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On the mean values of an entire Dirichlet series of order zero

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Abstract. In this paper, the mean values of entire functions of slow growth, defined by Dirichlet series have been considered. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ ($s = \sigma + it$, $\lambda_{n+1} > \lambda_n$, $\lambda_n \to \infty$ with n) define an entire function. The entire function $f(s)$ is said to be of *slow growth* if

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = 0,$$

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log M(\sigma)}{\log\sigma} = \frac{\varrho^*}{\lambda^*} \qquad (1 < \lambda^* \leqslant \varrho^* < \infty),$$

where $M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma+it)|$. The constants ϱ^* and λ^* are called *logarithmio order* and *lower logarithmic order* of $f(s)$, respectively. Define the following means of $f(s)$:

$$(1) \qquad A_\delta(\sigma) = A_\delta(\sigma, f) = [I_\delta(\sigma)]^\delta = [I_\delta(\sigma, f)]^\delta = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^\delta dt,$$

$$(2) \qquad m_{\delta,k}(\sigma) = m_{\delta,k}(\sigma, f) = e^{-k\sigma} \int_0^\sigma I_\delta(x) e^{kx} dx,$$

$$(3) \qquad m^*_{\delta,k}(\sigma) = m^*_{\delta,k}(\sigma, f) = \sigma^{-k-1} \int_0^\sigma I_\delta(x) x^k dx.$$

A few properties of above means and their derivatives have been studied in this paper. The growth properties of means of more than one entire function have also been studied. As an illustration we have the following theorem:

Let $f_1(s)$ and $f_2(s)$ be two entire functions of logarithmic orders ϱ_1^* and ϱ_2^* and lower logarithmic orders λ_1^* and λ_2^*, respectively. Then, if

$$\log\log m_{\delta,k}(\sigma, f) \sim \log[\{\log m_{\delta,k}(\sigma, f_1)\}\{\log m_{\delta,k}(\sigma, f_2)\}],$$

the logarithmic order ϱ^* and lower logarithmic order λ^* of $f(s)$ satisfy the inequalities

$$\lambda_1^* + \lambda_2^* \leqslant \lambda^* \leqslant \varrho^* \leqslant \varrho_1^* + \varrho_1^*.$$

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Further, if

$$\log\log m_{\delta,k}(\sigma,f) \sim [\{\log\log m_{\delta,k}(\sigma,f_1)\}\{\log\log m_{\delta,k}(\sigma,f_2)\}]^{1/2},$$

then

$$(\lambda_1^*\lambda_2^*)^{1/2} \leqslant \lambda^* \leqslant \varrho^* \leqslant (\varrho_1^*\varrho_2^*)^{1/2}.$$

1. Introduction. Consider a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ ($s = \sigma+it$, $\lambda_{n+1} > \lambda_n$, $\lambda_n \to \infty$ with n), which we shall assume to be absolutely convergent everywhere in the complex plane and is bounded in any left strip, and hence it defines an entire function. The order ϱ $(0 \leqslant \varrho \leqslant \infty)$ of $f(s)$ is defined as the limit superior of $\log\log M(\sigma)/\sigma$, as $\sigma \to \infty$, with $M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b}} |f(\sigma+it)|$.

For a class of functions of order zero, i.e. for which $\varrho = 0$, logarithmic order, ϱ^*, and lower logarithmic order, λ^*, are defined by [3];

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log M(\sigma)}{\log\sigma} = \frac{\varrho^*}{\lambda^*} \quad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty). \tag{1.1}$$

Also, for $0 < \delta < \infty$ and $0 < k < \infty$, we consider the following means of $f(s)$:

$$A_\delta(\sigma) = A_\delta(\sigma,f) = [I_\delta(\sigma)]^\delta = [I_\delta(\sigma,f)]^\delta = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^\delta dt, \tag{1.2}$$

where the integral in (1.2) exists on account of the absolute convergence of the series for $f(s)$;

$$m_{\delta,k}(\sigma) = m_{\delta,k}(\sigma,f) = e^{-k\sigma} \int_0^\sigma I_\delta(x) e^{kx} dx; \tag{1.3}$$

and

$$m^*_{\delta,k}(\sigma) = m^*_{\delta,k}(\sigma,f) = \sigma^{-k-1} \int_0^\sigma I_\delta(x) x^k dx. \tag{1.4}$$

Then the following results are known ([1], p. 277):

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log \Phi(\sigma)}{\log\sigma} = \begin{cases}\varrho^* \\ \lambda^*\end{cases} \quad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty), \tag{1.5}$$

where $\Phi(\sigma)$ stands for $I_\delta(\sigma)$ or $m_{\delta,k}(\sigma)$ or $m^*_{\delta,k}(\sigma)$.

In this paper we have obtained a few properties of $m_{\delta,k}(\sigma)$, $m^*_{\delta,k}(\sigma)$ and its derivatives from the mean values, defined by (1.2), (1.3), and (1.4). We shall always assume that $f(s)$ is of order zero and of logarithmic order ϱ^*, lower logarithmic order λ^*.

2. THEOREM 1. *Let $f(s)$ be of logarithmic order ϱ^* and lower logarithmic order λ^*. Then*

$$\lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log[m'_{\delta,k}(\sigma)/m_{\delta,k}(\sigma)]}{\log\sigma}=\frac{\varrho^*-1}{\lambda^*-1}\qquad(1\leqslant\lambda^*\leqslant\varrho^*\leqslant\infty),\tag{2.1}$$

where $m'_{\delta,k}(\sigma)$ is the derivative of $m_{\delta,k}(\sigma)$.

The proof of this theorem is based on the following lemma:

LEMMA 1. $\log m_{\delta,k}(\sigma)$ *is a convex increasing function of σ.*

Proof. Since

$$m_{\delta,k}(\sigma)=e^{-k\sigma}\int_0^\sigma I_\delta(x)e^{kx}dx,$$

we have

$$\frac{d[\log m_{\delta,k}(\sigma)]}{d[\sigma]}=\frac{I_\delta(\sigma)-ke^{-k\sigma}\int_0^\sigma I_\delta(x)e^{kx}dx}{m_{\delta,k}(\sigma)}=\frac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)}-k,$$

which increases, since for any finite positive k, $e^{k\sigma}I_\delta(\sigma)$ is a convex increasing function of $e^{k\sigma}m_{\delta,k}(\sigma)$ [2]. Hence we have

$$\frac{d^2[\log m_{\delta,k}(\sigma)]}{d[\sigma]^2}>0\qquad\text{for }\sigma>\sigma_0(^1),$$

and Lemma 1 follows.

Proof of Theorem 1. In the first lemma we have shown that $\log m_{\delta,k}(\sigma)$ is a convex increasing function of σ for $\sigma>\sigma_0$. This implies that $\log m_{\delta,k}(\sigma)$ is differentiable almost everywhere with an increasing derivative. This enables us to write $\log m_{\delta,k}(\sigma)$ in the form

$$\log m_{\delta,k}(\sigma)=\log m_{\delta,k}(\sigma_0)+\int_{\sigma_0}^{\sigma}\frac{m'_{\delta,k}(x)}{m_{\delta,k}(x)}dx,\qquad\sigma>\sigma_0.\tag{2.2}$$

Thus

$$\log m_{\delta,k}(\sigma)<\log m_{\delta,k}(\sigma_0)+\frac{m'_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)}(\sigma-\sigma_0).$$

Proceeding to limits as $\sigma\to\infty$, we get

$$\lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log\log m_{\delta,}(\sigma)}{\log\sigma}\leqslant\lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log[m'_{\delta,k}(\sigma)/m_{\delta,k}(\sigma)]}{\log\sigma}+1.\tag{2.3}$$

(1) σ_0 need not be the same at each occurrence.

Again, for an arbitrary $\eta > 1$, $\sigma > \sigma_0$, we have

$$\log m_{\delta,k}(\eta\sigma) = \log m_{\delta,k}(\sigma) + \int_{\sigma}^{\eta\delta} \frac{m'_{\delta,k}(x)}{m_{\delta,k}(x)}\,dx$$

$$> \frac{m'_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)}\sigma(\eta-1),$$

therefore,

$$\text{(2.4)}\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log m_{\delta,k}(\sigma)}{\log\sigma} \geqslant \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log[m'_{\delta,k}(\sigma)/m_{\delta,k}(\sigma)]}{\log\sigma}+1.$$

From (2.3) and (2.4), we have

$$\text{(2.5)}\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log m_{\delta,k}(\sigma)}{\log\sigma} = \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log[m'_{\delta,k}(\sigma)/m_{\delta,k}(\sigma)]}{\log\sigma}+1.$$

On using (1.5) in (2.5) for $\Phi(\sigma) = m_{\delta,k}(\sigma)$, we get (2.1).

COROLLARY. *For almost all values of* $\sigma > \sigma_0$,

$$m_{\delta,k}(\sigma)\sigma^{(\lambda^*-1-\varepsilon)} < m'_{\delta,k}(\sigma) < m_{\delta,k}(\sigma)\sigma^{(\varrho^*-1+\varepsilon)},$$

where ε *is an arbitrary small positive number.*

THEOREM 2. *Let* $f_1(s)$ *and* $f_2(s)$ *be two entire functions of logarithmic orders* ϱ_1^*, ϱ_2^* *and lower logarithmic orders* λ_1^*, λ_2^* *respectively; then if*

$$\text{(2.6)}\qquad \log\log m_{\delta,k}(\sigma) \sim \log[\{\log m_{\delta,k}(\sigma,f_1)\}\{\log m_{\delta,k}(\sigma,f_2)\}],$$

the logarithmic order ϱ^* *and lower logarithmic order* λ^* *of the entire function* $f(s)$ *are such that*

$$\text{(2.7)}\qquad \lambda_1^*+\lambda_2^* \leqslant \lambda^* \leqslant \varrho^* \leqslant \varrho_1^*+\varrho_2^*,$$

and if

$$\text{(2.8)}\qquad \log\log m_{\delta,k}(\sigma) \sim [\{\log\log m_{\delta,k}(\sigma,f_1)\}\{\log\log m_{\delta,k}(\sigma,f_2)\}]^{1/2},$$

then

$$\text{(2.9)}\qquad [\lambda_1^*\lambda_2^*]^{1/2} \leqslant \lambda^* \leqslant \varrho^* \leqslant [\varrho_1^*\varrho_2^*]^{1/2},$$

where $m_{\delta,k}(\sigma)$, $m_{\delta,k}(\sigma,f_1)$ *and* $m_{\delta,k}(\sigma,f_2)$ *are the mean values of* $f(s)$, $f_1(s)$ *and* $f_2(s)$, *respectively.*

Proof. Since the entire functions $f_1(s)$ and $f_2(s)$ are of logarithmic orders ϱ_1^* and ϱ_2^*, from (1.5) we have

$$\lim_{\sigma\to\infty}\sup\frac{\log\log m_{\delta,k}(\sigma,f_1)}{\log\sigma} = \varrho_1^*,$$

and

$$\lim_{\sigma\to\infty}\sup\frac{\log\log m_{\delta,k}(\sigma,f_2)}{\log\sigma} = \varrho_2^*.$$

Hence, for an arbitrary number $\varepsilon > 0$ and $\sigma > \sigma_0$,

$$\frac{\log\log m_{\delta,k}(\sigma, f_1)}{\log \sigma} < \left(\varrho_1^* + \frac{\varepsilon}{2}\right), \tag{2.10}$$

and

$$\frac{\log\log m_{\delta,k}(\sigma, f_2)}{\log \sigma} < \left(\varrho_2^* + \frac{\varepsilon}{2}\right). \tag{2.11}$$

Adding (2.10) and (2.11), we get

$$\frac{\log[\{\log m_{\delta,k}(\sigma, f_1)\}\{\log m_{\delta,k}(\sigma, f_2)\}]}{\log \sigma} < (\varrho_1^* + \varrho_2^* + \varepsilon). \tag{2.12}$$

Proceeding as above for the limit inferior, we get

$$\frac{\log[\{\log m_{\delta,k}(\sigma, f_1)\}\{\log m_{\delta,k}(\sigma, f_2)\}]}{\log \sigma} > (\lambda_1^* + \lambda_2^* - \varepsilon). \tag{2.13}$$

Now, if (2.6) holds, then from (2.12) and (2.13), for any $\varepsilon > 0$ and sufficiently large σ, we have

$$(\lambda_1^* + \lambda_2^* - \varepsilon) < \frac{\log\log m_{\delta,k}(\sigma)}{\log \sigma} < (\varrho_1^* + \varrho_2^* + \varepsilon).$$

Proceeding to limits, it leads to (2.7). Again, multiplying (2.10) and (2.11), we get

$$\frac{[\{\log\log m_{\delta,k}(\sigma, f_1)\}\{\log\log m_{\delta,k}(\sigma, f_2)\}]}{(\log \sigma)^2} < \left(\varrho_1^* + \frac{\varepsilon}{2}\right)\left(\varrho_2^* + \frac{\varepsilon}{2}\right) \tag{2.14}$$

for any $\varepsilon > 0$ and sufficiently large σ. Similarly, we have

$$\left(\lambda_1^* - \frac{\varepsilon}{2}\right)\left(\lambda_2^* - \frac{\varepsilon}{2}\right) < \frac{[\{\log\log m_{\delta,k}(\sigma, f_1)\}\{\log\log m_{\delta,k}(\sigma, f_2)\}]}{(\log \sigma)^2} \tag{2.15}$$

for any $\varepsilon > 0$ and sufficiently large σ. Further, if (2.8) holds, then from (2.14) and (2.15), on taking limits, (2.9) follows.

COROLLARY 1. *If $f_\xi(s)$ $(\xi = 1, \ldots, n)$ are n entire functions of logarithmic orders $\varrho_1^*, \ldots, \varrho_n^*$ and lower logarithmic orders $\lambda_1^*, \ldots, \lambda_n^*$ and having the mean values $m_{\delta,k}(\sigma, f_1), \ldots, m_{\delta,k}(\sigma, f_n)$ respectively, then if*

$$\log\log m_{\delta,k}(\sigma) \sim \log[\{\log m_{\delta,k}(\sigma, f_1)\} \ldots \{\log m_{\delta,k}(\sigma, f_n)\}],$$

the logarithmic order ϱ^* and lower logarithmic order λ^* of the entire function $f(s)$ having mean value $m_{\delta,k}(\sigma)$ are such that

$$\lambda_1^* + \ldots + \lambda_n^* \leqslant \lambda^* \leqslant \varrho^* \leqslant \varrho_1^* + \ldots + \varrho_n^*,$$

and if

$$\log\log m_{\delta,k}(\sigma) \sim [\{\log\log m_{\delta,k}(\sigma, f_1)\} \dots \{\log\log m_{\delta,k}(\sigma, f_n)\}]^{1/n},$$

then

$$(\lambda_1^* \dots \lambda_n^*)^{1/n} \leqslant \lambda^{*} \leqslant \varrho^* \leqslant (\varrho_1^* \dots \varrho_n^*)^{1/n}.$$

COROLLARY 2. *If $f_1(s)$ and $f_2(s)$ are two entire functions of regular logarithmic growth of logarithmic orders ϱ_1^* and ϱ_2^* respectively, then so is $f(s)$ of logarithmic order ϱ^* and*

$$\varrho^* = \varrho_1^* + \varrho_2^*.$$

COROLLARY 3. *If $f_\xi(s)$ $(\xi = 1, \dots, n)$ are n entire functions of regular logarithmic growth of logarithmic orders $\varrho_1^*, \dots, \varrho_n^*$ respectively, then so is $f(s)$ of logarithmic order ϱ^* and*

$$\varrho^* = \varrho_1^* + \dots + \varrho_n^*.$$

Now we know that $\log I_\delta(\sigma)$ is a convex increasing function of σ [2]. Hence, if we replace $m_{\delta,k}(\sigma)$, $m_{\delta,k}(\sigma, f_1), \dots, m_{\delta,k}(\sigma, f_n)$ by $I_\delta(\sigma)$, $I_\delta(\sigma, f_1), \dots, I_\delta(\sigma, f_n)$ respectively, in theorems first and second, then results remains the same in view of (1.5). The details are omitted.

THEOREM 3. *Let $f(s)$ be of logarithmic order ϱ^* and lower logarithmic order λ^*. Then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[m_{\delta,k}^{*\prime}(\sigma)/m_{\delta,k}^*(\sigma)]}{\log\sigma} = \begin{matrix}\varrho^*\\ \lambda^*\end{matrix} \qquad (1 \leqslant \lambda^* \leqslant \varrho^* \leqslant \infty),$$

where $m_{\delta,k}^{*\prime}(\sigma)$ is the derivative of $m_{\delta,k}^*(\sigma)$.

In order to prove this theorem we need the following lemma:

LEMMA 2. $\log m_{\delta,k}^*(\sigma)$ *is a convex increasing function of* $\log\sigma$.

Proof. We have

$$\frac{d[\log m_{\delta,k}^*(\sigma)]}{d[\log\sigma]} = \frac{\dfrac{d}{d\sigma}[\log m_{\delta,k}^*(\sigma)]}{\dfrac{d}{d\sigma}[\log\sigma]} = \frac{I_\delta(\sigma)}{m_{\delta,k}^*(\sigma)} - (k+1),$$

which increases, since for any finite positive k, $\sigma^{k+1} I_\delta(\sigma)$ is a convex increasing function of $\sigma^{k+1} m_{\delta,k}^*(\sigma)$ ([1], p. 277). Hence the lemma follows.

Proof of Theorem 3. In the second lemma we have shown that $\log m_{\delta,k}^*(\sigma)$ is a convex increasing function of $\log\sigma$ for $\sigma > \sigma_0$. This implies that $\log m_{\delta,k}^*(\sigma)$ is differentiable almost everywhere with an increasing derivative. This enables us to write $\log m_{\delta,k}^*(\sigma)$ in the following form:

$$\log m_{\delta,k}^*(\sigma) = \log m_{\delta,k}^*(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{m_{\delta,k}^{*\prime}(x)}{m_{\delta,k}^*(x)} \frac{dx}{x}, \qquad \sigma > \sigma_0,$$

or,

$$\log m^*_{\delta,k}(\sigma) < \log m^*_{\delta,k}(\sigma_0) + \frac{m^{*\prime}_{\delta,k}(\sigma)}{m^*_{\delta,k}(\sigma)} \log\left(\frac{\sigma}{\sigma_0}\right).$$

Proceeding to limits as $\sigma \to \infty$, we get

$$(2.16) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m^*_{\delta,k}(\sigma)}{\log \sigma} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[m^{*\prime}_{\delta,k}(\sigma)/m^*_{\delta,k}(\sigma)]}{\log \sigma}.$$

Again, for an arbitrary $\eta > 1$, $\sigma > \sigma_0$, we have

$$\log m^*_{\delta,k}(\eta\sigma) = \log m^*_{\delta,k}(\sigma) + \int_{\sigma}^{\eta\sigma} \frac{m^{*\prime}_{\delta,k}(x)}{m^*_{\delta,k}(x)} \frac{dx}{x} > \frac{m^{*\prime}_{\delta,k}(\sigma)}{m^*_{\delta,k}(\sigma)} \log \eta.$$

Hence,

$$(2.17) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m^*_{\delta,k}(\sigma)}{\log \sigma} \geqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[m^{*\prime}_{\delta,k}(\sigma)/m^*_{\delta,k}(\sigma)]}{\log \sigma}.$$

Combining (2.16) and (2.17), we get the result in view of (1.5).

A similar results as in Theorem 2 can also be proved easily for the means $m^*_{\delta,k}(\sigma)$, $m^*_{\delta,k}(\sigma, f_1), \ldots, m^*_{\delta,k}(\sigma, f_n)$ in place of $m_{\delta,k}(\sigma)$, $m_{\delta,k}(\sigma, f_1), \ldots$ $\ldots, m_{\delta,k}(\sigma, f_n)$ respectively.

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