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A CRITICAL GROWTH RATE FOR HARMONIC AND SUBHARMONIC FUNCTIONS IN THE OPEN BALL IN R^n

BY

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1. Introduction and main results. This paper presents certain technique of investigation of properties of harmonic or subharmonic functions h in the open unit ball in R^n ($n \geqslant 2$) that have restricted growth in the sense that

$$\sup\{h(x) \mid |x| = r\} \leqslant Ck(r) \quad (C = \text{const}) \text{ for some fixed function } k.$$

We assume that k is defined in $[0, 1)$ and continuous, positive, nondecreasing and unbounded therein.

In the following $B_n = \{x \in R^n \mid |x| < 1\}$ ($|x|$ denotes the Euclidean norm of x), n is always an integer greater than or equal to 2. P_n denotes the Poisson kernel:

$$P_n(x, y) = \frac{1-|x|^2}{|x-y|^n}, \quad |x| < 1, \ |y| = 1.$$

Here are the main results of the paper.

THEOREM 1. *Let* $\int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr = +\infty$ *and let* $M > 0$, $\varepsilon \in (0, 1)$, $e \in \partial B_n$ *be arbitrary. Then there exists a nonnegative, continuous function h in $\bar{B}_n$ that is harmonic in B_n and satisfies the following conditions*:

(a) $|h(x) - MP_n(x, e)| \leqslant \varepsilon$, *whenever* $|x| < 1-\varepsilon$

and

(b) $h(x) - MP_n(x, e) \leqslant \varepsilon k(|x|)$, *for every* $x \in B_n$.

Since every nonpositive harmonic function h in B_n can be expressed in the form:

$$h(x) = \int_{\partial B_n} [-P_n(x, y)]\, \nu(dy), \quad x \in B_n,$$

where ν is some nonnegative Borel measure on ∂B_n, the following corollary easily follows.

COROLLARY. *If* $\int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr = +\infty$ *and* u *is a bounded from above harmonic function in* B_n, *then for every* $\varepsilon \in (0, 1)$ *there exists a continuous function* h *in* $\bar{B}_n$ *that is harmonic in* B_n *and satisfies the following conditions*:

(a′) $|h(x)+u(x)| \leqslant \varepsilon$, *if* $|x| < 1-\varepsilon$,

and

(b′) $h(x)+u(x) \leqslant \varepsilon \cdot k(|x|)$, *whenever* $x \in B_n$.

Theorem 2 says a little more than a converse to Theorem 1. Theorem 2′ is a "subharmonic" version of Theorem 2.

THEOREM 2 (THEOREM 2′). *If* $\int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr < +\infty$, *then*

(a) *for every* $C > 0$

$$\lim_{M \to +\infty} \sup_{e \in \partial B_n, C' > 0} \{h(0) - MP_n(0, e) |\ h \text{ (sub-)harmonic in } B_n,\ h(x) \leqslant C'k(|x|)$$
$$\text{and } h(x) - MP_n(x, e) \leqslant Ck(|x|) \text{ for every } x \in B_n\} = -\infty,$$

(b) *for every* $C > 0$ *and* $M > 0$

$$\lim_{\varepsilon \to 0^+} \sup_{e \in \partial B_n, C' > 0} \{h(0) - MP_n(0, e) |\ h \text{ (sub-)harmonic in } B_n,\ h(x) - MP_n(x, e) < \varepsilon$$
$$\text{if } |x| < 1-\varepsilon,\ h(x) - MP_n(x, e) < Ck(|x|) \text{ and } h(x) < C'k(|x|) \text{ for all } x \in B_n\}$$
$$= -M.$$

From the celebrated Riesz's representation theorem for subharmonic functions it follows that for every function v ($v \not\equiv -\infty$) subharmonic in B_n:

$$\int_{B_n} (1-|x|)\,\mu(dx) < \infty$$

(where $\mu = \Delta v$) iff v has a harmonic majorant. Theorem 3 says that if v is a subharmonic function in B_n, $v \not\equiv -\infty$ and

$$\int_0^1 \left(\frac{\max(\{0\} \cup \{v(x) |\ |x| = r\})}{1-r}\right)^{1/n} dr < +\infty,$$

then

$$\int_D (1-|x|)\,\mu(dx) < +\infty \qquad (\mu = \Delta v)$$

for every region $D \subset B_n$ that touches ∂B_n nontangentially. Theorem 3 says also that the condition

$$\int_0^1 \left(\frac{\max(\{0\} \cup \{v(x) |\ |x| = r\})}{1-r}\right)^{1/n} dr < +\infty$$

cannot be relaxed.

THEOREM 3. (a) *If* $\int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr < +\infty$ *and* $v \not\equiv 0$ *is a function subharmonic in* B_n *such that* $v(x) \leqslant k(|x|)$ *for every* $x \in B_n$, *then for every* $\alpha > 0$ *and* $e \in \partial B_n$:

$$\int_{G(e,\alpha)} (1-|x|)\,\mu(dx) < +\infty,$$

where $G(e, \alpha) = \{|x| \in B_n|\ 1-|x| > \alpha\varphi(x, e)\}$ *and* $\mu = \Delta v$, $\varphi(x, e)$ *defined in part* 2.

(b) *If* $\int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr = +\infty$, *then for every* $e \in B_n$ *there exists a function subharmonic in* B_n *such that* $v(x) \leqslant k(|x|)$ *for every* $x \in B_n$, *and*

$$\int_{(0,e)} (1-|x|)\,\mu(dx) = \infty,$$

where $\mu = \Delta v$ *and* $(0, e) = \{te|\ t \in (0, 1)\}$, *and* v *is harmonic in* $B_n \setminus (0, e)$. v *can be chosen in such a manner that* Δv *is a discrete measure with all its atoms having masses of integer multiplicities of any prescribed positive number.*

For $n = 2$ these theorems are closely related to the problem of description of closed ideals in topological algebras

$$A_\infty(k) = \{f|\ f \text{ holomorphic function in } \{z \in \mathbb{C}|\ |z| < 1\},$$
$$|f(z)| \leqslant C_f \exp(a_f k(|z|))\}$$

(for topology on $A_\infty(k)$ see [5] or [7]). To each theorem proved here there corresponds, via the correspondence $f \to \log|f|$, a theorem about $A_\infty(k)$.

Theorem 1 implies, by a standard argument, that if $\int_0^1 \sqrt{\frac{k(r)}{1-r}}\,dr = \infty$, then for every nonvanishing, bounded function $f \in A_\infty(k)$ there exists a sequence of polynomials $\{W_n\}$ such that $W_n f$ tends to $f_0 \equiv 1$ in $A_\infty(k)$, as $n \to \infty$. In fact, the convergence implied by Theorem 1 is stronger than that. This result is related to Apresjan's result from [1] saying that, under some regularity conditions imposed on k, the condition $\int_0^1 \sqrt{\frac{k(r)}{1-r}}\,dr = +\infty$ implies that every closed ideal in $A_\infty(k)$ is local.

Theorem 2′ implies that if $\int_0^1 \sqrt{\frac{k(r)}{1-r}}\,dr < \infty$, then the closed ideal generated in $A_\infty(k)$ by the function $f(z) = \exp\left(\frac{z+1}{z-1}\right)$ is not equal to the whole of $A_\infty(k)$. This result was, under some regularity conditions (which we do not need in our proof), derived by Nikol'skiĭ in [7] from a Beurling's result ([2]). Our proof is much simpler than that in [7].

Theorem 3 implies a result due to Hayman and Korenblum ([3]), saying that the condition $\int_0^1 \sqrt{\frac{k(r)}{1-r}}\,dr < \infty$ implies that if $f \in A_\infty(k)$ and S is any Stolz angle in $\{z|\ |z|<1\}$, then $\sum(1-|z_j|) < \infty$, where the summation extends over zeros z_j of f lying in S; and that if $\int_0^1 \sqrt{\frac{k(r)}{1-r}}\,dr = \infty$, then there is a function $f \in A_\infty(k)$ such that $\sum(1-\varrho_j) = \infty$, where ϱ_j's are positive zeros of f. It seems that our proof of this result is simpler than that of [3], since it does not use a pretty advanced conformal mapping technique employed in [3]. Proofs presented here are elementary and they use nothing but basic properties of Poisson kernel and of Green function of the ball. Most of them are listed in lemmas: 2, 3, 10 and 11. Lemma 6 that relates Poisson kernel to the function k is interesting by itself. The idea of this lemma was communicated to me by Professor B. Korenblum.

2. Notation and basic lemmas. The angle $\varphi(x, y)$ between two nonzero elements $x, y \in \boldsymbol{R}^n$ is defined, as usual, to be the unique number $\varphi(x, y)$ for which $0 \leqslant \varphi(x, y) \leqslant \pi$ and $\cos\varphi(x, y) = \frac{(x, y)}{|x|\,|y|}$, where (x, y) is the inner product of x and y, and $|x| = \sqrt{(x, x)}$ is the Euclidean norm of x. σ_n denotes the normalized surface measure on ∂B_n, i.e. the unique rotation invariant Borel measure σ on ∂B_n such that $\sigma(\partial B_n) = 1$.

For $x \neq y$ the function:

$$g_n(x, y) = \begin{cases} \log\dfrac{|x-y'|\ |y|}{|x-y|} & \text{if} \quad y \neq 0,\ n = 2, \\ \log(1/|x|) & \text{if} \quad y = 0,\ n = 2, \\ |x-y|^{2-n} - \{|y|\ |x-y'|\}^{2-n} & \text{if} \quad y \neq 0,\ n \geqslant 3, \\ |x|^{2-n} - 1 & \text{if} \quad y = 0,\ n \geqslant 3, \end{cases}$$

where $y' = y/|y|^2$, is the Green function of B_n.

For k introduced in the introduction we set

$$J_n(k) \stackrel{\mathrm{df}}{=} \int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr.$$

We list below a few elementary lemmas that we will need in the paper. These lemmas are trivial; therefore we omit the proofs.

LEMMA 1. *There are positive constants a_n and b_n depending only on n such that for every nonnegative measurable function g defined in $[0, \pi/2]$ and for every $e \in \partial B_n$ we have*:

$$a_n \int_0^{\pi/2} g(\varphi)\varphi^{n-2}\,d\varphi \leqslant \int_{\{y \in \partial B_n|\ \varphi(y,e) \leqslant \pi/2\}} g(\varphi(y, e))\sigma_n(dy) \leqslant b_n \int_0^{\pi/2} g(\varphi)\varphi^{n-2}\,d\varphi.$$

Notice that integrals in the above lemma may have infinite values.

LEMMA 2. *Let* $e \in \partial B_n$ *and* $0 < 1-|x| \leqslant \frac{\varphi(x, e)}{2\pi}$. *Then*

$$2^{-n-1} P_n(x, e) \leqslant (1-|x|)[\varphi(x, e)]^{-n} \leqslant 2^n P_n(x, e).$$

LEMMA 3. *Let* $e \in \partial B_n$ *and* $0 < 1-|x| \leqslant \varphi(x, e)/2\pi$. *Then*

$$\int_{\{y \in \partial B_n | \varphi(y,e) \leqslant \varphi(x,e)\}} P_n(x, y)\sigma_n(dy) \geqslant c_n,$$

where c_n *is some positive constant depending only on* n.

LEMMA 4. *For every* $\alpha \leqslant 1/2\pi$ *there exists a number* $d_\alpha > 0$ *such that for any* $e \in \partial B_n$:

$$|x-y| \geqslant d_\alpha |x-e|,$$

whenever $1-|y| \geqslant \alpha\varphi(y, e)$ *and* $0 < 1-|x| \leqslant \alpha\varphi(x, e)/2$.

In our considerations a function that we denote by ϱ_M is crucial. It is defined as follows. Let for every number $M \geqslant k(0)\pi^n$:

$$\varphi_M(\varrho) = \begin{cases} \left[\dfrac{M(1-\varrho)}{k(\varrho)}\right]^{1/n} & \text{if} \quad 0 \leqslant \varrho < 1, \\ 0 & \text{if} \quad \varrho = 1. \end{cases}$$

φ_M is a "1-1", continuous mapping from $[0, 1]$ onto $\left[0, \left[\frac{M}{k(0)}\right]^{1/n}\right]$. Let $\varrho_M = \varphi_M^{-1}$. Notice that

$$(*) \qquad \varphi^n k(\varrho_M(\varphi)) = M(1-\varrho_M(\varphi)) \quad \text{for} \quad \varphi \in (0, \pi].$$

3. Proofs of Theorems 1, 2 and 2′. We start with two lemmas that relate the integral $J_n(k)$ to a behavior of $k \circ \varrho_M \circ \varphi(\cdot, e)$.

LEMMA 5. *If* $J_n(k) < \infty$, *then for any* $M \geqslant k(0)\pi^n$

$$\lim_{\varphi \to 0^+} (1-\varrho_M(\varphi))^{n-1} k(\varrho_M(\varphi)) = 0.$$

Proof. Let $M \geqslant k(0)\pi^n$. From the fact that $k(r)$ tends to ∞ as $r \to 1^-$ and from $(*)$ it follows that $\varrho_M(\varphi) \to 1^-$ as $\varphi \to 0^+$. Hence

$$0 \leqslant \varliminf_{\varphi \to 0^+} (1-\varrho_M(\varphi))^{n-1} k(\varrho_M(\varphi)) \leqslant \varlimsup_{\varphi \to 0^+} (1-\varrho_M(\varphi))^{n-1} k(\varrho_M(\varphi))$$
$$\leqslant \varlimsup_{\varrho \to 1^-} (1-\varrho)^{n-1} k(\varrho).$$

The last limit is equal to 0 whenever $J_n(k) < +\infty$ since

$$0 \leqslant [(1-\varrho)^{n-1} k(\varrho)]^{1/n} = \frac{n-1}{n} \int_\varrho^1 (1-r)^{-1/n} dr\, k(\varrho)^{1/n} \leqslant \frac{n-1}{n} \int_\varrho^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr.$$

LEMMA 6. *For any* $M \geqslant k(0)\pi^n$, *and any* $e \in \partial B_n$:

$$\int k\big(\varrho_M(\varphi(y, e))\big)\sigma_n(dy) \leqslant 2b_n M^{(n-1)/n} J_n(k).$$

Moreover, if $k(r) \leqslant C(1-r)^{1-n}$ *for some* $C > 0$ *and all* $r \in [0, 1)$, *and*

$$\int_{B_n} k\big(\varrho_M(\varphi(y, e))\big)\sigma_n(dy) < \infty \quad \textit{for some } M,$$

then

$$J_n(k) < \infty.$$

Proof. The first part of the lemma is trivial if $J_n(k) = \infty$. Assume then that $J_n(k) < \infty$. Then for any $M \geqslant k(0)\pi^n$ and any $e \in \partial B_n$ we have:

$$\int k\big(\varrho_M(\varphi\,(y, e))\big)\sigma_n(dy) \leqslant 2b_n \int_0^{\pi/2} k\big(\varrho_M(\varphi)\big)\varphi^{n-2}\,d\varphi$$

$$= 2b_n \int_0^{\pi/2} M\big(1-\varrho_M(\varphi)\big)d(-1/\varphi) = 2b_n\left\{-M\frac{1-\varrho_M(\varphi)}{\varphi}\bigg|_0^{\pi/2} + M\int_0^{\pi/2}\frac{1}{\varphi}\,d\big(\varrho_M(\varphi)\big)\right\}$$

$$= 2b_n\left\{-M^{(n-1)/n}\big(1-\varrho_M(\varphi)\big)^{n-1} k\big(\varrho_M(\varphi)\big)^{1/n}\bigg|_0^{\pi/2} + M^{(n-1)/n}\int_{\varrho_M(\pi/2)}^{1}\left(\frac{k(r)}{1-r}\right)^{1/n} dr\right\}$$

$$\leqslant 2b_n M^{(n-1)/n} J_n(k).$$

The inequalities follow from Lemma 1, (*), and Lemma 5, respectively.

To prove the second part of the lemma assume that $k(r) \leqslant C(1-r)^{1-n}$ for some $C > 0$ and

$$\int_{\partial B_n} k\big(\varrho_M(\varphi(y, e))\big)\sigma_n(dy) < \infty \quad \text{for some } M \geqslant k(0)\pi^n,\ e \in \partial B_n.$$

By Lemma 1 the latter implies that

$$\int_0^{\pi/2} k\big(\varrho_M(\varphi)\big)\varphi^{n-2}\,d\varphi < \infty.$$

Now, let $\delta \in (0, \pi/2)$ be arbitrary. A similar calculation as in the first part of the proof yields:

$$\int_\delta^{\pi/2} k\big(\varrho_M(\varphi)\big)\varphi^{n-2}\,d\varphi$$

$$= -M^{(n-1)/n}\Big[\big(1-\varrho_M(\varphi)\big)^{n-1} k\big(\varrho_M(\varphi)\big)\Big]^{1/n}\Big|_\delta^{\pi/2} + M^{(n-1)/n}\int_{\varrho_M(\pi/2)}^{\varrho_M(\delta)}\left(\frac{k(r)}{1-r}\right)^{1/n} dr.$$

Hence by (*)

$$\int_{\varrho_M(\pi/2)}^{\varrho_M(\delta)} \left(\frac{k(r)}{1-r}\right)^{1/n} dr \leqslant M^{(1-n)/n} \int_0^{\pi/2} k(\varrho_M(\varphi))\varphi^{n-2}d\varphi + 2C^{1/n}.$$

Since $\varrho_M(\delta) \to 1^-$ as $\delta \to 0^+$, letting $\delta \to 0^+$ in the above inequality we obtain

$$\int_{\varrho_M(\pi/2)}^{1} \left(\frac{k(r)}{1-r}\right)^{1/n} dr < \infty .$$

And since $\left(k(r)/(1-r)\right)^{1/n}$ is an increasing function of r:

$$J_n(k) = \int_0^1 \left(\frac{k(r)}{1-r}\right)^{1/n} dr < \infty .$$

The next lemma shows how the condition $k(r) \leqslant C(1-r)^{1-n}$ can be detoured.

LEMMA 7. *If for every* $r \in (0, 1)$ *there is* $r' \in (r, 1)$ *such that* $k(r') = (1-r')^{1-n}$, *then*

$$\int_0^1 \left(\frac{\min\{k(r), (1-r)^{1-n}\}}{1-r}\right)^{1/n} dr = \infty .$$

Proof. Let $0 = r_0 < r_1 < r_2 < \ldots < 1$ be such that $k(r_j) = (1-r_j)^{1-n}$, $j = 1, 2, \ldots$, and $\lim_j r_j = 1$. Let us define a function p: $[0, 1) \to [0, +\infty)$ by the following formula:

$$p(r) = \begin{cases} 0 & \text{if} \quad r \in [0, r_1), \\ 1/(1-r_j) & \text{if} \quad r \in [r_j, r_{j+1}), \quad j = 1, 2, \ldots \end{cases}$$

Then

$$\left(\frac{\min\{k(r), (1-r)^{1-n}\}}{1-r}\right)^{1/n} = \min\left\{\left(\frac{k(r)}{1-r}\right)^{1/n}, \frac{1}{1-r}\right\} \geqslant p(r) \geqslant 0,$$

since $\left(k(r)/(1-r)\right)^{1/n}$ and $1/(1-r)$ are both increasing functions of r.

Therefore

$$\int_0^1 \left(\frac{\min\{k(r), (1-r)^{1-n}\}}{1-r}\right)^{1/n} dr \geqslant \int_0^1 p(r)\,dr = \sum_{j=1}^{\infty} \left(1 - \frac{1-r_{j+1}}{1-r_j}\right).$$

But the last series is divergent, since

$$\prod_{j=1}^{\infty} \left(\frac{1-r_{j+1}}{1-r_j}\right) = \lim_j \left(\frac{1-r_j}{1-r_1}\right) = 0.$$

Proof of Theorem 1. We will first prove the following:

CLAIM. *If* $J_n(k) = \infty$ *and* $M \geqslant (2\pi)^n$, *then for each* $e \in \partial B_n$ *and each* $\varepsilon > 0$

there exists a function h that is defined and continuous in $\bar{B}_n$ and harmonic in B_n, such that

(a′) $|h(x)-MP_n(x, e)| \leqslant \varepsilon$ *whenever* $|x| \leqslant 1-\varepsilon$

and

(b′) $h(x)-MP_n(x, e) \leqslant (2^{2n+1}+1)k(|x|)$ *for each* $x \in B_n$.

It is easy to see that the above claim implies Theorem 1.

Proof of the Claim. By Lemma 7, we may and do assume that $k(r) \leqslant (1-r)^{1-n}$. Let M, ε, e be as in the formulation of the Claim. Observe that, by (*), the conditions: $M \geqslant (2\pi)^n$ and $k(r) \leqslant (1-r)^{1-n}$ imply that $1-\varrho_M(\varphi) \leqslant \varphi/2\pi$ for each $\varphi \in [0, \pi]$. Let $\delta > 0$ be such a number that for each $x \in B_n$ such that $|x| \leqslant 1-\varepsilon$ and each $y \in \partial B_n$ such that $\varphi(y, e) < \delta$ we have

$$|P_n(x, y)-P_n(x, e)| < \varepsilon/2M.$$

Let $\eta > 0$ be such a number that

$$\eta P_n(x, e) < \varepsilon/2, \quad \text{whenever} \quad |x| \leqslant 1-\varepsilon.$$

Let us construct a sequence $\{\varphi_j\}_{j=0}^{\infty}$ inductively putting first $\varphi_0 = \pi$, $\varphi_1 = \delta$, and if $\varphi_0, \ldots, \varphi_{j-1}$ are determined, let φ_j be such a number that $0 < \varphi_j \leqslant \varphi_{j-1}/2$ and $|P_n(x, e)-P_n(x, y)| \leqslant k(0)/M$ for each $y \in \partial B_n$ and $x \in B_n$ such that $\varphi(y, e) \leqslant \varphi_j$ and $\varphi(x, e) \geqslant \varphi_{j-1}$. Let f be a continuous nonnegative function defined in $[0, \pi]$, such that

(i) $f(\varphi) = 0$ if $\varphi \geqslant \delta$,

(ii) $\int_{\partial B_n} f(\varphi(y, e))\sigma_n(dy) = M-\eta$,

(iii) $\int_{\varphi_j \leqslant \varphi(y,e) \leqslant \varphi_{j-1}} f(\varphi(y, e))\sigma_n(dy) \leqslant \eta/2^{j+2}$, $j = 1, 2, \ldots$,

(iv) $f(\varphi) \leqslant \dfrac{\eta}{M \cdot 2^{2n+1}} k(\varrho_M(\varphi))$ for each $\varphi \in (0, \pi]$.

Existence of such an f follows from nonintegrability of $k(\varrho_M(\varphi(\cdot, e)))$ around e with respect to σ_n. That is assured by Lemma 6.

Let v be the Poisson integral of $f(\varphi(\cdot, e))$, i.e.

$$v(x) = \begin{cases} \int_{\partial B_n} f(\varphi(y, e))P_n(x, y)\sigma_n(dy) & \text{if} \quad x \in B_n, \\ f(\varphi(x, y)) & \text{if} \quad x \in \partial B_n. \end{cases}$$

It is obvious that v is a nonnegative function, continuous in $\bar{B}_n$ and harmonic in B_n. We will verify that v satisfies (a′) and (b′).

Check of (a′). Let us assume that $|x| \leqslant 1-\varepsilon$. Then, by (i) and (ii),

$$\begin{aligned}
|v(x)-P_n(x, e)| &= |v(x)-(M-\eta)P_n(x, e)-\eta P_n(x, e)| \\
&\leqslant \int_{\varphi(y,e)\leqslant\delta} f(\varphi(y, e))|P_n(x, y)-P_n(x, e)|\,\sigma_n(dy)+\eta P_n(x, e) \\
&\leqslant (M-\eta)\frac{\varepsilon}{2M}+\frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Check of (b′). First we will show that for each $x \in B_n$ such that $|x| = \varrho_M(\varphi(x, e))$ we have $v(x)-MP_n(x, e) \leqslant k(0)$. Let $|x| = \varrho_M(\varphi(x, e))$, $x \neq e$, and let $j_0 = \min\{j|\ \varphi_j \leqslant \varphi(x, e)\}$. Then

$$\begin{aligned}
&v(x)-M\cdot P_n(x, e) \\
&= \int_{\varphi(y,e)\geqslant\varphi(x,e)/2} f(\varphi(y, e))P_n(x, y)\,\sigma_n(dy)-(\eta/2)P_n(x, e) \\
&+ \int_{\varphi_{j_0+1}\leqslant\varphi(y,e)\leqslant\varphi(x,e)/2} f(\varphi(y, e))P_n(x, y)\,\sigma_n(dy)-(\eta/2)P_n(x, e) \\
&+ \int_{0\leqslant\varphi(y,e)\leqslant\varphi_{j_0+1}} f(\varphi(y, e))P_n(x, y)\,\sigma_n(dy)-(M-\eta)P_n(x, e) \\
&= A_1+A_2+A_3.
\end{aligned}$$

But by (iv), (∗), and Lemma 2,

$$\begin{aligned}
A_1 &\leqslant \sup\{f(\varphi(y, e))|\ \varphi(y, e) \geqslant \varphi(x, e)/2\}-(\eta/2)P_n(x, e) \\
&\leqslant \frac{\eta}{2^{2n+1}M}\, k\left(\varrho_M\left(\frac{\varphi(x, e)}{2}\right)\right)-(\eta/2)P_n(x, e) \\
&\leqslant \frac{\eta}{2^{n+1}}\,\frac{1-\varrho_M(\varphi(x, e))}{[\varphi(x, e)]^n}-(\eta/2)P_n(x, e) \leqslant 0.
\end{aligned}$$

Also, by (iii),

$$\begin{aligned}
A_2 \leqslant& \int_{\varphi_{j_0+1}\leqslant\varphi(y,e)\leqslant\varphi(x,e)/2} f(\varphi(y, e))\,\sigma_n(dy) \\
&\times\max\{P_n(x, y)|\ y \in \partial B_n,\ \varphi(y, e) \leqslant \varphi(x, e)/2\}-(\eta/2)P_n(x, e) \\
\leqslant& \int_{\varphi_{j_0+1}\leqslant\varphi(y,2)\leqslant\varphi_{j_0-1}} f(\varphi(y, e))\,\sigma_n(dy)\cdot 2^n P_n(x, e)-(\eta/2)P_n(x, e) \leqslant 0
\end{aligned}$$

and

$$A_3 \leqslant \int_{0 \leqslant \varphi(y,e) \leqslant \varphi_{j_0+1}} f(\varphi(y, e))[P_n(x, y) - P_n(x, e)]\sigma_n(dy)$$
$$- \int_{\varphi(y,e) \geqslant \varphi_{j_0+1}} f(\varphi(y, e))\sigma_n(dy) \cdot P_n(x, e)$$
$$\leqslant \sup\{P_n(w, y) - P_n(w, e)|\ y \in \partial B_n,\ \varphi(y, e) \leqslant \varphi_{j_0+1},\ w \in B_n,\ \varphi(w, e) \geqslant \varphi_{j_0}\}$$
$$\times \int_{\varphi(y,e) \leqslant \varphi_{j_0+1}} f(\varphi(y, e))\sigma_n(dy)$$
$$\leqslant \frac{k(0)}{M}(M - \eta) < k(0).$$

Hence for each $x \neq e$ such that $|x| = \varrho_M(\varphi(x, e))$ we have

$$v(x) - MP_n(x, e) \leqslant k(0).$$

Since $v(x) - MP_n(x, e)$ is a bounded from above continuous function of x in $\{x \in \boldsymbol{R}^n|\ |x| \leqslant \varrho_M(\varphi(x, e))\} \setminus \{e\}$, this and the maximum principle give:

$$v(x) - MP_n(x, e) \leqslant k(0) \leqslant k(|x|) \leqslant (2^{2n+1} + 1)k(|x|),$$

whenever $|x| \leqslant \varrho_M(\varphi(x, e))$, $x \neq e$.

It remains to show that $v(x) - MP_n(x, e) \leqslant (2^{2n+1} + 1)k(|x|)$, whenever $1 > |x| \geqslant \varrho_M(\varphi(x, e))$. Let then x be such that $1 > |x| \geqslant \varrho_M(\varphi(x, e))$. Then we have

$$v(x) - MP_n(x, e) \leqslant v(x)$$
$$= \Big(\int_{\varphi(y,e) \geqslant \varphi(x,e)/2} + \int_{\varphi(y,e) \leqslant \varphi(x,e)/2}\Big) f(\varphi(y, e))P_n(x, y)\sigma_n(dy)$$
$$\leqslant \max\{f(\varphi(y, e))|\ \varphi(y, e) \geqslant \varphi(x, e)/2\}$$
$$+ (M - \eta)\max\{P_n(x, y)|\ \varphi(y, e) \leqslant \varphi(x, e)/2\}$$
$$\leqslant \frac{\eta}{2^{2n+1}M} k(\varrho_M(\varphi(x, e)/2)) + (M - \eta) \cdot 2^n P_n(x, e)$$
$$\leqslant \frac{\eta}{2^{2n+1}M} 2^n k(\varrho_M(\varphi(x, e))) + 2^n \frac{M - \eta}{M} \cdot 2^{n+1} k(\varrho_M(\varphi(x, e)))$$
$$\leqslant (2^{2n+1} + 1)k(\varrho_M(\varphi(x, e))) \leqslant (2^{2n+1} + 1)k(|x|).$$

LEMMA 8. *Let* $J_n(k) < \infty$, $M \geqslant \pi^n k(0)$, $e \in \partial B_n$. *Let* v *be a subharmonic function in* B_n *such that* $v(x) \leqslant Ck(|x|)$ *for some constant* $C > 0$ *and each* $x \in B_n$, *and* $v(x) \leqslant 0$ *whenever* $|x| = \varrho_M(\varphi(x, e))$, $x \neq e$. *Then* $v(0) \leqslant 0$.

Proof. Let the assumption of the lemma be satisfied. Let $r \in (\varrho_M(\pi/2), 1)$ be arbitrary. Let w_r denote the harmonic function in $\{x \in \boldsymbol{R}^n|\ |x| < r\}$ whose boundary values are equal to 0 in $\{x \in \boldsymbol{R}^n|\ |x| = r,\ \varphi(x, e) > \varphi_M(r)\}$ and to

$Ck(r)$ in $\{x \in \mathbf{R}^n \mid |x| = r,\ \varphi(x, e) < \varphi_M(r)\}$, where $\varphi_M(r)$ is such a number that $\varrho_M(\varphi_M(r)) = r$. Then from the maximum principle it follows that $v(0) \leqslant w_r(0)$. Therefore

$$v(0) \leqslant w_r(0) = Ck(r)\sigma_n(\{y \in \partial B_n \mid \varphi(y, e) \leqslant \varphi_M(r)\})$$
$$\leqslant C \cdot \int_{\varphi(y,e) \leqslant \varphi_M(r)} k(\varrho_M(\varphi(y, e)))\sigma_n(dy).$$

By Lemma 6, the last term tends to 0, as $r \to 1^-$, since $\varphi_M(r) \to 0^+$, as $r \to 1^-$. Hence $v(0) \leqslant 0$.

Theorem 2 and Theorem 2′ are an immediate consequence of the following lemma.

LEMMA 9. *If $J_n(k) < \infty$, then for each M, $M \geqslant (2\pi)^n \sup_r [k(r)(1-r)^{1-n}]$, each $e \in \partial B_n$ and each function v subharmonic in B_n such that $v(x) \leqslant Ck(|x|)$ and $v(x) - MP_n(x, e) \leqslant k(|x|)$ for some $C > 0$ and all $x \in B_n$, we have the inequality*

$$v(0) \leqslant \frac{2b_n(2^{n+1}+1)}{c_n} J_n(k) M^{(n-1)/n},$$

where b_n and c_n are constants from Lemmas 2 *and* 3.

Notice that, by Lemma 7, $J_n(k) < \infty$ implies $\sup_r [k(r)(1-r)^{1-n}] < \infty$.

Proof. Let k, M, e, v be such as in the formulation of the lemma. Note that the inequality $M \geqslant (2\pi)^n \sup_r [k(r)(1-r)^{1-n}]$ implies $1 - \varrho_M(\varphi) \leqslant \varphi/2\pi$ for each $\varphi \in [0, \pi]$. Since $v(x) - MP_n(x, e) \leqslant k(|x|)$ for each $x \neq e$ such that $|x| = \varrho_M(\varphi(x, e))$ we have

$$v(x) \leqslant k(|x|) + 2^{n+1} M(1 - |x|)[\varphi(x, e)]^{-n} = (2^{n+1}+1)k(|x|).$$

Let for each $x \in B_n$

$$w(x) = \frac{2^{n+1}+1}{c_n} \int_{\partial B_n} k(\varrho_M(\varphi(y, e))) \cdot P_n(x, y)\sigma_n(dy).$$

If $|x| = \varrho_M(\varphi(x, e))$, $x \neq e$, then, by Lemma 3, we have

$$w(x) \geqslant \frac{2^{n+1}+1}{c_n} \inf\{k(\varrho_M(\varphi(y, e))) \mid \varphi(y, e) \leqslant \varphi(x, e)\} \int_{\varphi(y,e) \leqslant \varphi(x,e)} P_n(x, y)\sigma_n(dy)$$
$$\geqslant (2^{n+1}+1)k(|x|).$$

Therefore for x as above: $v(x) - w(x) \leqslant 0$. Since $v(x) - w(x) \leqslant v(x) \leqslant Ck(|x|)$ in B_n, Lemma 8 implies: $v(0) - w(0) \leqslant 0$. But, by Lemma 6,

$$w(0) = \frac{2^{n+1}+1}{c_n} \int k(\varrho_M(\varphi(y, e)))\sigma_n(dy) = \frac{2^{n+1}+1}{c_n} \cdot 2b_n M^{(n-1)/n} J_n(k).$$

4. Proof of Theorem 3. Before we start the proof we give two lemmas that we will need in the proof.

LEMMA 10. *For each* $x,\ y \in B_n,\ x \neq y$:

$$g_n(x, y) \leqslant 4m_n \frac{(1-|x|)(1-|y|)}{|x-y|^n},$$

where

$$m_n = \begin{cases} n-2 & \text{if} \quad n \geqslant 3, \\ 1 & \text{if} \quad n = 2. \end{cases}$$

Proof. Trivial.

LEMMA 11. *Let* $e \in \partial B_n$ *and* $x \in B_n$. *Then*

$$\lim_{\substack{y \to e \\ y \in B_n}} (1-|y|^2)^{-1} g_n(x, y) = (m_n/2) P_n(x, e),$$

where m_n *is the constant from Lemma* 10. *The convergence is uniform on sets of the form* $B_n \setminus V$, *for each neighborhood* V *of* e.

Proof. We skip the proof, since it is easy and the lemma itself is well known.

Proof of part (a) of Theorem 3. Let $J_n(k) < \infty$ and let v be as in the formulation of the part (a) of Theorem 3.

Suppose that

$$\int_{G(e,\alpha)} (1-|y|)\,\mu(dy) = \infty \qquad \text{for some } e \in \partial B_n \text{ and } \alpha > 0 \ (\mu = \Delta v).$$

Since $G(e, \alpha_1) \subset G(e, \alpha_2)$, if $\alpha_1 > \alpha_2$, we may and do assume that $\alpha < 1/2\pi$. We assume also that $v(0) \neq -\infty$. For, if this is not the case we can replace v by v_1,

$$v_1(x) = v(x) + \frac{1}{m_n \gamma_n} \int_{|y| \leqslant 1/4} g_n(x, y)\,\mu(dy), \qquad \text{where} \qquad \gamma_n = 2\pi^{n/2}/\Gamma(n/2).$$

v_1 is subharmonic in B_n, $\Delta v_1 = \Delta v$ outside $\{y|\ |y| \leqslant 1/4\}$, $v_1(0) \neq -\infty$ and $v_1(x) \leqslant C' k(|x|)$ for some constant C' and each $x \in B_n$.

By our supposition

$$M(r) = \int_{G(e,\alpha) \cap \{y|\ |y| \leqslant r\}} (1-|y|)\,\mu(dy)$$

tends to $+\infty$ as $r \to 1^-$. Hence $1-\varrho_{M(r)}(\varphi) \leqslant (\alpha/2)\varphi$ for each $\varphi \in [0, \pi]$ whenever r is close enough to 1, say, greater than some $r_0 < 1$. In the remaining part of the proof we assume that $r > r_0$.

Let us define a function u_r by the following formula:

$$u_r(x) = \frac{1}{m_n \gamma_n} \int_{G(e,\alpha) \cap \{y|\ |y| \leqslant r\}} g_n(x, y)\,\mu(dy), \qquad x \in B_n,$$

γ_n as above. Notice that $v + u_r$ is subharmonic in B_n and $v(x) + u_r(x) \leqslant Ck(|x|)$

for some constant C and all $x \in B_n$. Now, let us define a harmonic function w_r in B_n letting

$$w_r(x) = \left(1+\frac{4}{\gamma_n d_\alpha^n}\right)\frac{1}{c_n}\int k\left(\varrho_{M(r)}(\varphi(y, e))\right) P_n(x, y)\sigma_n(dy), \qquad x \in B_n.$$

If $1 > |x| \geqslant \varrho_{M(r)}(\varphi(x, e))$, then using Lemmas 10 and 4,

$$u_r(x) \leqslant \int_{G(e,\alpha)\cap\{y|\,|y|\leqslant r\}} \frac{4(1-|x|)(1-|y|)}{\gamma_n |x-y|^n}\mu(dy)$$

$$\leqslant \frac{4M(r)(1-|x|)}{\gamma_n d_\alpha^n |x-e|^n} \leqslant \frac{4}{\gamma_n d_\alpha^n} k(|x|).$$

By Lemma 3, for the same x we have

$$w_r(x) \geqslant \left(1+\frac{4}{\gamma_n d_\alpha^n}\right) k(|x|).$$

Hence for each such an x: $v(x)+u_r(x)-w_r(x) \leqslant 0$. Therefore, by Lemma 8 (since $v(x)+u_r(x) \leqslant Ck(|x|)$ and $w_r(x) \geqslant 0$ for each $x \in B_n$): $v(0)+u_r(0) - w_r(0) \leqslant 0$. But since $g_n(0, y) \geqslant m_n(1-|y|)$, we have $u_r(0) \geqslant M(r)/\gamma_n$ and, by Lemma 6

$$w_r(0) \leqslant \left(1+\frac{4}{\gamma_n d_\alpha^n}\right)\cdot\frac{2b_n}{c_n}\cdot M(r)^{(n-1)/n} J_n(k).$$

Thus

$$v(0)+\frac{M(r)}{\gamma_n}-\frac{2b_n}{c_n}\left(1+\frac{4}{\gamma_n d_\alpha^n}\right) J_n(k) M(r)^{(n-1)/n} \leqslant 0$$

for each r close enough to 1. This contradicts the fact that $\lim\limits_{r\to 1^-} M(r) = +\infty$. The contradiction completes the proof of part (a).

Proof of part (b) of Theorem 3. Assume $J_n(k) = +\infty$. Let $a > 0$, $e \in B_n$ be arbitrary. Let, for each integer $j \geqslant 1$, h_j be a function continuous in $\bar{B}_n$, harmonic in B_n and such that

$$|h_j(x) - P_n(x, e)| \leqslant k(0)/2^{j+1} \quad \text{whenever} \quad |x| \leqslant 1-1/j$$

and

$$h_j(x) - P_n(x, e) \leqslant k(|x|)/2^{j+1} \quad \text{for each } x \in B_n.$$

Existence of such an h_j follows from Theorem 1. Let r_j, $r_j < 1$, be so close to 1 that $h_j(x) \leqslant 2^{-j} k(r_j)$ for each $x \in B_n$. Existence of such an r_j follows from boundedness of h_j. Now, let r_j', $r_j' < 1$, be a number that is so close to 1, that for each r such that $r_j' < r < 1$ we have

$$\left|\frac{2}{m_n(1-r^2)} g_n(x, re) - P_n(x, e)\right| \leqslant \frac{k(0)}{2^{j+1}}$$

whenever $|x| \leqslant \max\{r_j, 1-1/j\}$. Next, let us choose a number r_j'' such that $r_j' < r_j'' < 1$ and $2/m_n(1-r_j''^2)$ is an integer multiplicity of a, and define a function v_j, letting

$$v_j(x) = h_j(x) - \frac{2}{m_n(1-r_j''^2)} g_n(x, r_j'' e) \quad \text{for} \quad x \in B_n.$$

Then, keeping in mind that

$$v_j(x) = [h_j(x) - P_n(x, e)] + \left[P_n(x, e) - \frac{2}{m_n(1-r_j''^2)} g_n(x, r_j'' e) \right],$$

we can see that

$$v_j(x) \leqslant 2^{-j-1} k(0) + 2^{-j-1} k(0) = 2^{-j} k(0) \quad \text{if} \quad |x| \leqslant 1 - 1/j,$$

$$v_j(x) \leqslant 2^{-j-1} k(|x|) + 2^{-j-1} k(0) \leqslant 2^{-j} k(|x|) \quad \text{if} \quad x < r_j,$$

and

$$v_j(x) \leqslant h_j(x) \leqslant 2^{-j} k(r_j) \leqslant 2^{-j} k(|x|) \quad \text{if} \quad r_j \leqslant |x| < 1.$$

Therefore the series $\sum_{j=1}^{\infty} v_j$ converges uniformly on compact subsets of B_n and, as is easy to check, its sum is the function needed to complete the proof of part (b) of Theorem 3.

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