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STIFF DERIVATIONS OF COMMUTATIVE RINGS

BY

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The paper is concerned with the study of the mapping θ from the set of two-sided ideals of the Ore extension $S = R[t, d]$ of a commutative d-ring R to the set of d-ideals of R, defined by $\theta(A) = A \cap R$. A derivation d is said to be *rigid* on R if θ is a bijection and d is said to be *stiff* in a d-ideal I of R if $\theta^{-1}(I) = \{SI\}$. If R is a non-commutative ring with no Z-torsion and $d(Z(R)) \not\subset \mathfrak{z}(R)$, then D. A. Jordan has shown that d is stiff in 0 (see [7], 2.1). In [2] (§ 4.8) it is shown that if R is a Ritt algebra and there exists a central element z of R such that $d(z)$ is a unit, then d is rigid on R.

In this paper we define, at first, some ideal $\Delta = \Delta(R, d, I)$ of $R[t, d]$ and we prove that d is stiff in I if and only if $\Delta = SI$ (Theorem 2.1).

In Section 3 we show basic properties of Δ and we prove (Theorem 3.1) that if R is an arbitrary commutative d-ring, then the minimal polynomial of d is of the form $r_n t^{p^n} + r_{n-1} t^{p^{n-1}} + \ldots + r_1 t$, where p is some prime. This theorem is well known in the case of d-fields of characteristic $p > 0$ (see [4], p. 190).

Using the ideal Δ, we show in Section 4 that if R is an integral domain of characteristic 0, then d is stiff in 0 iff $d \neq 0$ (Theorem 4.1), and if R is an integral domain of characteristic $p > 0$, then d is stiff in 0 iff $\dim_C R_{(0)} = \infty$ (Theorem 4.2).

In Section 5 we define some d-ideal $E(I)$ and we prove that if R/I has no Z-torsion, then d is stiff in I if and only if $E(I) = I$ (Proposition 5.2). If R/I has Z-torsion, then the condition is not sufficient in general (Example 5.1).

Moreover, the following results are proved:

(1) If R is noetherian with no Z-torsion, then d is stiff in 0 if and only if $d(R) \not\subset \mathfrak{z}(R)$ (Corollary 5.4). (If R is non-noetherian, see Example 5.2.)

(2) d is rigid on R if and only if $R = Rd(R)$, where R is a Ritt algebra (Corollary 5.5).

(3) The condition $d(R) \cap u(R) \neq \emptyset$ is not necessary for a derivation d of a Ritt algebra R to be rigid on R (Example 5.3).

1. Preliminaries. Throughout this paper R is a commutative ring with identity. We say that R has no Z-*torsion* if for all $r \in R$ and integers n we have $nr = 0$ if and only if $r = 0$, and we say that R is of *characteristic* n if the subring generated by 1 is isomorphic to $Z/(n)$. We denote by $\mathfrak{z}(R)$ the set of all zero divisors of R, by $U^{-1}R$ a quotient ring of R with respect to a multiplicative system $U \subset R$, and by $u(R)$ the group of all units of R. If I is an ideal of R and A is a subset of R, then we denote by $(I : A)$ the ideal $\{r \in R;\ rA \subset I\}$.

The term *d-ring* will refer to a ring R together with a specified additive mapping $d\colon R \to R$, called *derivation*, which satisfies the condition $d(ab) = ad(b)+d(a)b$ for any $a, b \in R$.

Let R be a d-ring. If R is a field, then R is called a *d-field*. The set $C(R, d)$ of all elements r of R such that $d(r) = 0$ is a subring of R, called the *ring of constants* of R. If R is a field, then $C(R, d)$ is a subfield of R, called the *field of constants* of R. We say that R is a *Ritt algebra* if R contains the field Q of rational numbers. An ideal I of R is called a *d-ideal* if $d(I) \subset I$.

Throughout this paper, S is the *Ore extension* $R[t, d]$ of R (see [8]), i.e. S is a non-commutative ring of polynomials over R in an indeterminate t with multiplication subject to the relation $tr = rt+d(r)$ for all $r \in R$.

If I is a d-ideal of R, then SI is an ideal of S and

$$SI = \{r_n t^n + \ldots + r_0 \in S;\ r_i \in I \text{ for } i = 0, 1, \ldots, n\}$$

(see [6], Lemma 1.3). If A is an ideal of S, then $A \cap R$ is a d-ideal of R ([6], Lemma 1.3 (i)).

A derivation d of R is said to be *rigid* on R (see [2] and [7]) if the mapping θ from the set of ideals of S to the set of d-ideals of R, defined by $\theta(A) = A \cap R$ for all ideals A of S, is a bijection. If I is a d-ideal of R, then a derivation d of R is said to be *stiff* in I (see [7]) if, for all ideals A of S, $A \cap R = I$ implies $A = SI$. It is clear that d is rigid on R if and only if d is stiff in I for all d-ideals I od R.

2. Stiff derivations. Let I be a d-ideal of a d-ring R. We denote by $\Delta(R, d, I)$ the set

$$\{r_n t^n + \ldots + r_1 t + r_0 \in S;\ r_n d^n(r) + \ldots + r_1 d(r) + r_0 r \in I \text{ for any } r \in R\}.$$

This set has the following properties:

LEMMA 2.1. (i) *$\Delta(R, d, I)$ is an ideal of S.*

(ii) $SI \subset \Delta(R, d, I)$.

(iii) *If $\bar{d}$ is a derivation of R/I such that $\bar{d}(r+I) = d(r)+I$, then in the factor ring S/SI the ideal $\Delta(R, d, I)/SI$ is equal to $\Delta(R/I, \bar{d}, 0)$.*

(iv) $\Delta(R, d, I) \cap R = I$.

Proof. (i) If M is a d-ideal of R, then M together with the multiplication

$$(r_n t^n + \ldots + r_0)m = r_n d^n(m) + \ldots + r_1 d(m) + r_0 m$$

is a left S-module (see [5]). Using this fact for d-ideals I and R we infer that $\Delta(R, d, I) = \{f \in S;\ fR \subset I\}$ and $\Delta(R, d, I)$ is an ideal in S.

The proof of (ii) and (iv) is straightforward.

Property (iii) is an immediate consequence of the ring isomorphism $R[t, d]/IR[t, d] = (R/I)[t, \bar{d}]$ (see [7]).

THEOREM 2.1. *Let I be a d-ideal in a d-ring R. A derivation d is stiff in I if and only if $\Delta(R, d, I) = SI$.*

Proof. The necessity of the condition follows from (i) and (iv) of Lemma 2.1. To prove the sufficiency we can assume, by Lemma 2.1 (iii), that $I = 0$. Let $\Delta(R, d, 0) = 0$. Suppose that there exists an ideal A in S such that $A \neq 0$, $A \cap R = 0$. Let $f(t) = r_n t^n + \ldots + r_1 t + r_0$ ($r_i \in R$, $i = 0, 1, \ldots, n$, $r_n \neq 0$) be of minimal degree among the non-zero polynomials in A. Then $n \geqslant 1$ and $fr - rf \in A$ for every $r \in R$. We have

$$fr - rf = b_{n-1} t^{n-1} + \ldots + b_1 t + b_0,$$

where $b_{n-1}, \ldots, b_0 \in R$ and

$$b_0 = r_n d^n(r) + r_{n-1} d^{n-1}(r) + \ldots + r_1 d(r).$$

Since $\deg(fr - rf) < n$, by the minimality of n we have $fr - rf = 0$. Therefore

$$r_n d^n(r) + r_{n-1} d^{n-1}(r) + \ldots + r_1 d(r) = 0$$

for every $r \in R$, i.e. $r_n t^n + \ldots + r_1 t \in \Delta(R, d, 0) = 0$. This contradicts the fact that $r_n \neq 0$.

3. Some properties of the ideal $\Delta(R, d, 0)$. In this section, R is a d-ring and $\Delta = \Delta(R, d, 0)$. It is clear that if $r_n t^n + \ldots + r_1 t + r_0 \in \Delta$, then

$$r_0 = r_n d^n(1) + \ldots + r_1 d(1) + r_0 = 0.$$

If $\Delta \neq 0$ and m is the least degree of non-zero polynomials in Δ, then we denote by Δ_0 the set

$$\{r_m t^m + r_{m-1} t^{m-1} + \ldots + r_1 t \in \Delta;\ r_m \neq 0\}.$$

LEMMA 3.1. *Let $f, g \in \Delta_0$ and*

$$f = a_m t^m + \ldots + a_1 t, \qquad g = b_m t^m + \ldots + b_1 t.$$

Then

(1) $d^k(a_m) t^m + \ldots + d^k(a_1) t \in \Delta$ *for* $k = 0, 1, \ldots$;

(2) $a_m g = b_m f$;

(3) *if* $a_m \in C$, *then* $a_{m-1}, \ldots, a_1 \in C$, *where* $C = C(R, d)$;

(4) *if* $ra_m = 0$ *for some* $r \in R$, *then* $ra_{m-1} = \ldots = ra_1 = 0$;

(5) *if* a_m *is a unit of* R, *then* $\Delta = Sf$.

Proof. (1) Since Δ is an ideal of S and $f \in S$, we have

$$d(a_m)t^m + \dots + d(a_1)t = tf - ft \in \Delta$$

and, by induction,

$$d^k(a_m)t_m + \dots + d^k(a_1)t \in \Delta.$$

(2) Observe that $\deg(a_m g - b_m f) \leqslant m-1$ and $a_m g - b_m f \in \Delta$. Hence, and by the minimality of m, $a_m g - b_m f = 0$.

(3) If $a_m \in C$, then, by (1), $d(a_{m-1})t^{m-1} + \dots + d(a_1)t \in \Delta$ and, by the minimality of m, we have

$$d(a_{m-1}) = \dots = d(a_1) = 0.$$

(4) Since $rf \in \Delta$ and $\deg(rf) \leqslant m-1$, we obtain $ra_{m-1} = \dots = ra_1 = 0$.

(5) Clearly, $Sf \subset \Delta$. Let $h \in \Delta$. There exist $u, v \in S$ such that $h = uf + v$ and $\deg v < \deg f = m$ (see [6], Lemma 3.1 (i)). But $v = h - uf \in \Delta$, and thus, by the minimality of m, $v = 0$, i.e. $h = uf \in Sf$.

LEMMA 3.2. *Let* $r_m t^m + \dots + r_1 t \in \Delta_0$. *If* $m > 1$, *then*

$$\binom{k}{i} r_k = 0$$

for $k = 2, 3, \dots, m$ *and* $1 \leqslant i \leqslant k-1$.

Proof. Observe that if $a, b \in R$, then

$$0 = \sum_{i=1}^{m} r_i d^i(ab) = \sum_{i=1}^{m-1} u_i(a) d^i(b),$$

where

$$u_i(a) = \sum_{k=i+1}^{m} \binom{k}{k-i} r_k d^{k-i}(a).$$

This fact implies that

$$\sum_{i=1}^{m-1} u_i(a) t^i \in \Delta.$$

Hence, by the minimality of m, $u_i(a) = 0$ for $i = 1, 2, \dots, m-1$ and any $a \in R$. Therefore

$$\sum_{k=i+1}^{m} \binom{k}{k-i} r_k t^{k-i} \in \Delta$$

and, again by the minimality of m, we have

$$\binom{k}{k-i} r_k = 0 \quad \text{for } i = 1, 2, \dots, m-1 \text{ and } k = i+1, \dots, m,$$

i.e.

$$\binom{k}{i} r_k = 0 \quad \text{for } k = 2, 3, \dots, m \text{ and } 1 \leqslant i \leqslant m-1.$$

Note the following well-known result:

LEMMA 3.3. *If* $m > 1$, *then*

$$\left(\binom{m}{1}, \ldots, \binom{m}{m-1}\right) = \begin{cases} (1) & \text{if } m \text{ is not a power of a prime,} \\ (p) & \text{if } m = p^k. \end{cases}$$

THEOREM 3.1. *If* $\Delta \neq 0$ *and* $r_1 t^{n_1} + \ldots + r_k t^{n_k} \in \Delta_0$, *where* $n_1 > \ldots > n_k$, $r_i \neq 0$ *for* $i = 1, \ldots, k$, *then there exists a prime number* p *such that*

$$n_1 = p^{u_1}, \ldots, n_k = p^{u_k} \quad \text{and} \quad u_1, \ldots, u_k \geqslant 0.$$

Moreover, if char $R = n \geqslant 0$, *then* $p|n$.

Proof. If, for some i, n_i is not a power of a prime number, then, by Lemma 3.3,

$$1 = a_1\binom{n_i}{1} + a_2\binom{n_i}{2} + \ldots + a_{n_i-1}\binom{n_i}{n_i-1}$$

for some integers $a_1, a_2, \ldots, a_{n_i-1}$, and hence, by Lemma 3.2, we have

$$r_i = a_1 r_i \binom{n_i}{1} + \ldots + a_{n_i-1} r_i \binom{n_i}{n_i-1} = 0.$$

Therefore $n_1 = p_1^{u_1}, \ldots, n_k = p_k^{u_k}$, where $p_1, \ldots, p_k$ are prime numbers. Assume that $p_i \neq p_1$ for some $i \neq 1$. Then

$$p_i^{u_i} r_i = \binom{n_i}{1} r_i = 0, \quad p_1^{u_1} r_1 = \binom{n_1}{1} r_1 = 0$$

and, by Lemma 3.1 (4), $p_1^{u_1} r_i = 0$. Thus, if $1 = b_1 p_1^{u_1} + b_i p_i^{u_i}$, we have

$$r_i = b_1 p_1^{u_1} r_i + b_i p_i^{u_i} r_i = 0.$$

This contradicts the fact that $r_i \neq 0$. Finally, $n_1 = p^{u_1}, \ldots, n_k = p^{u_k}$, where p is a prime.

Let now n be a characteristic of R. If $n = 0$, then, clearly, $p|n$. If $n > 0$ and $p \nmid n$, then $1 = ap^{u_1} + bn$ for some integers a, b and, consequently, $r_1 = ap^{u_1} r_1 + bnr_1 = 0$.

LEMMA 3.4. *Let* $U^{-1}R$ *be a quotient ring of* R *with respect to a multiplicative system* U *and let* d_U *be a derivation of* $U^{-1}R$ *such that*

$$d_U\left(\frac{r}{u}\right) = \frac{d(r)u - rd(u)}{u^2}.$$

Moreover, let p *be a prime number and* char $R = p$. *If*

$$r_n t^{p^n} + r_{n-1} t^{p^{n-1}} + \ldots + r_1 t^p + r_0 t \in \Delta,$$

then

$$(r_n/1) t^{p^n} + (r_{n-1}/1) t^{p^{n-1}} + \ldots + (r_0/1) t \in \Delta(U^{-1}R, d_U, 0).$$

The proof is straightforward.

4. A characterization of stiff derivations in integral domain. Now we give some necessary and sufficient conditions for a derivation d of an integral domain to be stiff in 0.

THEOREM 4.1. *If R is an integral domain of characteristic* 0, *then d is stiff in* 0 *if and only if* $d \neq 0$.

Proof. If $d=0$, then $S = R[t]$ is a polynomial ring over R. Consequently, $\Delta = \Delta(R, d, 0) = Rt \neq 0$, so, by Theorem 2.1, d is not stiff in 0. Assume now that $\Delta \neq 0$ and $d \neq 0$. Let $f = a_m t^m + \ldots + a_1 t \in \Delta_0$. If $m > 1$, then Lemma 3.2 implies that $ma_m = 0$, i.e. $a_m = 0$. Therefore, $m = 1$, i.e. $f = a_1 t$, hence $a_1 d(r) = 0$ for any $r \in R$. Since $d \neq 0$ and R is an integral domain, $a_1 = 0$. This contradicts the fact that $f \neq 0$.

THEOREM 4.2. *Let R be an integral domain of characteristic $p > 0$, d a derivation of R, K the quotient field of R, and $C(K)$ the field of constants of K. Then d is stiff in* 0 *if and only if* $\dim_{C(K)} K = \infty$.

Proof. Denote by D a derivation of K such that

$$D\left(\frac{a}{b}\right) = \frac{d(a)b - ad(b)}{b^2}.$$

Assume that $\dim_{C(K)} K < \infty$. If $\dim_{C(K)} K = 1$, then $d = 0$ and, consequently, $\Delta = Rt \neq 0$. Thus, by Theorem 2.1, d is not stiff in 0. Therefore, let $\dim_{C(K)} K > 1$. Since D is a $C(K)$-linear mapping of a vector space K over $C(K)$, there exists a non-zero polynomial $f(t) \in C(K)[t]$ such that $f(D) = 0$. Let

$$f(t) = t^n + (a_1/b_1)t^{n-1} + \ldots + a_n/b_n,$$

where $a_i, b_i \in R$ for $i = 1, 2, \ldots, n$, and let $b = b_1 b_2 \ldots b_n$. Then $bf \neq 0$, $bf \in \Delta$ and, by Theorem 2.1, d is not stiff in 0. Assume now that d is not stiff in 0. Then, by Theorem 2.1, $\Delta \neq 0$ and, by Theorem 3.1, there exists a polynomial $f \in \Delta_0$ such that

$$f(t) = r_k t^{p^k} + r_{k-1} t^{p^{k-1}} + \ldots + r_0 t, \quad \text{where } r_k \neq 0.$$

Hence and from Lemma 3.4 we have $f \in \Delta(K, D, 0)$. Since f is minimal in Δ, we infer that $(1/r_k)f$ is minimal in $\Delta(K, D, 0)$, and by Lemma 3.1 (3) we obtain $(1/r_k)f \in C(K)[t]$ and $(1/r_k)f(D) = 0$. Therefore $\dim_{C(K)} K < \infty$.

5. Stiff derivations in d-rings with no Z-torsion. If I is a d-ideal of the d-ring R, then we denote by $E(I)$ the d-ideal $(I : d(R))$.

PROPOSITION 5.1. *If d is stiff in I, then* $E(I) = I$.

Proof. Let $b \in E(I)$, $b \notin I$, and let W be a left ideal of S generated by the set $\{bt, bt^2, bt^3, \ldots\}$. Then $B = W + SI$ is an ideal of S such that $B \supsetneq SI$, $B \cap R = I$. Hence d is not stiff in I.

COROLLARY 5.1. *If d is rigid on R, then $Rd(R) = R$.*

Proof. Observe that d is rigid on R if and only if d is stiff in I for every d-ideal I. Since $Rd(R)$ is a d-ideal of R, by Proposition 5.1 we have $E(RdR) = RdR$. Therefore

$$Rd(R) = E(Rd(R)) = (Rd(R): d(R)) = R.$$

PROPOSITION 5.2. *Assume that a d-ring R/I has no Z-torsion. Then d is stiff in I if and only if $E(I) = I$.*

Proof. The necessity follows from Proposition 5.1. Assume now that $E(I) = I$. Let A be an ideal of S such that $A \cap R = I$, $A \supsetneqq SI$, and let $f(t) = at^n + a_{n-1}t^{n-1} + \ldots + a_0$, where $a, a_{n-1}, \ldots, a_0 \in R$, $a \notin I$, be of minimal degree among the polynomials in A such that $a \notin I$. Since $A \cap I = I$, we obtain $n > 0$. Then, for every $r \in R$, we have

$$f(t)r - rf(t) = nd(r)at^{n-1} + g(t),$$

where $\deg(g(t)) < n-1$. Hence, by the minimality of n, $nd(r)a \in I$ for any $r \in R$. Since R/I has no Z-torsion, we get $ad(R) \subset I$, i.e. $a \in E(I) = I$, a contradiction.

If R/I has Z-torsion, then the condition of Proposition 5.2 is not sufficient in general.

Example 5.1. Let $R = Z_2[x]/(x^2)$, $d(x) = 1$. Then $E(0) = 0$ and d is not stiff in 0 (since $0 \neq t^2 \in \Delta(R, d, 0)$).

COROLLARY 5.2. *Let $\{I_j\}$ be the set of d-ideals of a Ritt algebra R. If, for every j, d is stiff in I_j, then d is stiff in $\bigcap_j I_j$.*

Proof. We know from Proposition 5.2 that $E(I_j) = I_j$. Then

$$E(\bigcap_j I_j) = (\bigcap_j I_j: d(R)) = \bigcap_j (I_j: d(R)) = \bigcap_j E(I_j) = \bigcap_j I_j$$

and, again by Proposition 5.2, d is stiff in $\bigcap_j I_j$.

COROLLARY 5.3 (Jordan [7]). *Let R be a d-ring with no Z-torsion. If $d(R) \not\subset \mathfrak{z}(R)$, then d is stiff in 0.*

For the proof apply Proposition 5.2.

COROLLARY 5.4. *Let R be a noetherian d-ring with no Z-torsion. Then d is stiff in 0 if and only if $d(R) \not\subset \mathfrak{z}(R)$.*

Proof. If $d(R) \not\subset \mathfrak{z}(R)$, then, by Corollary 5.3, d is stiff in 0. Assume now that d is stiff in 0 and $d(R) \subset \mathfrak{z}(R)$. Since R is a noetherian ring, we have

$$\mathfrak{z}(R) = \bigcup_{i=1}^{n} P_i,$$

where $P_1, P_2, \ldots, P_n$ are prime ideals such that $P_i = (0: x_i)$ for some $x_i \neq 0$ and $i = 1, 2, \ldots, n$ (see [1]). Since R has no Z-torsion, every ideal P_i does not contain an integer different from 0. Therefore, by Corollary 6.3 (see Appendix), $d(R) \subset P_i$ for some i, i.e. $x_i \in E(0) = 0$, which gives a contradiction since $x_i \neq 0$.

If R is a non-noetherian d-ring with no Z-torsion, then Corollary 5.4 is not true in general.

Example 5.2. Let $T = K[x_1, x_2, \ldots]$ be a polynomial ring over a field K of characteristic 0 and let d be a derivation of T such that $d(K) = 0$, $d(x_n) = x_n$ for any natural n. Then $I = (x_1^2, x_2^2, \ldots)$ is a d-ideal of T. Consider the d-ring $R = T/I$. Clearly, $\bar{d}(R) \subset \mathfrak{z}(R)$ and R is a non-noetherian d-ring with no Z-torsion. But $\bar{d}$ is stiff in 0 since $E(0) = 0$.

COROLLARY 5.5. *In a Ritt algebra the following conditions are equivalent*:

(1) *d is rigid on R*;

(2) *$Rd(R) = R$.*

Proof. (1)$\Rightarrow$(2) by Corollary 5.1.

(2)$\Rightarrow$(1). If I is a d-ideal in R, then

$$I = (I: R) = \big(I: Rd(R)\big) = \big(I: d(R)\big) = E(I).$$

Hence d is stiff in I for every d-ideal I of R and, consequently, d is rigid on R.

COROLLARY 5.6 ([2], § 4.8). *If R is a Ritt algebra and $d(R) \cap u(R) \neq \emptyset$, then d is rigid on R.*

The corollary is an immediate consequence of Corollary 5.5.

The condition $d(R) \cap u(R) \neq \emptyset$ from Corollary 5.6 is not necessary for a derivation d of a Ritt algebra R to be rigid on R.

Example 5.3. If $R = Q[x, y]$ and $d(x) = xy + 1$, $d(y) = y$, then one can show that $d(R) \cap u(R) = \emptyset$ and $Rd(R) = R$.

COROLLARY 5.7. *If R is a Ritt algebra admitting only a finite number of maximal ideals, then the following conditions are equivalent*:

(1) *d is rigid on R*;

(2) *$u(R) \cap d(R) \neq \emptyset$.*

Proof. (2)$\Rightarrow$(1) by Corollary 5.5.

(1)$\Rightarrow$(2). Let $d(R) \cap u(R) = \emptyset$. Then $d(R) \subset M_1 \cup \ldots \cup M_n$, where $M_1, \ldots, M_n$ are all maximal ideals in R, and thus, by Corollary 6.3, $d(R) \subset M_i$ for some i. Hence $Rd(R) \subset M_i$, i.e. $Rd(R) \neq R$, which gives a contradiction since, by Corollary 5.5, $Rd(R) = R$.

The next two corollaries are immediate consequences of Corollary 5.5.

COROLLARY 5.8. *Let K be a d-field of characteristic 0. Then d is rigid on K if and only if $d \neq 0$.*

COROLLARY 5.9 ([3], p. 43). *Let R be a Ritt algebra with $d \neq 0$. Then R is a d-simple d-ring (i.e. R has no d-ideals other than 0 and R) if and only if S is simple.*

6. Appendix. A note on a finite union of ideals. It is well known that if R is a commutative ring with identity and A is an ideal contained in the union of prime ideals $P_1, \ldots, P_n$ of R, then $A \subset P_i$ for some i (see [1]). In this appendix we show that if R is an algebra over the rational number field Q, then this theorem is also true without the assumption that $P_1, \ldots, P_n$ are prime. We use this generalization in the proofs of Corollaries 5.4 and 5.7.

THEOREM 6.1. *Let R be a non-commutative algebra over the rational number field Q and suppose that $B, A_1, \ldots, A_n$ are submodules of a left R-module M. If $B \subset \bigcup_{i=1}^{n} A_i$, then $B \subset A_i$ for some i.*

Proof (by induction on n). The case $n = 1$ is trivial. Let $n = 2$. Suppose that $B \subset A_1 \cup A_2$ and $B \not\subset A_1$, $B \not\subset A_2$. Choose $x_1 \in B \backslash A_2$ and $x_2 \in B \backslash A_1$. Then $x_1 \in A_1$ and $x_2 \in A_2$. Consider the element $x_1 + x_2$. If $x_1 + x_2 \in A_1$, then

$$x_2 = -x_1 + (x_1 + x_2) \in A_1,$$

and if $x_1 + x_2 \in A_2$, then

$$x_1 = (x_1 + x_2) - x_2 \in A_2.$$

Therefore, $x_1 + x_2 \notin A_1 \cup A_2$, which gives a contradiction since

$$x_1 + x_2 \in B \subset A_1 \cup A_2.$$

Suppose now that the result is true for $n-1$, where $n > 2$. Assume that $B \subset \bigcup_{i=1}^{n} A_i$ and denote by B_i the set

$$A_1 \cup \ldots \cup A_{i-1} \cup A_{i+1} \cup \ldots \cup A_n, \qquad i = 1, \ldots, n.$$

If $B \subset B_i$ for some i, then, by induction, $B \subset A_j$ for some j. Now assume that $B \not\subset B_i$ for $i = 1, \ldots, n$. Let $x_1 \in B \backslash B_1$ and $x_2 \in B \backslash B_2$. Then $x_1 \in A_1$ and $x_2 \in A_2$. Consider the set

$$T = \{x_1 + x_2,\ x_1 + 2x_2,\ \ldots,\ x_1 + (n-1)x_2\}.$$

If $x_1 + kx_2 \in A_1$ for some k, then $kx_2 = (x_1 + kx_2) - x_1 \in A_1$, hence $x_2 \in A_1$. If $x_1 + kx_2 \in A_2$ for some k, then $x_1 = (x_1 + kx_2) - kx_2 \in A_2$. Therefore, every element of T belongs to $A_3 \cup \ldots \cup A_n$. Since T has $n-1$ elements, there exist $k \in \{3, 4, \ldots, n\}$ and $x_1 + ix_2$, $x_1 + jx_2 \in T$ such that $i > j$ and $x_1 + ix_2$, $x_1 + jx_2 \in A_k$. Hence

$$(i-j)\, x_2 = (x_1 + ix_2) - (x_1 + jx_2) \in A_k,$$

and therefore $x_2 \in A_k$. This contradicts the fact that $x_2 \notin A_k$ and completes the proof.

COROLLARY 6.1 (to the proof). *Let $B, A_1, \ldots, A_n$ be ideals of a ring R. If $(n!)1$ is invertible in R and $B \subset \bigcup_{i=1}^{n} A_i$, then $B \subset A_i$ for some i.*

COROLLARY 6.2 (to the proof for $n = 2$). *Let B, A_1, A_2 be subgroups of a group G. If $B \subset A_1 \cup A_2$, then $B \subset A_1$ or $B \subset A_2$.*

COROLLARY 6.3 (to the proof). *Let $P_1, \ldots, P_n$ be prime ideals in a commutative ring R of characteristic 0 and let B be a subgroup of an additive group R contained in $\bigcup_{i=1}^{n} P_i$. If every P_i does not contain an integer different from 0, then $B \subset P_i$ for some i.*

Note also the following

Example 6.1. Let $R = Z_2[x, y]/A$, where $A = (x^2, xy, y^2)$, and let $\bar{x} = x+A$, $\bar{y} = y+A$. Then $(\bar{x}, \bar{y}) = (\bar{x}+\bar{y}) \cup (\bar{x}) \cup (\bar{y})$ and $(\bar{x}, \bar{y}) \not\subset (\bar{x}+\bar{y})$, $(\bar{x}, \bar{y}) \not\subset (\bar{x})$, $(\bar{x}, \bar{y}) \not\subset (\bar{y})$.

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