$. In order to check (12) write $pk - j = q^s m$, $s \geqslant 0$, $q \nmid m$ and obviously also $p \nmid m$. Then by (15)

$$c(pk-j) = \sum_{i=1}^{s+1} \varepsilon^{i-1} a(p^i q^{s+1-i} m) = -\varepsilon^{-1} a(q^{s+1}m) = -\varepsilon^{-1} a(pqk - jq)$$

in virtue of (17).

The "only then" part as well as the uniqueness statement result from the considerations preceding the theorem.

Remark. If $p = 1$, then conditions (17) disappear (since there are no m such that $1 \nmid m$) and the existence of a formal solution of (1) depends only on condition (9). On the other hand, if $p > 1$, then the existence of a formal solution of (1) depends also on infinitely many conditions (17). Let us note that in this case every coefficient $a(n)$ occurs in exactly one of relations (17).

2. Radius of convergence. Now we assume that series (3) converges for some z so that it represents an analytic function. We assume also that the conditions of Theorem 1 are fulfilled, i.e., equation (1) has a formal solution (4). The following theorem asserts the convergence of series (4):

THEOREM 2. *Let $1 \leqslant p < q$ be relatively prime integers, let ε be a complex constant fulfilling* (2), *and let series* (3) *have a positive radius of convergence $R > 0$. If* (4) *is a formal solution of equation* (1), *then the radius of convergence r of series* (4) *fulfils the relation*

$$r = \begin{cases} R^p & \text{if } R \leqslant 1, \\ 1 \text{ or } R^q & \text{if } R > 1. \end{cases} \tag{18}$$

Proof. If the function $\varphi(z)$ is regular in the disc $|z| < r$, then $\varphi(z^p)$ is regular for $|z| < r^{1/p}$, $\varphi(z^q)$ is regular for $|z| < r^{1/q}$, and consequently $h(z)$ is regular at least for $|z| < \min(r^{1/p}, r^{1/q})$. This means that

$$R \geqslant \min(r^{1/p}, r^{1/q}). \tag{19}$$

Now we must distinguish two cases.

I. $R \leqslant 1$. For any $\gamma > 1/R \geqslant 1$ there is a constant $M > 0$ such that

$$|a(n)| \leqslant M\gamma^n, \quad n = 0, 1, 2, \ldots \tag{20}$$

Writing $n = q^s m$ we have by (15), (2) and (20)

$$|c(n)| \leqslant \sum_{i=1}^{s+1} |a(p^i q^{s+1-i} m)| \leqslant M \sum_{i=1}^{s+1} \gamma^{p^i q^{s+1-i} m} \leqslant M(s+1)\gamma^{pn} \leqslant Mn\gamma^{pn}.$$

Hence

$$r^{-1} = \limsup_{n\to\infty} \sqrt[n]{|c(n)|} \leqslant \gamma^p,$$

and with $\gamma \to 1/R$ we obtain $r \geqslant R^p$. Now, (19) implies that $r \leqslant 1$, whence $\min(r^{1/p}, r^{1/q}) = r^{1/p}$. Consequently (19) yields $R^p \geqslant r$ and (18) follows.

II. $R > 1$. Again for any γ such that $1/R < \gamma < 1$ there is a constant $M > 0$ such that (20) holds, and by (15), (2) and (20) we have for $n = q^s m$

$$|c(n)| \leqslant \sum_{i=1}^{s+1} |a(p^i q^{s+1-i} m)| \leqslant M \sum_{i=1}^{s+1} \gamma^{p^i q^{s+1-i} m} < M(s+1) < Mn.$$

Hence

$$r^{-1} = \limsup_{n\to\infty} \sqrt[n]{|c(n)|} \leqslant 1,$$

i.e., $r \geqslant 1$. On the other hand, relation (19) implies in this case $R^q \geqslant r$ so that

$$1 \leqslant r \leqslant R^q. \tag{21}$$

We shall show that in (21) only the extreme values are possible. Let us suppose that

$$1 < r < R^q. \tag{22}$$

The function $\varphi(z)$ must have a singularity at a point z_0 with $|z_0| = r$. Let us choose a point ζ_0 such that $\zeta_0^q = z_0$ and a positive number δ,

$$0 < \delta < \min\left(r^{1/q}\sin\frac{\pi}{q},\ r^{1/p}-r^{1/q},\ R-r^{1/q}\right). \tag{23}$$

Then the disc

$$D_0\colon |\zeta-\zeta_0| < \delta$$

is contained in the sector

$$|\zeta| < R, \qquad |\arg\zeta-\arg\zeta_0| < \frac{\pi}{q}, \tag{24}$$

and consequently the function $z = \zeta$ is a schlicht map of D_0 onto a neighbourhood U of z_0. Hence also ζ is an analytic function of z in U (a branch of the q-th root).

Let us observe that in view of (23) we have for $\zeta \in D_0$

$$|\zeta^p| < r. \tag{25}$$

By (24) and (25) the function

$$\psi(z) = \psi(\zeta^q) = \varepsilon^{-1}\{\varphi(\zeta^p)-h(\zeta)\}$$

is regular in U. On the other hand, we have by (1) for those $\zeta \in D_0$ for which $|\zeta^q| < r$

$$\varphi(\zeta^q) = \varepsilon^{-1}\{\varphi(\zeta^p)-h(\zeta)\}.$$

Thus $\psi(z) = \varphi(z)$ for $z \in U$, $|z| < r$, which means that $\psi(z)$ is an analytic continuation of $\varphi(z)$ over z_0. This contradicts the supposition that z_0 is a singularity of φ and thus shows that (22) is impossible. Consequently (18) follows from (21).

COROLLARY. *Under the conditions of Theorem 2 every formal solution of equation* (1) *is actual, i.e. has a positive radius of convergence.*

Remark. Theorem 2 is an improvement on a result of Ganapathy Iyer [2] (cf. also [3], p. 187), where for the equation

$$\varphi(z)-\varphi(z^2) = h(z) \tag{26}$$

only the inequality $r \geqslant \min(1, R)$ has been proved.

3. Case $r = 1 < R$. In case $R > 1$ both possibilities $r = 1$ and $r = R^q$ can actually happen. For the equation

$$\varphi(z^2) - \varphi(z^3) = \frac{z^2(1-z)}{(2-z^3)(2-z^2)}$$

we have $R = \sqrt[3]{2}$, whereas the radius of convergence of the solution

$$\varphi(z) = c + \frac{1}{2-z} = c + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

is $r = 2$. Thus $r = R^3$. On the other hand, in the case of the equation

$$\varphi(z) - \varphi(z^2) = z$$

the right-hand side is entire ($R = \infty$), whereas the solution

$$\varphi(z) = c + \sum_{n=0}^{\infty} z^{2^n} \tag{27}$$

has the radius of convergence $r = 1$. In the latter case the circle $|z| = 1$ is the natural boundary of function (27). We shall show that this is a general situation.

THEOREM 3. *Let $1 \leqslant p < q$ be relatively prime integers, let ε be a complex constant fulfilling* (2) *and let series* (3) *have a radius of convergence $R > 1$. If* (4) *is a formal solution of equation* (1) *and its radius of convergence is $r = 1$, then the circle $|z| = 1$ is the natural boundary of function* (4).

Proof. The function $\varphi(z)$ must have a singularity on the circle $|z| = 1$. We shall show that, for every integer $k \geqslant 0$, $\varphi(z)$ has p^k singularities equidistributed on the circle C: $|z| = 1$. As we have just remarked, this is certainly true for $k = 0$. Further we proceed by induction. Let us assume that z_i, $i = 1, \ldots, p^k$, $k \geqslant 0$, are singularities of $\varphi(z)$ equidistributed on C. For each i there are exactly p points ζ_{ij}, $i = 1, \ldots, p^k$; $j = 1, \ldots, p$, such that $\zeta_{ij}^p = z_i$. The points ζ_{ij} are equidistributed on C, since they are p^2-th roots of the (common) value $z_0 = z_i^p$. Each of the points ζ_{ij} is a singularity of the function $\varphi(z^p)$. Consequently each ζ_{ij} must be a singularity of $\varphi(z^q)$, since $h(z)$ has no singularities on C. Hence each ζ_{ij}^q is a singularity of $\varphi(z)$. Since $(p, q) = 1$, the points ζ_{ij}^q are all distinct and, being p^2-th roots of z_0^q, are equidistributed on C. Thus φ has p^{k+1} singularities equidistributed on C.

This implies that φ has a dense set of singularities on C. Thus C is a natural boundary of φ, which was to be proved.

Analogous results have been obtained by Matkowski [4] for the equation

$$\varphi(z^q) - g(z)\varphi(z) = h(z),$$

and by Ganapathy Iyer [2] for equation (26) with the polynomial right-hand side.

References

[1] V. Ganapathy Iyer, *On a functional equation II*, Indian J. Math. 2 (1960), p. 1-7.

[2] — *On a functional equation III*, J. Indian Math. Soc. (N. S.) 26 (1962), p. 69-75.

[3] M. Kuczma, *Functional equations in a single variable*, Monografie Mat. 46, Warszawa 1968.

[4] J. Matkowski, *On meromorphic solutions of a linear functional equation* (to appear).

[5] W. Smajdor, *On the existence and uniqueness of analytic solutions of the functional equation* $\varphi(z) = h(z, \varphi[f(z)])$, Ann. Polon. Math. 19 (1967), p. 37-45.

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