INDEPENDENCE WITH RESPECT TO FAMILIES
OF CHARACTERS

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Introduction. The set of functions

\[ f_n(x) = nx \pmod{1}, \quad n = 1, 2, \ldots, \]

defined on \([0, 1]\) and regarded as a subspace of the Tychonoff cube \([0, 1]^{[0,1]}\) becomes dense when projected (by restriction) into \([0, 1]^E\), where \(E \subseteq [0, 1]\) and \(E \cup \{1\}\) is independent over the rationals. This is a consequence of the Kronecker multidimensional theorem, according to which if \(1, x_1, \ldots, x_r\) are independent over the rationals and \(U_1, \ldots, U_r\) are intervals on \([0, 1]\), then there exists an integer \(n\) such that

\[ nx_1 \in U_1, \ldots, nx_r \in U_r \pmod{1}. \]

It was observed by Priestley [7] that if \(E\) is, in addition, of positive outer measure, then the sequence \(f_n|E, n = 1, 2, \ldots,\) contains no nontrivial convergent subsequences. Indeed, if \(n_0 < n_1 < \ldots\) is a sequence of integers, then according to the Hardy–Littlewood theorem (or an even stronger theorem of Weyl asserting a.e. uniform distribution of \(n_k x \pmod{1}\)) the sequence \(n_0 x, n_1 x, \ldots \pmod{1}\) is dense in \([0, 1]\) for almost all \(x\), and clearly such an \(x\) can be found in \(E\), \(E\) being not of measure zero. This gives a strengthened version of the known Hewitt–Marczewski–Pondiczery theorem, namely the existence of a countable dense subset having no nontrivial convergent sequences in the Tychonoff cube of weight continuum.

A question arises whether subsequences of the sequence (1) have an analogous property, i.e., whether for any subsequence

\[ f_{n_k}(x) = n_k x \pmod{1}, \quad n_0 < n_1 < \ldots, \]

of (1) there exists an uncountable subset \(E\) of \([0, 1]\) such that the set of functions \(f_{n_k}|E, k = 0, 1, \ldots,\) is dense in \([0, 1]^E\). In particular, this concerns
the sequences
\[ f_{p^k}(x) = p^k x \pmod{1}, \quad k = 0, 1, \ldots, \]
where \( p \) is an integer, \( p \geq 2 \).

A more natural setting of the problem is to consider, instead of (1), the functions
\[ f_n(z) = z^n, \quad n = 0, \pm 1, \pm 2, \ldots, \]
defined on the unit circle \( T = \{ z : |z| = 1 \} \), i.e., the continuous characters of \( T \).

It is known from the work of Ryll-Nardzewski [3] that the set of limit functions of a sufficiently thin subsequence of (2) is homeomorphic to the remainder \( \beta \omega \setminus \omega \) in the Čech–Stone compactification \( \beta \omega \) of the set \( \omega \) of natural numbers. This is true, in particular, for any sequence
\[ f_{p^k}(z) = z^{p^k}, \quad k = 0, 1, \ldots, \]
where \( p \) is an integer, \( p \geq 2 \). More generally, it was shown by Strzelecki [11] that this is the case if the sequence \( n_0 < n_1 < \ldots \) is lacunary, i.e., \( n_{k+1}/n_k \geq \gamma > 1 \).

The whole sequence (2) is far from having the property considered above since it is dense in the set \( b\mathbb{Z} \) of all characters of \( T \) and, in particular, dense in itself. In fact, the closure of the sequence (2) in \( T^T \) coincides with \( b\mathbb{Z} \) (see, e.g., [4], Corollary (26.16)). Moreover, \( b\mathbb{Z} \) is nowhere dense in \( T^T \) and it is well known that all the characters except the functions \( f_n(z) = z^n \) are non-measurable. The corresponding nonmeasurable functions on the unit interval were originally investigated by Sierpiński [9], [10].

In Section 3 it will be proved that for each set
\[ \Phi = \{ f_{n_k} : n_0 < n_1 < \ldots \} \]
of continuous characters of \( T \) there exists a subset \( E \) of \( T \) of cardinality continuum such that the set
\[ \{ f_{n_k} \mid E : k = 0, 1, \ldots \} \]
is dense in \( T^E \). This means that the closure \( \bar{\Phi} \) of \( \Phi \) in \( T^T \) (or, equivalently, in \( b\mathbb{Z} \)) is projected onto \( T^E \) (by restriction). In other words, each function \( f : E \to T \) extends to a character in \( \bar{\Phi} \subset b\mathbb{Z} \).

Subsets of \( T \) having this property with respect to a given set \( \Phi \) of continuous characters of \( T \) will be called \( \Phi \)-independent. The \( \Phi \)-independent sets are the smaller the thinner the sets \( \Phi \). The problem of the existence of \( \Phi \)-independent sets which are “large” in the sense of measure or topology is dealt with in Section 3. For
\[ \Phi_p = \{ f_{p^k} : k = 0, 1, \ldots \} \]
the existence of \( \Phi_p \)-independent sets of cardinality continuum can be obtain-
ed directly by modifying the theorem of Fichtenholz and Kantorovitch [1], but our proof presented in Section 3 is based on a different method and works for arbitrary infinite \( \Phi \), answering the question stated at the beginning.

1. Comments concerning the position of \( \Phi_p \) in \( b \mathbb{Z} \). We shall describe in some detail the position, in \( b \mathbb{Z} \), of the set \( \Phi_p \) consisting of the functions (3) as well as the set \( \Phi_p^* = \Phi_p \setminus \Phi_p \) of limit functions of (3).

Denoting by \( \chi \), the character of \( \mathbb{Z} \) given by

\[
\chi_t(n) = \exp(2\pi i nt), \quad n \in \mathbb{Z},
\]

consider the set \( B_p \) of those functions in \( b \mathbb{Z} \) which are equal to 1 on all characters \( \chi_{j/l} \), \( l \geq 0, j \in \mathbb{Z} \). The set \( B_p \) has the structure of a closed subgroup of \( b \mathbb{Z} \) (in fact, the annihilator of the \( l \)-th roots of unity, \( l \geq 0 \)) and coincides with the intersection of the sets \( B_{p,j,l}, \ l \geq 0, \ j \in \mathbb{Z}, \) consisting of those functions equal to one on a single character \( \chi_{j/l} \). The function \( f_n \) from (2) is in \( B_{p,j,l} \) iff \( \exp(2\pi i jn/p) = 1 \). This implies that \( f_{p,k} \in B_{p,j,l} \) iff \( k \geq l \).

Thus, the set \( \Phi_p^* \) is contained in \( B_p \) and \( B_p \cap \Phi_p = \emptyset \). Since \( B_p \) is a group, this clearly implies that \( B_p \) is nowhere dense in \( b \mathbb{Z} \).

The set \( B = \bigcap B_p \) of all those functions from \( b \mathbb{Z} \) which are equal to 1 on all characters (4) with \( t \) rational is the connected component of the neutral element of \( b \mathbb{Z} \) (see, e.g., [4]).

If there exists a prime factor of \( q \) which is not a factor of \( p \), then the set \( B_q \) does not contain any limit functions of the sequence (3), i.e. \( B_q \cap \Phi_p^* = \emptyset \).

To see this, project \( T^t \) onto the \( \chi_{1/q} \)-axis. Then all functions in \( B_q \) are projected into 1, while the functions \( f_{p,k} \) are projected into the set of \( q \)-th roots of unity from which 1 is removed (the fraction \( p/k/q \) cannot be an integer).

Since \( B \subset B_p \cap B_q \) for any two integers \( p \) and \( q \), as a consequence of the above observation we have

\[
B \cap \Phi_p^* = \emptyset
\]

for all integers \( p, p \geq 2 \).

We recall that a continuous map \( g: M \to N \) is called irreducible if \( g(A) = N \) implies \( A = M \) for each closed subset of \( A \) of \( M \). We note that if \( E \) is such that \( \Phi_p^* \) projects onto \( T^E \) (by restriction), then the projection \( \pi: \Phi_p^* \to T^E \) is far from being irreducible. Indeed, suppose \( A \) is a closed subset of \( \Phi_p^* \) such that \( \pi(A) = T^E \) and \( \pi|A: A \to T^E \) is irreducible (the existence follows from the Zorn lemma). Since \( T^E \) is separable, i.e., it has a countable dense subset, the set \( A \) is also separable. Thus \( A \) is a separable subspace of the space \( \Phi_p^* \) which is nowhere separable as a homeomorphic copy of \( \beta \omega \setminus \omega \). Thus, \( A \) is nowhere dense in \( \Phi_p^* \).
2. Definition of $\Phi$-independent sets. Let $G$ be an abelian group. A subset $E$ of $G$ is called independent if for any distinct elements $x_1, \ldots, x_n$ of $E$ and any integers $k_1, \ldots, k_n$ the equality

$$x_1^{k_1} \cdots x_n^{k_n} = 1$$

can only occur if $k_1 = \ldots = k_n = 0$. This definition implies that an independent set contains no elements of finite order, which makes our notion slightly different from the independence considered in [4] (but consistent with [2], V). Note that $\{x\}$ is independent in $G$ iff $x$ has infinite order.

The independent sets can be characterized by the following extension property:

(i) $E$ is independent iff any function $f : E \rightarrow T$ extends to a character on $G$.

Indeed, if $E$ is independent, then the formula

$$\chi(x_1^{k_1} \cdots x_n^{k_n}) = f(x_1)^{k_1} \cdots f(x_n)^{k_n}$$

extends $f$ to a character $\chi$ on the group generated by $E$, and next $\chi$ can be extended to a character on $G$ (see, e.g., [4], 24.12). The converse is clear.

The following observation will also be useful:

(ii) If $E$ is independent and $a \neq 1$, then $\text{card}(E \cap Ea) \leq 1$.

To see this suppose $x = ua$, $y = va$, $x \neq y$; $x$, $y$, $u$, $v \in E$. This implies $xy^{-1}vu^{-1} = 1$. If $u \neq y$ and $v \neq x$, then the four elements are distinct and we arrive at a contradiction. If, e.g., $u = y$, then $xuv^{-2} = 1$, which is also impossible.

Throughout the rest of the paper we consider a locally compact abelian group (LCA group) $G$. By $\hat{G}$ we denote the dual LCA group of all continuous characters on $G$ and by $b\hat{G}$ the group of all characters on $G$ endowed with the compact topology of pointwise convergence inherited from $T^G$. It is well known that $\hat{G}$ is dense in $b\hat{G}$, the Bohr compactification of $\hat{G}$ ([4], 26.16).

We say that a subset $\Phi$ of $\hat{G}$ is unbounded if it is not relatively compact for the locally compact (Pontryagin) topology on $\hat{G}$.

**Definition.** Let $\Phi \subset \hat{G}$ be unbounded. A subset $E$ of $G$ is said to be $\Phi$-independent if any function $f : E \rightarrow T$ extends to a character contained in the (pointwise) closure of $\Phi$ in $b\hat{G}$.

A subset of a $\Phi$-independent set is $\Phi$-independent. Also, if $\Psi \subset \Phi \subset \hat{G}$, then every $\Psi$-independent set is $\Phi$-independent. In particular, any $\Phi$-independent set is independent (see (i)). The converse fails in general.

**Example 1.** Consider

$$\Phi_2 = \{z^{2^n} : n = 0, 1, \ldots\} \subset \hat{T}$$
and let
\[ z = \exp(2\pi i \sum_{n=1}^{\infty} 2^{-n^2}). \]
The singleton \( \{z\} \) is not \( \Phi_2 \)-independent since \( \text{Arg} \varphi(z) < 5\pi/4 \) for all \( \varphi \in \Phi_2 \). On the other hand, \( \sum 2^{-n^2} \) is irrational, so \( z \) is of infinite order, hence independent in \( T \).

The following example shows that the union of the \( \Phi_p \)-independent sets (\( p \) fixed) in the unit circle is not too large.

**Example.** Let \( p \geq 2 \) be an integer. Then there exists a Borel subset \( A_p \) of \( T \) such that
(a) the Hausdorff dimension of \( A_p \) is equal to one,
(b) \( A_p \) is disjoint with every \( \Phi_p \)-independent set.

In fact, let \( A_p \) be the set of all \( z \in T \) such that the sequence \( z^{p^n} \), \( n \geq 0 \), is not dense in \( T \). Given \( \varepsilon > 0 \) we fix \( k \geq 1 \) satisfying the condition
\[ \log (p^k - 1)/\log p^k > 1 - \varepsilon \]
and define \( D_p \) to be the set of all \( z = e^{2\pi it} \) such that \( 0 < t < 1 \) and there is no \( 2k \)-block of zeroes in the expansion of \( t \) to the base \( p \),
\[ t = \sum_{j=1}^{\infty} t_j/p^j, \quad 0 \leq t_j < p, \]
where in the case of two different representations the one with, say, infinitely many zeroes is chosen. Clearly, \( D_p \subset A_p \). We let \( C_p \) be the set of all numbers \( e^{2\pi it} \), \( 0 < t < 1 \), containing no block of zeroes of the form
\[ t_{nk+1} \cdots t_{(n+1)k} \quad (n \geq 0). \]
We have \( C_p \subset D_p \) and \( C_p \) can be identified with the set of all numbers in the unit interval in whose \( p^k \)-expansion the digit 0 does not occur. Therefore
\[ \dim C_p = \log (p^k - 1)/\log p^k > 1 - \varepsilon \]
and, consequently, \( \dim A_p = 1 \).

The sets \( C_p \) belong to the class of Rajchman's \((H)\)-sets, which play an important role in the problem of uniqueness of trigonometric expansions (Rajchman [8]).

**3. Existence of \( \Phi \)-independent sets.** Our aim is to obtain uncountable \( \Phi \)-independent sets. The two methods presented below parallel the construction of independent sets of transitive points in dynamical systems [5]. The first is an application of Mycielski's independence theorem in topological relational structures [6]. Using his theorem, Mycielski obtained algebraically independent sets (a notion slightly different from our \( \hat{G} \)-independence in the case of
abelian groups) of cardinality continuum in any connected locally compact
group $G$, card $G > 1$ ([6], p. 144). Our case of $\Phi$-independence seems to be
less straightforward because of the lack of a purely algebraic condition for $\Phi$-
independence. The second method is based on transfinte induction and
resembles the well-known construction of the Bernstein set.

In order to obtain uncountable $\Phi$-independent sets for each $\Phi$, we
restrict the class of groups by imposing the following condition on $G$:

$(\ast)$ For every $m \neq 0$ the mapping $f_m: \hat{G} \rightarrow \hat{G}$ defined by $f_m(\chi) = \chi^m$ is
continuous at infinity.

An equivalent wording is the following:

$(\ast\ast)$ For every $m \neq 0$ the set $f_m^{-1}(K)$ is compact whenever $K$ is a
compact subset of $\hat{G}$.

If $G$ is compact, then $\hat{G}$ is discrete, so “compact” means “finite” in $(\ast\ast)$. Therefore, for any compact $G$, $(\ast)$ is equivalent to

$(\ast\ast\ast)$ For every $m \neq 0$ there are only finitely many characters of order $m$
in $\hat{G}$.

The class of groups satisfying $(\ast)$ is vast. First note that every compact
monothetic group satisfies $(\ast)$. In fact, $\hat{G}$ is a subgroup of $T$ ([4], 24.32), so
$(\ast\ast\ast)$ holds. If $G$ is any connected compact abelian group, then $\hat{G}$ is torsion
free ([4], 24.25), whence $G$ satisfies $(\ast\ast\ast)$. It is also easy to see that the class
of LCA groups satisfying $(\ast\ast)$ is closed under direct products. Since $(\ast)$
clearly holds for $\mathbb{R}^n$, we infer by the structural theorem ([4], 9.14) that any
connected LCA group satisfies $(\ast)$.

Note that if $G$ is compact, then $(\ast\ast\ast)$ is necessary for the existence of a
nonempty $\Phi$-independent set for all unbounded $\Phi$. In fact, suppose

$$\Phi(m) = \{\chi \in \hat{G}: \chi^m = 1\}$$

is infinite. Since

$$\text{card} \{\chi(x): \chi \in \Phi(m)\} \leq m$$

for each $x \in G$, the singleton $\{x\}$ is never $\Phi(m)$-independent.

We shall prove that $(\ast\ast\ast)$ is also sufficient for the existence of uncount-
able $\Phi$-independent sets (Theorems 1 and 2).

From now on, $G$ is an LCA group satisfying $(\ast)$ and $\Phi$ an unbounded
subset of $\hat{G}$. We denote by $dx$ the Haar measure on $G$. The measure of a set
$A \subset G$ will be denoted by $|A|$.

The assertion of the following lemma is reminiscent of the mixing
conditions considered in [5].

**Lemma 1.** If $f \in C(T)$ and $g \in L^1(G)$, then

$$\lim_{x \to \infty} \int f(\chi(x))g(x)dx = \int f \int g.$$
**Proof.** For a fixed $g$, the integral on the left can be viewed as a bounded linear functional $F_x$ on $C(T)$. Since $\|F_x\| \leq \|g\|_1$, it suffices to prove
\[
\lim_{x \to \infty} F_x(f) = \int f \hat{g}
\]
for the linearly dense set of characters $f(z) = z^m, m \in \mathbb{Z}$. If $m = 0$, then clearly
\[
F_x(f) = \int g = \int f \hat{g}.
\]
If $m \neq 0$, then
\[
F_x(f) = \hat{g}(\chi^m),
\]
where $\hat{g} \in C_0(G)$ is the Fourier transform of $g$. Since
\[
\lim_{x \to \infty} \chi^m = \infty
\]
by (*), we obtain
\[
\lim_{x \to \infty} F_x(f) = 0 = \int f \hat{g}
\]
as required.

The following lemma can be viewed as an extension of the Hardy–Littlewood theorem:

**Lemma 2.** Let $U$ be an open neighborhood in $T$. The set
\[
F(U) = \{x \in G: (\forall \chi \in \Phi) \chi(x) \notin U\}
\]
is nowhere dense and of Haar measure zero.

**Proof.** Clearly, $F(U)$ is closed. Suppose $V \subset F(U), 0 < |V| < \infty$. We let $g = 1_U$, the indicator of $V$, and $0 \leq f \leq 1_U, 0 \neq f \in C(T)$. Then
\[
\int f(\chi(x))g(x) = 0
\]
for all $\chi \in \Phi$. On the other hand,
\[
\lim_{x \to \infty} \int f(\chi(x))g(x) \, dx = |V| \int f > 0
\]
by Lemma 1. Since we may choose $\chi \in \Phi, \chi \to \infty$, this is a contradiction.

**Lemma 3.** Let $n \geq 1$. The set of all $(x_1, \ldots, x_n) \in G^n$ such that some $f: [x_1, \ldots, x_n] \to T$ cannot be extended to a character contained in the pointwise closure of $\Phi$ is of the first category in $G^n$.

**Proof.** It suffices to prove that for any neighborhoods $U_1, \ldots, U_n$ from a countable basis of $T$ the set
\[
F(U_1, \ldots, U_n)
\]
\[
= \{(x_1, \ldots, x_n) \in G^n: (\forall \chi \in \Phi)(\chi(x_1), \ldots, \chi(x_n)) \notin U_1 \times \cdots \times U_n\}
\]
is nowhere dense. Suppose, to the contrary, that there are open sets
$V_1, \ldots, V_n$ in $G$ such that

$$0 < |V_j| < \infty \quad \text{and} \quad V_1 \times \ldots \times V_n \subset F(U_1, \ldots, U_n).$$

We let $g_j = 1_{V_j}$ and $0 \leq f_j \leq 1_{U_j}$, $0 \neq f_j \in C(T)$, $j = 1, \ldots, n$. Now argue as in Lemma 2. If $\chi \in \Phi$, then

$$\prod_{j=1}^{n} f_j(\chi(x_j))g_j(x_j) \equiv 0,$$

while

$$\int \prod_{j=1}^{n} f_j(\chi(x_j))g_j(x_j) \, dx_1 \ldots dx_n = \prod_{j=1}^{n} \int f_j(\chi(x))g_j(x) \, dx \to \prod_{j=1}^{n} |V_j| \int f_j > 0$$

as $\chi \to \infty$, a contradiction.

Now we are in a position to apply Mycielski's theorem on independent sets in topological relation structures [6]. First define for each $n \geq 1$ an $n$-ary relation $R_n$ on $G$ by letting $(x_1, \ldots, x_n) \in R_n$ iff some function

$$f: \{x_1, \ldots, x_n\} \to T$$

does not extend to a character belonging to the pointwise closure of $\Phi$. By Lemma 3, $R_n$ is of the first category in $G^n$. Moreover, the relational structure $\mathcal{R} = (G, \{R_1, R_2, \ldots\})$ is closed under identification of variables since

$$(x_1, \ldots, x_i, x_j, x_{i+1}, \ldots, x_n) \in R_{n+1} \iff (x_1, \ldots, x_n) \in R_n$$

for any $j$ ($1 \leq j \leq n$, $n \geq 1$). Besides, $G$ is dense in itself if an unbounded $\Phi$ exists. Consequently, $\mathcal{R}$ satisfies the assumptions of Mycielski's theorem [6] asserting the existence of an independent set of cardinality continuum (which can be chosen to be a dense countable union of Cantor sets if $G$ is second countable). Since independence in $\mathcal{R}$ coincides with our independence defined in Section 2, we have

**Theorem 1.** Let $G$ be an LCA group satisfying (*) Then for every unbounded $\Phi \subset \hat{G}$ there exists a $\Phi$-independent set of cardinality continuum. If, in addition, $G$ is second countable, then the independent set can be chosen to be a dense countable union of Cantor sets.

**Remark.** If $\Phi^k, k = 1, 2, \ldots$, is a sequence of unbounded subsets of $\hat{G}$, then by a slight modification of the above argument we may obtain a set of cardinality continuum which is $\Phi^k$-independent for all $k \geq 1$ simultaneously. Indeed, it suffices to consider the relational structure $(G, \{R^k_n: n \geq 1, k \geq 1\})$, where the relations $R^1_n, R^2_n, \ldots$ correspond to the family $\Phi^k$. In particular, there exists a Borel uncountable set $E \subset T$, $\Phi_p$-independent for all $p \geq 2$. An analogous remark applies to Theorem 2 below.

Another method of constructing uncountable $\Phi$-independent sets is
based on transfinite induction. For the remaining part of the paper, \( G \) is a compact abelian group satisfying (\(*\)) and \( \Phi \) is an infinite subset of \( \hat{G} \).

First note that if \( E \) is \( \Phi \)-independent, then it is either of measure zero or nonmeasurable. Indeed, the translations \( Ea, a \in G \), are almost disjoint by (ii), so the inner measure of \( E \) must be zero. Our aim is now to produce nonmeasurable \( \Phi \)-independent sets. This will be achieved under additional conditions, e.g., \( G \) metrizable plus the continuum hypothesis.

Denote by \( \mathcal{B} \), \( \mathcal{M} \), and \( \mathcal{N} \) the Borel \( \sigma \)-algebra, the ideal of first category sets, and the ideal of sets of Haar measure zero in \( G \), respectively. If \( \mathcal{I} = \mathcal{M} \) or \( \mathcal{I} = \mathcal{N} \), then we consider the cardinal number

\[
\text{cov} \mathcal{I} = \min \{ \text{card} \mathcal{F} : \mathcal{F} \subseteq \mathcal{I}, (\exists C \in \mathcal{B} \setminus \mathcal{I}) C \subseteq \bigcup_{D \in \mathcal{F}} D \}.
\]

Clearly, \( \aleph_1 \leq \text{cov} \mathcal{I} \leq \text{card} G \). Also note that if \( G \) is metrizable, then by the well-known isomorphism theorems \( \text{cov} \mathcal{I} \) is independent of \( G \).

**Theorem 2.** Let \( G \) be a compact abelian group satisfying (\(*\)) and \( \Phi \subset \hat{G} \) be infinite. For any family

\[
\mathcal{C} \subset \mathcal{B} \setminus (\mathcal{M} \cap \mathcal{N}) \quad \text{with} \quad \text{card} \mathcal{C} = \min (\text{cov} \mathcal{M}, \text{cov} \mathcal{N})
\]

there exists an uncountable \( \Phi \)-independent set \( E \) such that \( E \cap C \neq \emptyset \) for each \( C \in \mathcal{C} \).

**Proof.** Put \( \gamma = \min (\text{cov} \mathcal{M}, \text{cov} \mathcal{N}) \) and order \( \mathcal{C} \) as \( \{ C_\alpha : \alpha < \gamma \} \). We define a transfinite sequence \( x_\alpha, \alpha < \gamma \), such that \( x_\alpha \in C_\alpha \) and \( \{ x_\alpha : \alpha < \beta \} \) is \( \Phi \)-independent for each \( \beta < \gamma \). To carry out the induction suppose \( \{ x_\alpha : \alpha < \beta \} \) is \( \Phi \)-independent for some \( \beta < \gamma \). We shall find an element \( x_\beta \in C_\beta \) such that \( \{ x_\alpha : \alpha \leq \beta \} \) is still \( \Phi \)-independent. First choose any \( 0 \leq \alpha_1 < \ldots < \alpha_n < \beta \) (\( n \) finite) and neighborhoods \( U_1, \ldots, U_n \) from a countable basis in \( T \). By inductive assumption, the set

\[
\Psi = \{ \chi \in \Phi : (\chi(x_{\alpha_1}), \ldots, \chi(x_{\alpha_n})) \in U_1 \times \ldots \times U_n \}
\]

is infinite (we let \( \Psi = \Phi \) if \( \beta = 0 \)). By Lemma 2, the set

\[
Y(\alpha_1, \ldots, \alpha_n, U_1, \ldots, U_n) = \{ y \in G : \{ y \} \text{ is } \Psi \text{-independent} \}
\]

is a dense \( G_\delta \) of full measure. Since \( C_\beta \subset \mathcal{B} \setminus \mathcal{M} \) or \( C_\beta \subset \mathcal{B} \setminus \mathcal{N} \) and there are less than \( \gamma \) sets \( Y \), the intersection

\[
C_\beta \cap \bigcap Y(\alpha_1, \ldots, \alpha_n, U_1, \ldots, U_n)
\]

is nonempty. Choose any \( x_\beta \) in this intersection. It is clear that for any \( 0 \leq \alpha_1 < \ldots < \alpha_n < \beta \) and neighborhoods \( U_1, \ldots, U_n, U_{n+1} \) we have

\[
x_\beta \in Y(\alpha_1, \ldots, \alpha_n, U_1, \ldots, U_n),
\]
so the set
\[ \{ \chi \in \Phi : (\chi(x_\alpha), \ldots, \chi(x_\beta)) \in U_1 \times \cdots \times U_n \times U_{n+1} \} \]
is nonempty. This proves that \( x_\beta \notin \{ x_\alpha : \alpha < \beta \} \) and that \( \{ x_\alpha : \alpha \leq \beta \} \) is \( \Phi \)-independent.

**Corollary.** If \( G \) is compact abelian and metrizable, \( \Phi \subset \hat{G} \) is infinite, and \( \text{cov} \, \mathcal{M} = \text{cov} \, \mathcal{V} = 2^{\aleph_0} \), then there exists a \( \Phi \)-independent set \( E \) such that \( E \cap C \neq \emptyset \) whenever \( C \in \mathfrak{B}(\mathcal{M} \cap \mathcal{V}) \). In particular, \( E \) is nonmeasurable and does not belong to the Baire \( \sigma \)-algebra.

Note that \( \text{cov} \, \mathcal{M} = \text{cov} \, \mathcal{V} = 2^{\aleph_0} \) is implied by the continuum hypothesis.

**REFERENCES**


