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On a version of Littlewood–Paley function

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Abstract. If f is a locally integrable function on $\boldsymbol{R}^n$, then the Friedrichs extension of f to $\boldsymbol{R}^{n+1}$ is in the Sobolev space $H^{1/2}$ if and only if f is in L^2. Several analogues of this result are considered in the L^p-case, $1 < p < \infty$, as well as some partial results in the case when $p \leqslant 1$.

1. Introduction and notations. Various analogues of the classical function g of Littlewood–Paley were considered by several authors, see for instance [2]–[4]. The objective of this note is to introduce yet another analogue and discuss some of its properties. The form of the function we consider does not seem to fit into schemes considered by other authors and is motivated by the compatibility conditions in the theory of Bessel Potentials [1] in the exceptional case.

Throughout this paper we shall use the following notations. $\boldsymbol{R}_+^{n+1}$ is the upper half-space in $\boldsymbol{R}^{n+1}$; $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}_+ = \{(x, y);\ x \in \boldsymbol{R}^n,\ y \in (0, \infty)\}$. For a function u defined on $\boldsymbol{R}_+^{n+1}$, $t \in \boldsymbol{R}_+^{n+1}$, $\Delta_t u(x, y)$ is the forward difference of u at (x, y) with increment t; $\Delta_t u(x, y) = u(x+t', y+\tau) - u(x, y)$; $t = (t', \tau)$.

Let $\varphi \in L^1(\boldsymbol{R}^n)$ and for $y > 0$ denote $\varphi_y = y^{-n}\varphi(y^{-1}x)$. Denote by T_m the $m+1$ dimensional cone in $\boldsymbol{R}_+^{n+1}$: $T_m = \{(t', \tau);\ t' \in \boldsymbol{R}^m,\ |t'| \leqslant \varkappa\tau\}$, where $\boldsymbol{R}^m$ is identified in some way (not necessarily canonical) with a subspace of $\boldsymbol{R}^n$ and $\varkappa > 0$. In the case when $\varkappa = \infty$, $T_m = \boldsymbol{R}^m \times \boldsymbol{R}_+$. On $\boldsymbol{R}_+ \times T_m$ and T_m we consider the measures $d\mu_m(y, t) = |t|^{-m-2}\,dy\,dt$, $d\mu'_m(y, t') = |t'|^{-m-1}\,dy\,dt'$ which are clearly invariant under homotheties.

For any function or distribution f on $\boldsymbol{R}^n$ for which $\varphi_y * f(x)$ makes sense, at least a.e. on $\boldsymbol{R}_+^{n+1}$ we define

$$h_1(f; x) = \Big(\int_{T_m}\int_0^\infty |\Delta_t \varphi_y * f(x)|^2\, d\mu_m(y, t)\Big)^{1/2}, \tag{1.1}$$

$$h_2(f; x) = \Big(\int_{T_m} |\Delta_{t'} \varphi_y * f(x)|^2\, d\mu'_m(y, t')\Big)^{1/2}. \tag{1.1'}$$

The objective of the paper is to establish under some regularity conditions on φ the inequality

$$(1.2)\qquad \|h_j\|_p \leqslant C_p\|f\|_p, \quad 1 < p < \infty,\ j = 1, 2,$$

where $\| \ \|_p$ denotes the L^p norm. We are also interested in the opposite inequality, valid under suitable conditions on φ,

$$(1.3)\qquad \|h_j\|_p \geqslant c_p\|f\|_p.$$

It was shown in [1] that every measurable function on R^n is a restriction of a function in $P^{1/2}(R^{n+1}_+)$ and that for $f \in L^2$, φ Lipschitzian and satisfying $\int \varphi(x)\,dx = 1$, the function

$$(1.4)\qquad u(x, y) = \varphi_y * f(x)$$

is in $P^{1/2}(R^{n+1}_+)$ and has restriction f to R^n. (1.3) for $p = 2$ shows that a function $u \in P^{1/2}(R^{n+1}_+)$ with $f = u|_{R^n} \notin L^2(R^n)$ cannot be given by (1.4).

Throughout this paper C, possibly with subscripts and superscripts, will denote positive constants which may be different at different instances. ${}^\wedge$ denotes the Fourier transform and ${}^\vee$ its inverse.

2. The case $p = 2$. We consider inequalities (1.2), (1.3) for $p = 2$.

PROPOSITION 2.1. *Suppose that* φ *satisfies the following conditions*:

$$(2.1)\qquad \varphi \in L^1(R^n), \quad \int |\varphi(x)|^2(1+|x|^2)^{n/2}\,dx < \infty.$$

Then: (i) (1.2) *holds for all* $f \in L^2(R^n)$; (ii) *in order that* (1.3) *be valid for all* $f \in L^2(R^n)$ *it is necessary and sufficient that* $\varphi^\wedge(\xi)$ *be not identically* 0 *on any ray through the origin* $\xi = 0$.

Remarks. (2.1) implies that $\varphi^\wedge \in P^{n/2}(R^n) \cap BC(R^n)$, $BC(R^n)$ denoting the space of bounded continuous functions on R^n. The condition in (ii) is satisfied, for example, in the case when $\int_{R_n} \varphi(x)\,dx \neq 0$.

Proof of the proposition. By the Plancherel identity we can write, $\| \ \|$ denoting the norm in $L^2(R^n)$ and $\xi \cdot t'$ the scalar product in R^n:

$$(2.2)\qquad \|h_j\|^2 = \int_{R^n} a_j(\varphi, \xi)\,|f^\wedge(\xi)|^2\,d\xi, \quad j = 1, 2,$$

where

$$(2.3)\qquad \begin{aligned} a_1(\varphi, \xi) &= \int_{T_m}\int_0^\infty |e^{-it'\cdot\xi}\varphi^\wedge((y+\tau)\xi) - \varphi^\wedge(y\xi)|^2\,d\mu_m(y, t), \\ a_2(\varphi; \xi) &= \int_{T_m} |e^{-it'\cdot\xi} - 1|^2\,|\varphi^\wedge(y\xi)|^2\,d\mu'_m(y, t'). \end{aligned}$$

We observe that by the invariance property of the measures μ_m, μ'_m, $a_j(\xi)$

are homogeneous of degree 0. To verify (i) we have to show that $a_j(\xi)$ are bounded. We have $a_2(\varphi, \xi) \leqslant C\|\varphi(\cdot\theta)\|_{L^2(R_+)}$, $\theta = |\xi|^{-1}\xi$ and

$$a_1(\varphi;\xi) \leqslant 2\int_{T_m}\int_0^\infty |\varphi^\wedge((y+\tau)\xi)-\varphi^\wedge(y\xi)|\,d\mu_m(y,t)+$$
$$+2\int_{T_m}\int_0^\infty |e^{-i\xi\cdot t'}-1|^2\varphi^\wedge((y+\tau)\xi)\,d\mu_m(y,t).$$

In the first term we integrate first with respect to t', $|t'| \leqslant \varkappa\tau$ to obtain the estimate of the form

$$C\int_0^\infty\int_0^\infty |\varphi^\wedge((y+\tau)\xi)-\varphi^\wedge(y\xi)|^2\,d\mu_0(y,\tau)$$
$$= C\int_0^\infty\int_0^\infty |\varphi^\wedge((y+\tau)\theta)-\varphi^\wedge(y\theta)|\,d\mu_0(y,\tau),$$

where $\theta = |\xi|^{-1}\xi$. Integrating in the second term with respect to τ first and then with respect to t' we get an estimate of the form $C(\theta)\int_0^\infty |\varphi^\wedge(y\theta)|^2\,dy$, where $C(\theta)$ is a bounded function of θ. Hence

$$a_1(\varphi;\xi) \leqslant C\Big[\int_0^\infty\int_0^\infty |\varphi^\wedge((y+\tau)\theta)-\varphi^\wedge(y\theta)|^2\,d\mu_0(y,\tau)+\int_0^\infty |\varphi^\wedge(y\theta)|^2\,dy\Big].$$

The expression on the right is equivalent to the square of the norm in $P^{1,2}$ of the restriction of $\varphi^\wedge$ to the ray $y\theta$, $y > 0$ and by the known restriction theorem and the remark preceding the proof we get

$$a_j(\varphi;\xi) \leqslant C\|\varphi^\wedge\|^2_{P^{n/2}}, \quad j = 1, 2. \tag{2.4}$$

This proves (i). To prove (ii) it suffices to show that the condition

$$\varphi^\wedge(\xi) \not\equiv 0 \quad \text{on any ray through } 0 \tag{2.5}$$

is equivalent to the condition that

$$a_j(\varphi;\xi) \geqslant c > 0, \tag{2.6}$$

c is a constant. To show this we first notice that $a_j(\varphi;\xi)$ is continuous on $|\xi| = 1$. This is immediate if $\varphi^\wedge$ is sufficiently regular and with compact support. Also if $\varphi^\wedge, \varphi_1^\wedge \in P^{n/2}(R^n)$, then

$$|a_j(\varphi;\xi)-a(\varphi_1;\xi)| \leqslant a_j(\varphi+\varphi_1,\xi)^{1/2}a_j(\varphi-\varphi_1;\xi)^{1/2}$$

which together with (2.4) shows that $a(\varphi_l,\xi) \underset{l\to\infty}{\to} a(\varphi,\xi)$ uniformly if $\varphi^\wedge \underset{l\to\infty}{\to} \varphi^\wedge$ in $P^{n/2}(R^n)$. Since smooth functions with compact support are dense in $P^{n/2}$ it follows that $a(\varphi;\xi)$ is continuous.

For $j = 1$ (2.6) fails if and only if $a(\varphi;\theta) = 0$ for some $|\theta| = 1$ and this occurs if and only if $\varphi^\wedge(y\theta) = 0$ for all $y > 0$. In fact, if $a(\theta) = 0$, then by (2.3) $e^{-i\theta\cdot t'}\varphi^\wedge((y+\tau)\theta) - \varphi^\wedge(y\theta) = 0$ for a.e. (y, t) and since $\varphi^\wedge \in BC(R^n)$ the identity is valid for all y, t. With $\tau = 0$ we conclude that either $e^{-i\theta\cdot t'} = 1$ or $\varphi^\wedge(y\theta) = 0$ for all $y > 0$. In the first case $\varphi^\wedge((y+\tau)\theta) = \varphi^\wedge(y\theta)$ for all $y, \tau > 0$ and since $\varphi^\wedge(y\theta) \in L^2(R_+)$ (as a function of y) this also implies $\varphi^\wedge(y\theta) \equiv 0$. On the other hand, it is trivial that if $\varphi^\wedge(y\theta) \equiv 0$, then $a_1(\varphi;\theta) = 0$. For $j = 2$ the result is obvious.

We remark that by the argument leading to (2.2) we also get for any $f_1, f_2 \in L^2(R^n)$

$$\int_{R^n}\int_{T_m}\int_0^\infty (\Delta_t\varphi_y * f_1(x))\overline{(\Delta_t\varphi_y * f_2(x))}\,d\mu_m(y,t)\,dx = \int_{R^n} a_1(\varphi,\xi) f_1^\wedge(\xi) f_2^\wedge(\xi), \tag{2.7}$$

$$\int_{R^n}\int_{T_m} (\Delta_{t'}\varphi_y * f_1(x))(\Delta_{t'}\varphi_y * f_2(x))\,d\mu'_m(y,t')\,dx = \int a_2(\varphi,\xi) f_1^\wedge(\xi) f_2^\wedge(\xi)\,d\xi. \tag{2.7'}$$

3. $a_j(\xi)$ as multipliers of $\mathfrak{F}L^p$. We consider here some sufficient conditions on φ which guarantee that $a_j(\varphi;\xi)$ are multipliers of $\mathfrak{F}L^p$, i.e. the mappings $\psi \in \mathcal{S} \to (a_j(\varphi;\xi)\psi^\wedge(\xi))^\vee$ can be extended to a bounded linear operators in L^p. The Banach algebra of all multipliers of $\mathfrak{F}L^p$ we denote by $\mathfrak{M}_p$.

We use the following version of Marcinkiewicz–Mihlin–Lizorkin–Hörmander theorem (see [3]):

PROPOSITION 3.1. *If $|\xi^\alpha||D^\alpha a(\xi)| \leqslant M$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i = 0, 1$, of length $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant k(n)$, $k(n)$ being the least integer $> n/2$, then $m_a \in \mathfrak{M}_p$, $1 < p < \infty$, and $\|m_a\|_{\mathfrak{M}_p} \leqslant cM$.*

Since in our case $a_j(\varphi,\xi)$ given by (2.3) is homogeneous of degree 0, the condition of the above theorem will obviously be satisfied if $D^\alpha a_j(\varphi,\xi)$ exist for all multi-indices as in the theorem and are bounded for $|\xi| = 1$. Also if $a(\xi)$ satisfies the hypotheses of Proposition 3.1 and $|a(\xi)| \geqslant c > 0$ for all ξ, then $a(\xi)^{-1}$ also satisfies the hypotheses of the Proposition and therefore $m_{a^{-1}}$ is in $\mathfrak{M}_p$; $m_{a^{-1}} = m_a^{-1}$.

We give next an example of conditions one can impose on φ in order that $a_j(\varphi;\xi)$ satisfy the hypotheses of Proposition 3.1. We still assume that φ satisfies (2.1), and for all β, $|\beta| \leqslant k(n)$, $\beta_j = 0, 1$,

$$\text{(3.2)}\quad \begin{aligned} &\text{(i)}\quad |\xi|^{|\beta|} D^\beta \varphi^\wedge(\xi) \in P^{n\,2},\\ &\text{(ii)}\quad \int_0^\infty |r^l D^\beta \varphi^\wedge(r\theta)|^2\,dr \leqslant C \quad \text{for all } \theta \in R^n,\ |\theta| = 1,\ 0 \leqslant l \leqslant k(n). \end{aligned}$$

To verify that (3.2) imply the hypotheses of Proposition 3.1, we compute $D^\alpha a_j(\varphi;\xi)$. We note that for any m index β' with $\beta'_j = 0, 1$ we have:

$$(3.3)\qquad \int_{T_m} (t')^{\beta'} f(y)\,d\mu'(y, t') = 0, \quad \int_0^\infty \int_{T_m} (t')^{\beta'} f(y, \tau)\,d\mu(y, t) = 0$$

because $(t')^{\beta'}$ is odd. Differentiating under the integral sign:

$$D^\alpha a_2(\varphi, \xi) = \sum_{\beta+\gamma=\alpha} \int_{T_m} \sum_{\beta'+\beta''=\beta} [D^{\beta'}(e^{-it'\cdot\xi}-1)D^{\beta''}\varphi^\wedge(y\xi)],$$

$$\sum_{\gamma'+\gamma''=\gamma} D^{\gamma'}(e^{it'\cdot\xi}-1)D^{\gamma''}\varphi^\wedge(y\xi)\,d\mu'_m(y, t') = \sum_{\beta+\gamma=\alpha}\sum_{\beta'+\beta''=\beta}\sum_{\gamma'+\gamma''=\gamma} I_{\beta',\beta'';\gamma',\gamma''}.$$

We have the estimates: $|I_{0,\beta;0,\gamma}| \leqslant a_2(\varphi_\beta;\xi)^{1/2} a_2(\varphi_\gamma;\xi)^{1/2}$, where $\varphi_\beta(\xi) = |\xi|^{|\beta|} D^\beta \varphi^\wedge(\xi)$. (3.2) (i) implies that $a_j(\varphi_\alpha;\xi)$ are bounded for $|\alpha| \leqslant k$. If $\beta', \gamma' \neq 0$, then by (3.3) $I_{\beta',\beta'';\gamma',\gamma''} = 0$ and for $\beta' \neq 0$

$$|I_{\beta',\beta'';0,\gamma}| = \Big|\int_{T_m} e^{-it'\cdot\xi}(-it')^{\beta'} y^{|\beta''|}(D^{\beta''}\varphi^\wedge)(y\xi)(e^{it'\cdot\xi}-1)\,y^{|\gamma|}(D^\gamma\varphi^\wedge)(y\xi)\,d\mu'_m(y, t')\Big|$$

which can be estimated by $Ca_2(\varphi_\gamma, \xi)^{1/2} \int_0^\infty y^{2|\beta|-1}|(D^{\beta''}\varphi^\wedge)(y\xi)|^2\,dy$.

The same argument remains valid for $a_1(\varphi, \xi)$: $D^\alpha a_1(\varphi, \xi) = \sum_{\beta+\gamma=\alpha}\sum_{\beta'+\beta''=\beta}\sum_{\gamma'+\gamma''=\gamma} I_{\beta',\beta'';\gamma',\gamma''}$, where

$$|I^{(1)}_{0,\beta,0,\gamma}| = \Big|\int_0^\infty\int_{T_m} [e^{-it\cdot\xi}(D^\beta\varphi^\wedge(y+\tau)\xi) - D^\beta\varphi(y\xi)]\,[e^{it'\cdot\xi}\overline{D^\gamma\varphi^\wedge((y+\tau)\xi)} - \overline{D^\gamma\varphi(y\xi)}]\,d\mu_m(y, t)\Big| \leqslant a_2(\varphi_\beta;\xi)^{1/2} a_2(\varphi_\gamma;\xi)^{1/2},$$

$$I^{(1)}_{\beta',\beta'';\gamma',\gamma''} = 0 \quad \text{if} \quad \beta' \neq 0,\ \gamma' \neq 0$$

and

$$|I^{(1)}_{\beta',\beta'',0,\gamma}| = \Big|\int_0^\infty\int_{T_m} e^{-it'\cdot\xi}(-it')^{\beta'}(y+\tau)^{|\beta''|}(D^{\beta''}\varphi(y+\tau)\xi)[e^{it'\cdot\xi}\overline{D^\gamma\varphi^\wedge((y+\tau)\xi)} - \overline{D^\gamma\varphi^\wedge(y\xi)}]\,d\mu_m(y, t)\Big|$$

$$\leqslant Ca_1(\varphi_\gamma, \xi)^{1/2}\Big[\int_0^\infty \tau^{2|\beta'|-2}\int_\tau^\infty y^{2|\beta''|}(D^{|\beta''|}\varphi^\wedge)(y\xi)\,dy\,d\tau\Big]^{1/2}.$$

The term in the square brackets is now estimated by

$$\int_0^\infty (1+\tau^2)^{-|\beta'|}\tau^{2|\beta'|-2}\int_0^\infty (1+y^2)^{|\beta|}|(D^{\beta''}\varphi^\wedge)(y\xi)|^2\,dy\,d\tau$$

which is bounded by (3.2), (ii).

4. The case $1 < p < \infty$. In this section we are going to discuss inequality (1.2) and its converse for $1 < p < \infty$.

PROPOSITION 4.1. *Suppose that $l > 0$ is an integer and φ satisfies the following conditions*:

$$(4.1)\qquad \nabla^j\varphi\in L^\infty(R^n)\cap C(R^n)\quad \textit{and}\quad \int_0^\infty |\nabla^j\varphi(r,\theta)|^2 r^{2n+2l}\,dr\in L^\infty(S_1),$$

$$\textit{for } j = l,\ l+1,\ S_1 = \{\theta\in R^n,\ |\theta| = 1\}.$$

Then

$$\text{(i)}\qquad \int_{T_m} |\nabla^l\Delta_{t'}\varphi_y(x)|^2|t'|^{-m-1}\,dt'\,dy \leqslant C|x|^{-2(n+l)},$$

$$\text{(ii)}\qquad \int_0^\infty\int_{T_m} |\nabla^l\Delta_t\varphi_y(x)|^2|t|^{-m-2}\,dt\,dy \leqslant C|x|^{-2(n+l)},$$

where C is a constant.

Proof. (i) We note that $\nabla^l\Delta_{t'}\varphi_y(x) = y^{n-l}\Delta_{t'}(\nabla^l\varphi)(x/y)$ for $x\neq 0$ and decompose that integral in (i) into two parts: I_1 — the integral over $T'_m = \{(t', y)\in T_m,\ |t'|\leqslant 2|x|\}$ and I_2-the integral over $T''_m = \{(t', y)\in T_m,\ |t'|\geqslant 2|x|\}$. In I_1 we use the estimate:

$$|\Delta_{t'}(\nabla^l\varphi)(x/y)| = \left|\frac{1}{y}\int_0^1(\nabla^{l+1}\varphi)\left(\frac{x+st'}{y}\right)\cdot t'\,ds\right| \leqslant \frac{1}{y}|t'|\left(\int_0^1\left|(\nabla^{l+1}\varphi)\left(\frac{x+st'}{y}\right)\right|^2 ds\right)^{1/2},$$

thus

$$(4.2)\qquad I_1 \leqslant \int_0^1\int_{|t'|\leqslant\sigma|x|}\left(\int_{x^{-1}|t'|}^\infty y^{-2(n+l+1)}\left|(\nabla^{l+1}\varphi)\left(\frac{x+st'}{y}\right)\right|^2 dy\right)|t'|^{-m+1}\,dt'\,ds$$
$$\equiv \int_0^1\int_{|t'|\leqslant\sigma|x|} F(t',s)|t'|^{-m+1}\,dt'\,ds.$$

For $x+st'\neq 0$ we make the change of variables: $|x+st'|/y = r$ which leads to the following expression for the inner integral in (4.2)

$$(4.3)\qquad F(t',s) = \int_0^{x|t'|^{-1}|x+st'|}\frac{r}{|x+st'|^{2n+2l+1}}|\nabla^{l+1}\varphi(r\theta)|^2\,dr,\qquad \theta = \frac{x+st'}{|x+st'|}.$$

We consider now two cases, (a) $|t'|\leqslant\frac{1}{2}|x|$, (b) $|t'|\geqslant\frac{1}{2}|x|$.

In the first case we replace the upper limit in (4.3) by ∞ and note that $|x+st'| \geqslant |x|-s|t'| \geqslant \frac{1}{2}|x|$

$$F(t, s) \leqslant 2^{2n+2l+1}|x|^{-2n-2l-1} \int_0^\infty r^{2n+2l}|\nabla^{l+1}\varphi(r\theta)|^2\,dr.$$

In case (b) we have the estimate

$$F(t', s) \leqslant \|\nabla^{l+1}\varphi\|_\infty^2 (2n+2l+1)^{-1}|t'|^{-2n-2l-1} \leqslant \|\nabla^{l+1}\varphi\|_\infty^2 2^{n+2l+1}|x|^{-2n-2l-1}.$$

Substituting into (4.2) and integrating with respect to t' first we get

$$I_1 \leqslant C\int_0^1 |x|^{-2n-2l-1} \int_{|t'|\leqslant\sigma|x|} |t'|^{-m+1}\,dt'\,ds = C_1\sigma|x|^{-2n-2l}$$

which is the desired estimate.

We consider now I_2; we have $|x+t'| \geqslant |x|$ and we replace $\Delta_{t'}(\nabla^l\varphi)(x/y)$ by $|(\nabla^l\varphi)((x+t')/y)|+|(\nabla^l\varphi)(x/y)|$. The integral corresponding to the second term is

$$\int_0^1 \int_{|t'|\geqslant 2|x|} \Big(\int_{x^{-1}|t'|}^{\infty} y^{-2n-2l}|(\nabla^l\varphi)(x/y)|^2\,dy\Big)|t'|^{-m-1}\,dt'\,ds.$$

Substituting again $|x|/y = r$ we get the estimate for the inner integral

$$|x|^{-2n-2l+1} \int_0^{x|t'|^{-1}} r^{2n+2l-2}|(\nabla^l\varphi)(r\theta)|^2\,dr \leqslant C|x|^{-2n-2l+1}$$

and integration with respect to t' gives the desired inequality.

The same argument applies to the integral corresponding to the first term, leading to the bound $C|x+st'|^{-2n-2l+1} \leqslant C_1|x|^{-2n-2l+1}$.

(ii) In this case the procedure is quite similar to the proof of (i) except that we have to account for the increment in the variable y.

We divide the integration over T_m in (ii) into two parts, the first over $T_m' = \{(t', \tau)\in T_m;\ |t'| \leqslant 2|x|\}$, and the second over $T_m'' = \{(t', \tau)\in T_m;\ |t'| \geqslant 2|x|\}$; denote the corresponding integrals by I_1, I_2. In the first integral we use as in (i) the mean value theorem to get $\nabla^l\Delta_t\varphi_y(x) = \int_0^1 \frac{d}{ds}\nabla^l\varphi_{y+s\tau}(x+st')\,ds$.

Now

$$\frac{d}{ds}\nabla^l\varphi_{y+s\tau}(x+st') = (y+s\tau)^{-n-l-1}(\nabla^{l+1}\varphi)\Big(\frac{x+st'}{y+s\tau}\Big)\cdot t' -$$

$$-(n+l)\tau(y+s\tau)^{-n-l-1}(\nabla^l\varphi)\Big(\frac{x+st'}{y+s\tau}\Big)+$$

$$+(y+s\tau)^{-n-l-2}\tau(\nabla^{l+1}\varphi)\Big(\frac{x+st'}{y+s\tau}\Big)\cdot(x+st).$$

Thus I_1 can be estimated by the sum of three integrals, each corresponding to one of the terms above, which we denote by I_{11}, I_{12}, I_{13}. We have

$$
\begin{aligned}
I_{11} &\leqslant C\int_0^1\int_0^\infty\int_{T'_m} (y+s\tau)^{-2(n+l+1)}\left|(\nabla^{l+1}\varphi)\left(\frac{x+st'}{y+s\tau}\right)\right|^2 |t'|\,|t|^{-m-2}\,dt\,dy\,ds,\\
(4.7)\qquad I_{12} &\leqslant C\int_0^1\int_0^\infty\int_{T'_m} (y+s\tau)^{-2(n+l+1)}\left|(\nabla^{l}\varphi)\left(\frac{x+st'}{y+s\tau}\right)\right|^2 \tau^2|t|^{-m-2}\,dt\,dy\,ds,\\
I_{13} &\leqslant C\int_0^1\int_0^\infty\int_{T'_m} (y+s\tau)^{-2(n+l+2)}\left|(\nabla^{l+1}\varphi)\left(\frac{x+st'}{y+s\tau}\right)\right|^2 |x+st'|^2\tau^2|t|^{-m-2}.
\end{aligned}
$$

In each of the above we integrate first with respect to y making a substitution $r = |x+st'|/(y+s\tau)$, $x+st' \neq 0$ to obtain respectively expressions:

$$
\int_0^{(s\tau)^{-1}|s+st'|} r^{2n+2l}|(\nabla^{l+1}\varphi)(r\theta)|^2\,dr\,|x+st'|^{-2n-2l-1},
$$

$$
\int_0^{(s\tau)^{-1}|x+st'|} r^{2n+2l}|(\nabla^{l}\varphi)(r\theta)|^2\,dr\,|x+st'|^{-2n-2l-1},
$$

$$
\int_0^{(s\tau)^{-1}|x+st'|} r^{2n+2l+2}|(\nabla^{l+1}\varphi)(r\theta)|^2\,dr\,|x+st'|^{-2n-2l-1}.
$$

We consider separately cases (a) $|x+st'| \geqslant \frac{1}{2}|x|$ and (b) $|x+st'| \leqslant \frac{1}{2}|x|$. In case (a) we replace the upper limits in the integrals by ∞ and obtain bounds of the form $C|x+st'|^{-2n-2l-1} \leqslant C_1|x|^{-2n-2l-1}$. In case (b) we have

$$
s|t'| \geqslant \tfrac{1}{2}|x| \geqslant \tfrac{1}{4}|t| = \tfrac{1}{4}(|t'|^2+\tau^2)^{1/2} \geqslant \tfrac{1}{4}(|t'|^2+\varkappa^{-2}|t'|^2)^{1/2} = \frac{1}{4\varkappa}(1+\varkappa^2)^{1/2}|t'|,
$$

in particular, $s \geqslant \dfrac{1}{4\varkappa}(1+\varkappa^2)^{1/2}$. Also $\tau \geqslant \varkappa^{-1}|t'| \geqslant \dfrac{1}{2\varkappa}|x|$. Replacing $|\nabla^j\varphi(r\theta)|$ by $\|\nabla^j\varphi\|_\infty$, $j = l, l+1$, we get the bounds for the first two integrals

$$
C\big((s\tau)^{-1}|x+st'|\big)^{2n+2l+1}|x+st'|^{-2n-2l-1} = C(s\tau)^{-2n-2l-1} \leqslant C_1|x|^{-2n-2l-1}
$$

and for the third integral

$$
\begin{aligned}
C\big((s\tau)^{-1}|x+st'|\big)^{2n+2l+3}|x+st'|^{-2n-2l-1} &= C(s\tau)^{-2n-2l-3}\cdot|x+st'|^2\\
&\leqslant C_1|x|^{-2n-2l-3}\cdot|x|^2 \leqslant C_1|x|^{-2n-2l-1}.
\end{aligned}
$$

Substituting it into (4.4) and observing that

$$
\begin{aligned}
\int_{T'_m}|t|^{-m}\,dt &= \int_{|t'|\leqslant 2|x|}\ \int_{\varkappa^{-1}|t'|}^{\infty}(|t'|^2+\tau^2)^{-m/2}\,dt\,dt'\\
&\leqslant C\int_{|t'|\leqslant 2|x|}|t'|^{-m+1}\,dt' = C_1|x|,
\end{aligned}
$$

we get the desired estimate

$$I_{1j} \leqslant C|x|^{-2n-2l}, \quad j = 1, 2, 3.$$

The integral I_2 is written as a sum of

$$I_{21} = \int_0^{\infty} \int_{T''_m} y^{-2n-2l} |\nabla^l \varphi(x/y)|^2 |t|^{-m-2} dt dy$$

and

$$I_{22} = \int_{\tau}^{\infty} \int_{T''_m} y^{-2n-2l} \left|\nabla^l \varphi((x+t')/y)\right|^2 |t|^{-m-2} dt dy.$$

In I_{21} we substitute $|x|/y = r$ to get the bound for the integral with respect to y, of the form $|x|^{-2n-2l+1} \int_0^{\infty} r^{2n+2l-2} |\nabla^l \varphi(r\theta)|^2 dr$ and the desired estimate follows from the inequality

$$\int_{T''_m} |t|^{-m-2} dt = \int_{|t'| \geqslant 2|x|} \int_{x^{-1}|t'|}^{\infty} (|t'|^2 + \tau^2)^{-(m+2)/2} dt dt' \leqslant C|x|^{-1}.$$

The substitution $|x+t'|/y = r$ gives for the integral with respect to y in I_{21} a bound of the form $C|x+t'|^{-2n-2l+1} \leqslant C(2|x|)^{-2n-2l+1}$ and the proof is complete.

THEOREM 4.1. *If φ satisfies (4.1) with $l = 1$ and $|\xi| |\nabla \varphi^{\wedge}(\xi)|$, $|\xi|^{1/2} \varphi^{\wedge}(\xi) \in L^2(\boldsymbol{R}^n)$, then 1.2 holds, i.e. for every p, $1 < p < \infty$, there is a constant A_p such that $\|h_j(f; \cdot)\|_p \leqslant A_p \|f\|_p$ for every $f \in L^p(\boldsymbol{R}^n)$.*

In the proof we use Theorem 5, Chapter 2, § 5 of [4] in the following setting: for $j = 1$: $H_1 = \boldsymbol{C}$, $H_2 = H = L^2(\boldsymbol{R}_+ \times T_m, d\mu_m)$ and the operator valued kernel $K^1(x)$: $\boldsymbol{C} \to H$ is given by $x \to K^1(x) = \Delta_t \varphi_y(x)$, $K(x)$: $\lambda \in \boldsymbol{C} \to \lambda K(x) \in H$. For $j = 2$, $H = L^2(T_m, d\mu'_m)$ and $K^2(x) = \Delta_{t'} \varphi_y(x)$. The objective is to show that $f \to K^j * f$ is a bounded transformation form $L^p(\boldsymbol{R}^n)$ into $L^p(\boldsymbol{R}^n, H)$. We will actually consider the families of operators $K^1_\varepsilon(x) = \Delta_t \varphi_{y+\varepsilon}(x)$, $K^2_\varepsilon = \Delta_{t'} \varphi_{y+\varepsilon}(x)$, and show that

$$\|K_\varepsilon * f\| \leqslant A_p \|f\|_p, \quad 1 < p < \infty, \tag{4.2}$$

with a constant A_p independent of ε.

Since it is easy to check that $K^j_\varepsilon * f \underset{\varepsilon \searrow 0}{\to} K^j * f$ in $L^p(\boldsymbol{R}^n, H)$, provided f is sufficiently smooth and with compact support, it will follow that $\|K * f\|_p \leqslant A_p \|f\|_p$, $f \in L^p(\boldsymbol{R}^n)$.

To prove (4.2) we have to verify, according to the theorem referred to above, the following conditions:

$$\begin{aligned} &\text{(i)} \quad \|K_\varepsilon(x)\| \in L^2, \\ &\text{(ii)} \quad \|K^{\wedge}_\varepsilon(\xi)\| \leqslant B, \quad \xi \in \boldsymbol{R}^n, \\ &\text{(iii)} \quad \|\nabla K_\varepsilon(x)\| \leqslant B|x|^{-n-1}, \end{aligned} \tag{4.3}$$

where $\| \ \|$ denotes the norm of a transformation from C to H. If (i), (ii), (iii) are satisfied, then the constant A_p in (4.2) depends on B and p alone.

Verification of (i) and (ii). We have:

$$\|K^{1\wedge}(\xi)\|^2 \leqslant \int_0^\infty \int_{T_m} |e^{-it'\cdot\xi}\varphi^\wedge((y+\tau)\xi)-\varphi^\wedge(y\xi)|^2\, d\mu_m(y,t) = a_1(\varphi,\xi),$$

$$\|K^{2\wedge}(\xi)\|^2 \leqslant \int_{T_m} |(e^{-it'\cdot\xi}-1)\varphi(y\xi)|^2\, d\mu'_m(y,t') = a_2(\varphi;\xi)$$

and $\|K_\varepsilon^{\wedge j}(\xi)\|$, $j = 1, 2$ are bounded independently of ε (see Section 2).

On the other hand, using the same inequality as in the estimate of $a(\xi)$

$$\begin{aligned}\int_{R^n} \|K_\varepsilon^1(x)\|^2\,dx &= \int_{R^n} \|K_\varepsilon^{1\wedge}(\xi)\|^2\,d\xi \\ &= \int_{R^n}\int_\varepsilon^\infty\int_{T_m} |e^{-it'\cdot\xi}\varphi^\wedge((y+\tau)\xi)-\varphi^\wedge(y\xi)|^2\,d\mu_m(y,t)\,d\xi \\ &\leqslant 2\Big[C\int_{R^n}\int_\varepsilon^\infty\int_0^\infty |\varphi^\wedge((y+\tau)\xi)-\varphi^\wedge(y\xi)|^2\tau^{-2}\,d\tau\,dy\,d\xi+ \\ &\quad+\int_{R^n}\int_0^\infty\int_{T_m} |(e^{-i\xi\cdot t'}-1)\varphi^\wedge(y\xi)|^2\,d\mu_m(y,t)\,d\xi\Big].\end{aligned}$$

In the first integral we use Hardy's inequality

$$\begin{aligned}&\int_0^\infty |\varphi^\wedge((y+\tau)\xi)-\varphi^\wedge(y\xi)|^2\tau^{-2}\,d\tau \\ &\qquad = \int_y^\infty |\varphi^\wedge(s\xi)-\varphi^\wedge(y\xi)|^2(s-y)^{-2}\,ds \leqslant 4\int_y^\infty |\nabla\varphi^\wedge(s\cdot\xi)\cdot\xi|^2\,ds,\end{aligned}$$

and

$$4\int_\varepsilon^\infty\int_y^\infty s^{-n-2}\,ds\int_{R^n} |\nabla\varphi^\wedge(\xi)|^2|\xi|^2\,d\xi = C\varepsilon^{-n}\|\,|\nabla\varphi^\wedge(\xi)|\,|\xi|\,\|_{L^2(R^n)}.$$

The second term is estimated in the same way as in the proof of boundedness of $a(\xi)$ in Section 1; here we get the bound $C\varepsilon^{-n}\int_{R^n} |\varphi^\wedge(\xi)|^2|\xi|\,d\xi$. The same estimate is also valid for $\int \|K_\varepsilon^{2\wedge}(\xi)\|^2\,d\xi$.

(iii) is an immediate consequence of Proposition 4.1: it suffices to notice that $\|\nabla K_\varepsilon^j(x)\| \leqslant \|\nabla K^j(x)\|$. QED.

We next consider the validity of the converse inequality to (1.2).

THEOREM 4.2. *If* $1 < p < \infty$, φ *satisfies the assumptions of Theorem* 4.1, $a_j(\varphi,\xi) > 0$ *for all* $\xi \in R^n$ *and* $a_j(\varphi;\xi)^{-1}$ *is a multiplier of* $\mathfrak{F}L^p$, $1 < p < \infty$, *then there is a constant* $B_p > 0$ *such that* $\|h_j(f,\cdot)\|_p \geqslant B_p\|f\|_p$ *for all* $f \in L^p(R^n)$.

Proof. If $f \in L^p \cap L^2$, $f_1 \cap L^{p'} \cap L^2$, $1/p+1/p' = 1$, then with $a(\xi) = a_1(\varphi, \xi)$

$$(f, f_1) = \int f^{\wedge}(\xi)\overline{f_1^{\wedge}(\xi)}\,d\xi = \int a(\xi) f^{\wedge}(\xi) a^{-1}(\xi)\overline{f_1^{\wedge}(\xi)}\,d\xi$$

$$= \int_{R^n}\int_0^{\infty}\int_{T_m} (\Delta_t \varphi_y * f_1(x))\overline{(\Delta_t \varphi_y * f_2(x))}\,d\mu_m(y, t)\,dx,$$

where $f_2^{\wedge}(\xi) = a(\xi)^{-1} f_1^{\wedge}(\xi)$, $f_2 \in L^2 \cap L^{p'}$. Using the Cauchy–Schwartz and Hölder inequalities and Theorem 4.1, we get

$$|(f, f_1)| \leqslant \|h(f, \cdot)\|_p \|h(f_2, \cdot)\|_{p'} \leqslant \|h(f, \cdot)\|_p A_{p'} \|f_2\|_{p'}$$
$$\leqslant \|h(f, \cdot)\| A_{p'} \|m_a\|_{\mathfrak{M}_p} \|f_1\|_{p'};$$

we have used the fact that $\mathfrak{M}_p = \mathfrak{M}_{p'}$.

The last inequality with the usual density argument completes the proof. When $a(\xi) = a_2(\varphi, \xi)$ the proof is identical.

The hypotheses of Theorem 4.2 are satisfied under conditions discussed in Sections 2 and 3 which we shall not repeat here.

Theorem 4.1 remains valid in a suitable formulation for $0 < p \leqslant 1$. Using the terminology of [2], chapter V, we have

THEOREM 4.3. *Suppose that* $0 < p \leqslant 1$. *Than there is an integer* $l(p)$ *such that for* φ *satisfying* (4.1) *for all* l, $0 < l \leqslant l(p)$ *there is a constant* C *such that* $\|h_j(f, \cdot)\|_p \leqslant C\|f\|_{H^p}$ *for all sufficiently smooth* $f \in H^p$.

Proof. We apply a vector valued version of Theorem 12 in [2] in the same setting as in the proof of Theorem 4.1, Proposition 4.1 and the hypothesis imply that the conditions of this theorem are satisfied and we conclude that $\Delta_t \varphi_y * f \in H^p(R^n, H)$, where, as before, $H = L^2(R_+ \times T_m, d\mu_m)$ or $H = L^2(T_m, d\mu_m')$ with $\|\Delta_t \varphi_y * f\|_{H^p(R^n, H)} \leqslant C\|f\|_{H^p}$. In particular for any $\psi \in C_0^{\infty}(R^n)$ with $\int \psi = 1$ we have

$$\sup\{\|\psi_z * K^j * f(x)\|_H ; z > 0\} = (K^j f)^+(x) \in L^p, \quad \|(K^j f)^+\|_p \leqslant C\|f\|_{H^p}.$$

If f is sufficiently regular, then for $z \to 0$, $\psi_z * K^j * f \to K^j * f$ pointwise and, by Fatou's lemma, $\|K^j * f(x)\|_H \leqslant (K^j f)^+(x)$. QED.

We don't know if an analogue of (1.3) is valid for $0 < p \leqslant 1$.

The following remarks seem to be in order:

1° Some of the arguments in this paper could be somewhat simplified if we assumed that $\varphi \in \mathcal{S}$; in particular, it would not be necessary to impose various smoothness and integrability conditions on φ and $\varphi^{\wedge}$. This would eliminate, however, the function $\varphi(x) = 1/(1+|x|^2)^{(n+1)/2}$ corresponding to the Poisson kernel.

2° The results of Section 2 give rise to the following consequence. A harmonic function u which is $L^2(R_+^{n+1})$ is in $P^{1/2}(R_+^{n+1})$ if and only if its boundary value $u|_{R^n}$ is in L^2. The same is true for a disk or n-dimensional ball. It would be of interest to see an analogue of this result for more general domains and for $p \neq 2$.

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