Some remarks on a new pseudo-differential metric

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Abstract. Let $S(\Delta, M)$ be the family of injective holomorphic mappings of the unit disc $\Delta \subset C$ into a complex manifold $M$. Following the method of the Kobayashi–Royden pseudodifferential metric $K_M$ and using the family $S(\Delta, M)$, the author introduces a new pseudodifferential metric $S_M$ on the complex manifold $M$ and studies some basic properties of this metric. The metric $S_M$ has a distance-decreasing property under injective holomorphic mappings and coincides with the Carathéodory–Reiffen and Kobayashi–Royden differential metrics on any bounded symmetric domain. One of the interesting features of $S_M$ is that it is larger than or equal to $K_M$, but differs from $K_M$ on some spaces. For example, $S_M$ defines a complete proper metric on $M = C - \{0\}$, while $K_M$ is trivial. More generally, the $n$-dimensional complex projective space $P_n(C)$ minus $(n+1)$ hyperplanes in general position is hyperbolic with respect to $S$-metric, but the space $P_n(C)$ minus $n$ hyperplanes in general position is not. An immediate consequence of this fact is that every injective holomorphic function in $C^m$ must take every complex number. Other peculiarities of the metric $S_M$ are also given.

1. Introduction. Let $S(\Delta, M)$ be the family of (1-1) holomorphic mappings of the unit disc $\Delta \subset C$ into complex manifold $M$. In this note we introduce, using the family $S$, a new pseudo-differential metric $S_M$ analogous to the Kobayashi–Royden pseudo-differential metric $K_M$ and study some basic properties of this metric. The metric $S_M$ has a distance-decreasing property under (1-1) holomorphic mappings and coincides with the Carathéodory–Reiffen and Kobayashi–Royden differential metrics on any bounded symmetric domain. One of the interesting features of $S_M$ lies in the fact that it is larger than or equal to $K_M$, but differs from $K_M$ on some spaces. For example, $S_M$ defines a complete proper metric on $M = C - \{0\}$, while $K_M$ is trivial. More generally, the $n$-dimensional complex projective space $P_n(C)$ minus $(n+1)$ hyperplanes in general position is hyperbolic with respect to $S$-metric, but the space $P_n(C)$ minus $n$ hyperplanes in general position is not. An immediate consequence of this fact is that there is no (1-1) holomorphic mapping of $C^m$ into $P_n(C)$ minus $(n+1)$ hyperplanes in general position.
2. Preliminaries. Let $M$ be a complex manifold of dimension $n$ and $T(M)$ the complex tangent bundle on $M$. Following Grauert and Reckziegel ([1]), we define a differential metric on $M$ by an upper-semicontinuous function

$$F_M: T(M) \to R^+ \cup \{0\}$$

such that for each $(z, \xi) \in T(M)$

(1) \[ F_M(z, \lambda \xi) = |\lambda| F_M(z, \xi), \quad \lambda \in C, \]

and

(2a) \[ F_M(z, \xi) > 0 \quad \text{for} \quad \xi \neq 0. \]

We say that $F_M$ is a pseudo-differential metric if, instead of condition (2a), it satisfies

(2b) \[ F_M(z, \xi) \geq 0 \quad \text{for} \quad (z, \xi) \in T(M). \]

The Carathéodory-Reiffen and Kobayashi-Royden pseudo-differential metrics are two well-known examples of pseudo-differential metrics.

Let $H(M, A)$ be the family of holomorphic mappings of $M$ into $A$. The Carathéodory-Reiffen metric (CR-metric) is defined by

(3) \[ C_M(z, \xi) = \sup \{ |df(z) \xi| : f \in H(M, A) \ \exists \ f(z) = 0 \} \]

([6]), while the Kobayashi-Royden metric (KR-metric) is

(4a) \[ K_M(z, \xi) = \inf \{ |v| : \exists f \in H(A, M) \ \exists f(0) = z, f'(0)v = \xi \} \]

([7]), where $H(A, M)$ is the family of holomorphic mappings of $A$ into $M$.

In terms of a differential metric $F_M$, the KR-metric may be written in the form:

(4b) \[ K_M(z, \xi) = \inf \left\{ \frac{F_M(z, \xi)}{F_M(f(0), f'(0))} : \exists f \in H(A, M) \ \exists f(0) = z \right\}. \]

If, in (4b), $F_M$ is replaced by $C_M$, then

(5) \[ K_M(z, \xi) \geq C_M(z, \xi), \quad (z, \xi) \in T(M). \]

For,

(6) \[ C_M(f(0), f'(0)) = \sup \{ |dg(f(0)) f'(0)| : g \in H(M, A) \ \exists \ g(z) = 0 \}
\]

\[ = \sup \{ |d(g \circ f)(0)| : g \circ f \in H(A, M) \ \exists (g \circ f)(0) = 0 \} \leq 1, \]

by the classical Schwarz lemma on $g \circ f$.

3. Basic properties of $S$-metric. Let $\mathcal{P}(A, M)$ be the family of (1-1) holomorphic mappings of $A$ into $M$. Analogous to the KR-metric, we define

(7) \[ S_M(z, \xi) = \inf \{ |v| : \exists f \in \mathcal{P}(A, M) \ \exists f(0) = z, f'(0)v = \xi \}
\]

\[ = \inf \left\{ \frac{F_M(z, \xi)}{F_M(f(0), f'(0))} : \exists f \in \mathcal{P}(A, M) \ \exists f(0) = z \right\}. \]
for \((z, \xi) \in T(M)\) and for any differential metric \(F_M\) on \(M\). Using the argument of H. Royden ([7]) we see that \(S_M\) is a pseudo-differential metric. Furthermore, \(S_M\) satisfies the following:

**Proposition 1.** Let \(f : M \to N\) be a \((1-1)\) holomorphic mapping of \(M\) into another complex manifold \(N\). Then

\[
S_N(f(z), df(z) \xi) \leq S_M(z, \xi), \quad (z, \xi) \in T(M).
\]

If, in particular, \(f : M \to N\) is onto, then

\[
S_N(f(z), df(z) \xi) = S_M(z, \xi).
\]

**Proposition 2.** For any complex manifold \(M\),

\[
K_M(z, \xi) \leq S_M(z, \xi), \quad (z, \xi) \in T(M).
\]

Let \(z\) and \(w\) be any two points in \(M\). As usual, we define the integrated metric by

\[
\mathcal{S}_M(z, w) = \inf \int_{\gamma} S_M(z, dz),
\]

where the inf runs over all piecewise regular curves \(\gamma\) joining \(z\) and \(w\) in \(M\).

Following S. Kobayashi ([3]), we define a pseudo-distance \(\tau_M\) on \(M\) as follows: For any two points \(z\) and \(w\) in \(M\), choose a chain of points \(z = z_0, z_1, \ldots, z_k = w\) of \(M\), points \(a_1, \ldots, a_k, b_1, \ldots, b_k\) of \(\Delta\), and functions \(f_1, \ldots, f_k \in \mathcal{S}(\Delta, M)\) such that

\[
f_i(a_i) = z_{i-1} \quad \text{and} \quad f_i(b_i) = z_i, \quad i = 1, 2, \ldots, k.
\]

Then

\[
\tau_M(z, w) = \inf \sum_{i=1}^{k} \mathcal{L}_\Delta(a_i, b_i),
\]

where the inf runs over all possible chains of the points connecting \(z\) and \(w\) in the manner described above and \(\mathcal{L}_\Delta\) is the Poincaré metric on \(\Delta\). Then \(\tau_M\) is an inner metric in the sense of Rinow, i.e., \(\tau_M = \mathcal{S}_M\), as it easily follows from the method of Royden ([7]).

We note that the notion of pseudo-differential metric may as well be defined on a domain \(D\) in any infinite dimensional normed linear space \(X\) exactly in the same way as before.

We state the following theorem in a general normed linear space setting.

**Theorem 1.** Let \(B\) be the unit ball in any normed linear space \(X\) which is homogeneous. Then

\[
S_B(z, \xi) = K_B(z, \xi) = C_B(z, \xi), \quad z \in B, \quad \xi \in X.
\]

In particular, (10) holds for any bounded symmetric domain of finite dimension.

**Proof.** By the Schwarz lemma, for all \(f \in \mathcal{S}(\Delta, B)\) with \(f(0) = 0\),
|f'(0)| \leq 1$, where $| \cdot |$ denotes the norm in $X$. See [2] for example. Hence,

$$|v| = \frac{|\xi|}{|f'(0)|} \geq |\xi|$$

for all $f \in \mathcal{H}(A, B)$ with $f(0) = 0$ and $f'(0)v = \xi$. Therefore,

$$S_B(0, \xi) \geq |\xi|.$$  

The mapping $h(z) = (x/|x|)z, x \in B, x \neq 0$, belongs to $\mathcal{H}(A, B)$ and satisfies:

$h(0) = 0, h'(0) = x/|x|$. Since $|h'(0)v| = |\xi|$ implies $|v| = |\xi|$, we have

$$S_B(0, \xi) = |\xi|. \tag{11}$$

Here $h(z)$ serves as an extremal map. So,

$$S_B(0, \xi) = K_B(0, \xi) = C_B(0, \xi) = |\xi|. \tag{12}$$

Since $B$ is homogeneous, any point of $B$ can be mapped by a (1-1) holomorphic mapping to the origin. By the invariant property of $S_B$, $K_B$ and $C_B$, we have

$$S_B(z, \xi) = K_B(z, \xi) = C_B(z, \xi), \quad z \in B, \xi \in X. \tag{13}$$

Equalities (13) hold for any bounded symmetric domains of finite dimension when we observe that the homogeneous unit ball of any normed linear space includes all bounded symmetric domains of finite dimension including two exceptional cases.

In spite of this similarity of $S_M$ and $K_M$, these two metrics behave differently.

Let $M = C - \{0\}$. Since $C$ is a covering surface of $M$, $K_M(z, \xi) = 0$ for $(z, \xi) \in T(M)$. On the other hand, $S_M$ is not identically zero. In fact, we have

**Theorem 2.** Let $M$ be any domain in $C$ with $M \neq C$. Then

$$S_M(z, \xi) \geq |\xi|/4\delta(z) \tag{14}$$

and the integrated metric $\mathcal{S}_M$ satisfies:

$$\mathcal{S}_M(z_1, z_2) \geq \frac{1}{2} \left| \log \frac{\delta(z_1)}{\delta(z_2)} \right|, \quad z_1, z_2 \in M, \tag{15}$$

where $\delta(z)$ denotes the distance from $z \in M$ to $C \setminus M$. Furthermore, $\mathcal{S}_M$ is a complete metric.

**Proof.** Since $M$ is a proper subset of $C$, $\delta(z)$ is finite for every $z \in M$. It is well known ([5]) that if $f$ is a (1-1) holomorphic mapping of $A$ into $C$, then for $z \in A$,

$$\frac{1}{2}(1 - |z|^2)|f'(z)| \leq \text{dist}(f(z), \partial f(A)). \tag{16}$$

In particular, for $z = 0,$
New pseudo-differential metric

\[ \frac{1}{2} |f'(0)| \leq \text{dist} \left( f(0), \partial f(\Delta) \right). \]

If \( f(0) = z \), then
\[ \frac{1}{|f'(0)|} \geq \frac{1}{4\delta(z)} \]
and, hence,
\[ S_M(z, \xi) = \inf \left\{ \frac{\|\xi\|}{|f'(0)|} : f \in \mathcal{D}(\Delta, M) \exists f(0) = z \right\} \geq \frac{\|\xi\|}{4\delta(z)}. \]

Therefore, the integrated metric satisfies
\[ \mathcal{S}_M(z_1, z_2) = \inf \int_\gamma S_M(z, dz) \geq \inf \int_\gamma \frac{|dz|}{4\delta(z)}, \]
where the \( \inf \) is taken over all piecewise regular curves \( \gamma \) connecting \( z_1 \) and \( z_2 \) in \( M \). Since
\[ |z_1 - z_2| \geq |\delta(z_1) - \delta(z_2)| \]
for any \( z_1 \) and \( z_2 \) in \( M \), we have
\[ \left| \frac{d\delta(z)}{dz} \right| \leq 1. \]

Therefore,
\[ \mathcal{S}_M(z_1, z_2) \geq \frac{1}{2} \inf \left| \frac{d\delta(z)}{\delta(z)} \right| \geq \frac{1}{2} \inf \left| \int_\gamma \frac{d\delta(z)}{\delta(z)} \right| \geq \frac{1}{2} \left| \log \frac{\delta(z_1)}{\delta(z_2)} \right|, \]
which completes the proof.

**Corollary 1.** There is no (1-1) holomorphic mapping of \( C \) into itself which omits at least one point.

Let \( M \) and \( N \) be two complex manifolds. Applying the distance decreasing property of CR- and KR-metrics to the projections \( p: M \times N \to M \) and \( q: M \times N \to N \), we have
\[ C_{M \times N} \geq \max (C_M, C_N), \]
and
\[ K_{M \times N} \geq \max (K_M, K_N). \]

The opposite inequality also holds for the KR-metric. In fact, we prove:

**Lemma 1.** Let \( M \) and \( N \) be two complex manifolds. Then
\[ K_{M \times N} \leq \max (K_M, K_N) \]
and
\[ S_{M \times N} \leq \max (S_M, S_N). \]
Proof. Let $F_M$ and $F_N$ be differential metrics of $M$ and $N$, respectively. Then $\max (F_M, F_N)$ defines a differential metric on $M \times N$. Therefore, for each $(z, w) \in M \times N$.

\begin{equation}
K_{M \times N}((z, w), (\xi, \eta)) = \inf \left\{ \frac{\max \left\{ F_M(z, \xi), F(w, \eta) \right\}}{\max \left\{ F_M(f_0, f'(0)), F_N(g(0), g'(0)) \right\}} : \exists h = (f, g) \in H(\Delta, M \times N) \ni h(0) = (z, w) \right\}
\end{equation}

for $(\xi, \eta) \in T_\xi(M) \times T_\eta(N)$. Let $\varepsilon > 0$. By the definition of $K_M(z, \xi)$ there exists an $f_1 \in H(\Delta, M)$ with $f_1(0) = z$ such that

\begin{equation}
\frac{F_M(z, \xi)}{F_M(f_1(0), f_1'(0))} < K_M(z, \xi) + \varepsilon.
\end{equation}

Similarly, there exists a $g_1 \in H(\Delta, N)$ with $g_1(0) = w$ such that

\begin{equation}
\frac{F_N(w, \eta)}{F_N(g_1(0), g_1'(0))} < K_N(w, \eta) + \varepsilon.
\end{equation}

Therefore, there exists an $h_1 = (f_1, g_1) \in H(\Delta, M \times N)$ with $h_1(0) = (z, w)$ such that

\begin{equation}
K_{M \times N}((z, w), (\xi, \eta)) \leq \frac{\max \left\{ F_M(z, \xi), F_N(w, \eta) \right\}}{\max \left\{ F_M(f_1(0), f_1'(0)), F_N(g_1(0), g_1'(0)) \right\}} \leq \max \left\{ \frac{F_M(z, \xi)}{F_M(f_1(0), f_1'(0))}, \frac{F_N(w, \eta)}{F_N(g_1(0), g_1'(0))} \right\} < \max \left\{ K_M(z, \xi) + \varepsilon, K_N(w, \eta) + \varepsilon \right\}
\end{equation}

for every $\varepsilon > 0$. This proves (25). It is clear that the same argument works for $S_{M \times N}$ when the family $H(\Delta, M \times N)$ is replaced by $H(\Delta, M \times N)$.

It is not likely that the same inequality in Lemma 1 holds for the CR-metric. Combining (24) and (25), we have

**Corollary 2.** Let $M$ and $N$ be two complex manifolds. Then

$$K_{M \times N} = \max \{K_M, K_N\}.$$

The same result is, however, not true for the $S$-metric. It can be shown by the following example: Let $M = C - \{0\}$ and $N = C$. Then $K_M = K_N = S_N = 0$ and $S_M(z, \xi) \geq |\xi|/4\delta(z)$. Therefore,

\begin{equation}
\max (S_M, S_N)((z, w), (\xi, \eta)) \geq |\xi|/4\delta(z).
\end{equation}

But, $S_{M \times N} = 0$. To see this, let $(z, w) \in M \times N$ and $(\xi, \eta) \in C^2$. Let $h_n \in H(\Delta, M \times N)$ be given by

\begin{equation}
h_n(\lambda) = (f_n, g_n)(\lambda) = (z^2 \lambda^2, n\lambda + w), \quad \lambda \in \Delta.
\end{equation}
Then \( h_n(0) = (z, w) \) and \( \hat{h}'_n(0) = (2\pi i z, n) \). From definition,

\[
S_{M \times N}((z, w), (\xi, \eta)) \leq \frac{\sqrt{|\xi|^2 + |\eta|^2}}{\sqrt{|f_n'(0)|^2 + |g_n'(0)|^2}} = \frac{\sqrt{|\xi|^2 + |\eta|^2}}{\sqrt{4\pi^2 |z|^2 + n^2}},
\]

which shows: \( S_{M \times N} = 0 \). But, we can prove:

**Theorem 3.** Let \( M \) and \( N \) be two complex manifolds. Then

\[
\min \{ \max \{K_M, S_N\}, \max \{S_M, K_N\}\} \leq S_{M \times N} \leq \max \{S_M, S_N\}.
\]

**Proof.** The second inequality of (34) was proved in Lemma 1. Therefore, we only need to prove the first inequality. By definition,

\[
S_{M \times N}((z, w), (\xi, \eta)) = \max \left\{ \frac{F_M(z, \xi), F_N(w, \eta)}{\sup \max \{F_M(f(0), f'(0)), F_N(g(0), g'(0))\}} \right\},
\]

where the sup is taken for \((f, g) \in \mathcal{S}(\Delta, M \times N)\) such that \((f, g)(0) = (z, w)\). For \((f, g) \in \mathcal{S}(\Delta, M \times N)\), the following three cases are possible.

1° \( f \in \mathcal{S}(\Delta, M) \) and \( g \in \mathcal{S}(\Delta, N) \);

2° \( f \in \mathcal{S}(\Delta, M) \) and \( g \in H(\Delta, N) \);

3° \( f \in H(\Delta, M) \) and \( g \in \mathcal{S}(\Delta, N) \).

Case 1°. Replacing \( F_M \) by \( S_M \) and \( F_N \) by \( S_N \) in (35), we have

\[
S_{M \times N}((z, w), (\xi, \eta)) \geq \max \{S_M(z, \xi), S_N(w, \eta)\}.
\]

Case 2°. Replacing \( F_M \) by \( S_M \) and \( F_N \) by \( K_N \) in (35), we have

\[
S_{M \times N}((z, w), (\xi, \eta)) \geq \max \{S_M(z, \xi), K_N(z, \eta)\}.
\]

Case 3°. Replacing \( F_M \) by \( K_M \) and \( F_N \) by \( S_N \) in (35), we have

\[
S_{M \times N}((z, w), (\xi, \eta)) \geq \max \{K_M(z, \xi), S_N(w, \eta)\}.
\]

In any case we have the first inequality of (35), since \( S_M \geq K_M \) and \( S_N \geq K_N \).

4. **Hyperbolicity in S-metric.**

**Definition.** Let \( M \) be a complex manifold furnished with a pseudo-differential metric \( F_M \). \( M \) is said to be **hyperbolic** with respect to \( F_M \) if for each \( z_0 \in M \) there exists a neighbourhood \( U(z_0) \) and a constant \( c > 0 \) such that

\[
F_M(z, \xi) \geq c|\xi|^2 \quad \text{for } z \in U(z_0) \text{ and } \xi \in T_z(M).
\]

From Theorem 2, \( M = C - \{0\} \) is hyperbolic with respect to \( S_M \)-metric. Moreover, we have

**Theorem 4.** Let \( \overline{M} = C - \{0\} \times \ldots \times C - \{0\} \) be the Cartesian product of \( n \) copies of \( C - \{0\} \). Then \( \overline{M} \) is hyperbolic with respect to \( S_{\overline{M}} \).

**Proof.** Repeated use of the first inequality of Theorem 3 and inequality (14) imply
\[ S_M(z, \xi) \geq \frac{1}{4} \min \left\{ \frac{|\xi_j|}{\delta(z_j)} : j = 1, 2, \ldots, n \right\}, \]

where \( z = (z_1, \ldots, z_n) \in \tilde{M} \), \( \xi = (\xi_1, \ldots, \xi_n) \in C^n \) and \( \delta(z_j) \) denotes the distance from \( z_j \) to the boundary of \( C - \{0\} \). The hyperbolicity of \( \tilde{M} \) is now clear from (37).

**Theorem 5.** Let \( M \) be the \( n \)-dimensional complex projective space which omits \( (n+1) \) hyperplanes in general position. Then \( M \) is hyperbolic with respect to \( S_M \).

**Proof.** Without loss of any generality we may assume that

\[ M = \pi \{ (z_0, \ldots, z_n) \in C^{n+1} - \{0\} : z_j \neq 0, j = 0, 1, \ldots, n \}, \]

where \( \pi : C^{n+1} - \{0\} \rightarrow P_n(C) \) is the canonical projection. That is,

\[ M = P_n(C) - H_0 \cup H_1 \cup \ldots \cup H_n, \]

where \( H_j = \pi \{ (z_0, \ldots, z_n) \in C^{n+1} - \{0\} : z_j = 0 \}, j = 0, 1, \ldots, n \). Since \( M \) is biholomorphically equivalent to \( \tilde{M} = C - \{0\} \times \ldots \times C - \{0\} \) which is hyperbolic with respect to \( S_M \) by Theorem 4, \( M \) is also hyperbolic with respect to \( S_M \).

In view of \( S_{cm} = 0 \) for all \( m \in N \), we have the generalization of Corollary 1.

**Corollary 3.** There cannot be any \((1,1)\) holomorphic mapping from \( C^m \) into \( P_n(C) \) minus \( (n+1) \) hyperplanes in general position. Equivalently, there is no \((1,1)\) holomorphic mapping of \( C^m \) into \( C^n \) minus \( n \) hyperplanes in general position.

**Theorem 6.** Let \( M = C \times C - \{0\} \times \ldots \times C - \{0\} \) be the Cartesian product of \( C \) with \((n-1)\) copies of \( C - \{0\} \). Then \( M \) is not hyperbolic with respect to \( S_M \).

**Proof.** Let \( z^0 \) be any point in \( M \). We want to know if there exists a neighbourhood \( U(z^0) \) and \( c > 0 \) such that

\[ S_M(z, \xi) \geq c \max_{1 \leq j \leq n} |\xi_j| \]

or

\[ \max_{2 \leq j \leq n} \frac{|\xi_j|}{\delta(z_j)} \geq 4c \max_{1 \leq j \leq n} |\xi_j|. \]

For definiteness, we assume that

\[ \frac{|\xi_2|}{\delta(z_2)} = \max_{2 \leq j \leq n} \frac{|\xi_j|}{\delta(z_j)}. \]

If \( \max_{1 \leq j \leq n} |\xi_j| = |\xi_2| \), then
\[ \frac{|\xi_2|}{\delta(z_2)} \geq 4c |\xi_2|. \]

or

(42)
\[ \frac{1}{\delta(z_2)} \geq 4c. \]

Since \( \delta(z_2) < \infty \), there exists a neighbourhood \( U(z^0) \) of \( z^0 \) and \( c > 0 \) so that (42) holds for all \( z \in U(z^0) \). If \( \max |\xi_j| = |\xi_k| \) for \( k \neq 2 \), then

\[ \frac{|\xi_2|}{\delta(z_2)} \geq 4c |\xi_k| \]

or

(43)
\[ \frac{|\xi_2|}{|\xi_k|} > 4c \delta(z_2). \]

Keeping \( \xi_2 \) finite and taking \( |\xi_k| \) sufficiently large we may let \( \epsilon > |\xi_2|/|\xi_k| \) for any given \( \epsilon > 0 \). From (43), \( c < \epsilon/4\delta(z_2) \) and there is no \( c > 0 \) which satisfies (40).

**Corollary 4.** The \( n \) dimensional complex projective space \( P_n(C) \) minus \( n \) hyperplanes in general position is not hyperbolic with respect to \( S \)-metric.

In this connection, we show by an example that the hyperbolicity in the sense of Wong ([8]) differs from that of Royden ([7]).

**Example.** Let \( M = C - \{0, 1\} \) and \( N = C. \) Since \( K_C = 0 \), from Corollary 2,

(44)
\[ K_{M \times N}((z, w), (\xi, \eta)) = K_M(z, \xi) > 0 \]

for \( (z, w) \in M \times N \) and \( (\xi, \eta) \in C^2. \) Therefore, \( M \times N \) is hyperbolic with respect to KR-metric in the sense of Wong, but it fails to be hyperbolic in the sense of Royden.

To show this, we may assume that \( M = \Delta \) and \( N = C. \) Then

(45)
\[ K_{\Delta \times C}((z, w), (\xi, \eta)) = K_{\Delta}(z, \xi) = \frac{|\xi|}{1-|z|^2} \]

for \( (z, w) \in \Delta \times C \) and \( (\xi, \eta) \in C^2. \) Let \( (z_0, w_0) \) be any point in \( \Delta \times C. \) We need to check if there exists a neighbourhood \( U \) of \( (z_0, w_0) \) and a constant \( c > 0 \) such that

(46)
\[ \frac{|\xi|}{1-|z|^2} \geq n \max (|\xi|, |\eta|) \quad \text{for} \quad (\xi, \eta) \in C^2 \quad \text{and} \quad (z, w) \in U. \]

If \( |\xi| \geq |\eta| \), it clearly holds, while if \( |\eta| > |\xi| \),

(47)
\[ \frac{|\xi|}{|\eta|} \geq c (1-|z_0|^2). \]

Since \( \eta \in C \) may be chosen arbitrarily for each finite \( \xi, \epsilon = |\xi|/|\eta| \) can be made
arbitrarily small. Since \( z_0 \in A \), there exists an \( r \) such that \( |z_0| \leq r < 1 \).

From (47),
\[
|c| < \frac{\varepsilon}{1 - |z_0|^2} < \frac{\varepsilon}{1 - r^2}.
\]
This shows that there is no positive constant \( c \) which satisfies (46).

Concerning the continuity of \( S \)-metric we can only prove:

**Theorem 7.** Let \( M \) be a Riemann surface. Then \( S_M(z,\xi) \) is continuous on \( T(M) \).

**Proof.** If \( M \) is of genus 0, then \( S_M = 0 \) and, hence, it is continuous. If \( M \) is of genus greater than 0, then \( S_M \) defines a proper metric in which case
\[
S_M(z,\xi) = \inf \left\{ \frac{|\xi|}{|f^*(0)|} : \exists f \in \mathcal{F}(A, M) \ni f(0) = z \right\}.
\]

Let \( g(\zeta) = \frac{f(\zeta) - f(0)}{f'(0)} \). Then \( g \) is univalent in \( A \) and satisfies: \( g(0) = 0 \), \( g'(0) = 1 \). Since the class of univalent functions \( g \), normalized as above, forms a normal family, so does the class \( \mathcal{F}(A, M) \). Let \( (z, \xi) \in T(M) \) and let \( (z_k, \xi_k) \to (z, \xi) \). Let \( \varepsilon > 0 \). Suppose that \( f_k \in \mathcal{F}(A, M) \) is a sequence such that \( f_k(0) = z_k \), \( f'_k(0) = \xi_k \) and

\[
|\xi_k| < S_M(z_k, \xi_k) + \varepsilon
\]

for sufficiently large \( k \). Since \( f_k \in \mathcal{F}(A, M) \) is a normal family, there exists a subsequence \( \{f_{k_j}\} \) such that \( f_{k_j}(z) \to f(z) \) uniformly in \( A \). The limit function \( f \) is either univalent or constant in \( A \). We may assume that \( \xi \neq 0 \). Then \( f \) can not be constant. Therefore, \( f \) is in \( \mathcal{F}(A, M) \) and satisfies: \( f(0) = z \) and \( f'(0)v = \xi \). Thus,
\[
S_M(z, \xi) = \frac{|\xi|}{|f'(0)|} < S_M(z_k, \xi_k) + \varepsilon.
\]

This proves the continuity of \( S_M(z, \xi) \), since \( S_M \) is always upper-semi-continuous.

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**References**


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