ANNALES
POLONICI MATHEMATICI
XVII (1965)

A construction of continuous functions without the usual, the approximative and the distributional derivatives

by H. Świątak (Kraków)

There are numerous examples of continuous functions for which the derivative in the usual sense does not exist at any point. The proofs of the non-existence of the usual derivative are in general founded upon the choice of the very special sequence of difference quotients: the nth difference quotient is the sum of the difference quotients of the first n components of the series of functions. Such a method is applicable in the case of neither the approximative nor the distributional derivative.

The first example of a continuous function for which the distributional derivative does not exist at any point was given by Z. Zieleźny ([2]). Starting immediately from the definition of the value of a distribution at a point (given by S. Łojasiewicz in [1]), he proved that van der Waerden's function satisfied these conditions.

This paper gives a general method of the construction of continuous functions without the distributional derivative at any point.

The non-existence of the approximative derivative is proved under the same assumptions as the non-existence of the distributional one. Obviously, the non-existence of the approximative or the distributional derivative implies the non-existence of the usual one.

We say that f has at x_0 the *distributional derivative* C if the distribution f' has at x_0 the value C, i.e. (cf. [1]) if

$$\lim_{a\to 0} f'(x_0+ax) = C .$$

The following lemma is the starting point of the construction:

Lemma I. *Let* $0<|a|<1$ *and let the functions* $g_\nu(x)$ $(\nu = 1, 2, \ldots)$ *satisfy in the interval* $(-\infty, \infty)$ *the following conditions*:

(1) $$|g_\nu(x)| \leqslant 1 ,$$

(2) $|g_\nu(\bar{x})-g_\nu(\bar{\bar{x}})| \leqslant M|\bar{x}-\bar{\bar{x}}|^a$ *for certain* M *and* a *such that* $M>0$, $0<a<1$.

Let us consider the function

$$f(x) = \sum_{\nu=1}^{\infty} a^{\nu} g_{\nu}(b^{\nu} x) ;$$

it is defined and continuous for $x \in (-\infty, \infty)$ *and for any value of the constant* b.

Let d', d'', γ *be constants such that* $0 < d' < d''$, $0 < \gamma < 1$.

If a *and* b *satisfy the conditions*

$$0 < |a| < \frac{1}{1+2/\gamma}, \tag{3}$$

$$|b| > \left[\frac{M\left(\frac{d''^{a}}{|a|} - d'^{a}\right)}{\gamma\left(1 - |a| - \frac{2|a|}{\gamma}\right)} + \frac{1}{|a|} \right]^{1/a}, \tag{4}$$

then for every sequence $\{h_n\}$ *such that*

$$\frac{d'}{|b|^n} \leqslant |h_n| \leqslant \frac{d''}{|b|^n}, \tag{5}$$

$$\left| g_n\big(b^n(x_0 + h_n)\big) - g_n(b^n x_0) \right| > \gamma \tag{6}$$

we have the inequality

$$\left| \frac{f(x_0+h_n) - f(x_0)}{h_n} \right| > W_n ,$$

where

$$W_n = W_n(|a|, |b|, M, a, \gamma, d', d'')$$

and

$$W_n \to \infty \quad \textit{as} \quad n \to \infty .$$

Proof. Let $\{h_n\}$ be a sequence satisfying (5) and (6).

$$\left| \frac{f(x_0+h_n) - f(x_0)}{h_n} \right| = \left| \sum_{\nu=1}^{\infty} a^{\nu} \frac{g_{\nu}\big(b^{\nu}(x_0+h_n)\big) - g_{\nu}(b^{\nu} x_0)}{h_n} \right|$$

$$\geqslant \frac{|a|^n \left| g_n\big(b^n(x_0+h_n)\big) - g_n(b^n x_0)\right|}{|h_n|} -$$

$$- \sum_{\nu=1}^{n-1} |a|^{\nu} \frac{\left|g_{\nu}\big(b^{\nu}(x_0+h_n)\big) - g_{\nu}(b^{\nu} x_0)\right|}{|h_n|} -$$

$$- \sum_{\nu=n+1}^{\infty} |a|^{\nu} \frac{\left|g_{\nu}\big(b^{n}(x_0+h_n)\big) - g_{\nu}(b^{\nu} x_0)\right|}{|h_n|} .$$

We estimate the first component by condition (6), the next $n-1$ components by condition (2) and the rest by (1), and we obtain

$$\left|\frac{f(x_0+h_n)-f(x_0)}{h_n}\right| > \frac{|a|^n\gamma}{|h_n|} - |h_n|^{\alpha-1} M |a|\,|b|^\alpha \frac{(|a|\,|b|^\alpha)^{n-1}-1}{|a|\,|b|^\alpha-1} - \frac{2\,|a|^{n+1}}{|h_n|(1-|a|)}$$

$$= \frac{|a|^n\gamma}{|h_n|(|a|\,|b|^\alpha-1)(1-|a|)}\Bigg[(|a|\,|b|^\alpha-1)(1-|a|) - \frac{|h_n|^\alpha M\,|b|^{\alpha n}}{\gamma} -$$

$$-\frac{|h_n|^\alpha M a^2 |b|^\alpha}{|a|^n\gamma} + \frac{|h_n|^\alpha M\,|a|\,|b|^\alpha}{|a^n|\,\gamma} + \frac{|h_n|^\alpha M\,|a|\,|b|^{\alpha n}}{\gamma} - \frac{2\,|a|}{\gamma}(|a|\,|b|^\alpha-1)\Bigg].$$

Let us denote the expression in brackets by L_n. By (5) we have

$$L_n > L + \varrho_n\,,$$

where

$$L = (|a|\,|b|^\alpha-1)\left(1-|a|-\frac{2\,|a|}{\gamma}\right) - \frac{M}{\gamma}(d''^\alpha - d'^\alpha|a|)\,,$$

$$\varrho_n = \frac{M(d'^\alpha - d''^\alpha|a|)}{(|a|\,|b|^\alpha)^{n-1}}\,.$$

According to (3)

(3′) $$1-|a|-\frac{2\,|a|}{\gamma} > 0 \quad \text{and} \quad |a| > 0\,,$$

and therefore by (4)

(4′) $$L > 0\,.$$

Since $d' < d''$, by virtue of (3′) it follows from (4′) that

(*) $$|a|\,|b|^\alpha > 1\,.$$

Hence it follows that

$$L + \varrho_n \to L \quad \text{as} \quad n \to \infty\,.$$

Since $L > 0$, there exists an N such that for $n \geqslant N$ we have

$$L + \varrho_n > 0 \quad \text{and} \quad L_n > 0\,.$$

By (5)

$$\frac{1}{|h_n|} \geqslant \frac{|b|^n}{d''}\,.$$

Hence, and from the fact that

$$L_n > 0\,, \quad |a|\,|b|^\alpha > 1\,, \quad |a| < 1$$

we have the estimate

$$\left|\frac{f(x_0+h_n)-f(x_0)}{h_n}\right| > \frac{|ab|^n\gamma L_n}{d''(|a|\,|b|^a-1)(1-|a|)} > \frac{|ab|^n\gamma(L+\varrho_n)}{d''(|a|\,|b|^a-1)(1-|a|)}\,.$$

Let us denote by W_n the right side of this inequality. Since $L = L(|a|, |b|, M, a, \gamma, d', d'')$ and $\varrho_n = \varrho_n(|a|, |b|, M, a, \gamma, d', d'')$, we have

$$W_n = W_n(|a|, |b|, M, a, \gamma, d', d'')\,.$$

Since $|a| < 1$, it follows from $(*)$ that $|b| > 1$. Since $0 < a < 1$, we obtain from $(*)$ $|ab| > 1$. Now, $|ab|^n \to \infty$ as $n \to \infty$ and since $L+\varrho_n \to L$ as $n \to \infty$, we have

$$W_n \to \infty \quad \text{as} \quad n \to \infty\,. \qquad \text{Q. E. D.}$$

Let us introduce for further considerations the notation

$$D = \{\varphi\colon \varphi(x) \in C^\infty \text{ for } x \in (-\infty, \infty),\ \operatorname{supp}\varphi \text{ compact}\}\,.$$

The following lemma, in addition to Lemma I, will be necessary in the proof of the non-existence of the distributional derivative:

LEMMA II. *If a continuous function $f(x)$ defined in $(-\infty, \infty)$ has the finite distributional derivative C at a point x_0, then $\big(f(x_0+\lambda x)-f(x_0)\big)/\lambda$ converges distributionally to C as $\lambda \to 0$. (That is*

$$\int_{-\infty}^{\infty} \frac{f(x_0+\lambda x)-f(x_0)}{\lambda}\,\varphi(x)\,dx \to \int_{-\infty}^{\infty} Cx\varphi(x)\,dx$$

as $\lambda \to 0$ for each function $\varphi \in D$.)

Proof. We shall prove this lemma starting from the following theorem given by S. Łojasiewicz ([1]):

A distribution T has the value $T(x_0) = C$ at a point x_0 if and only if there exist an integer $n \geqslant 0$ and a continuous function F such that

$$T = F^{(n)}\,, \tag{1}$$

and

$$\frac{F(x)}{(x-x_0)^n} \to \frac{C}{n!} \quad \text{as} \quad x \to x_0\,. \tag{2}$$

In our case $T = f'$. Therefore if this distribution has the value C at a point x_0, there exist an integer $n \geqslant 0$ and a continuous function F such that (1) and (2) fulfilled and, moreover,

$$\frac{1}{\lambda^{n-1}}\cdot\frac{\partial^{n-1}}{\partial x^{n-1}}\big(F(x_0+\lambda x)\big) = F^{(n-1)}(x_0+\lambda x) = f(x_0+\lambda x)-f(x_0)\,, \tag{3}$$

(since $F^{(n-1)}(x_0) = 0$ by virtue of (2)).

If the continuous function $F(x)$ defined in $(-\infty, \infty)$ satisfies condition (2), then

(4) $\dfrac{F(x_0+\lambda x)}{\lambda^n} \to \dfrac{C}{n!} x^n$ as $\lambda \to 0$, and this convergence is uniform for x from an arbitrary closed and bounded interval.

It follows from (4) that

$$(5)\quad \int_{-\infty}^{\infty} \frac{F(x_0+\lambda x)}{\lambda^n} \varphi(x)\,dx \to \int_{-\infty}^{\infty} \frac{C}{n!} x^n \varphi(x)\,dx \text{ as } \lambda \to 0 \text{ for each function } \varphi \in D.$$

If $\varphi \in D$, then $\varphi^{(n-1)} \in D$ as well and by (5) we have

$$(6)\quad \int_{-\infty}^{\infty} \frac{F(x_0+\lambda x)}{\lambda^n} \varphi^{(n-1)}(x)\,dx \to \int_{-\infty}^{\infty} \frac{C}{n!} x^n \varphi^{(n-1)}(x)\,dx \quad \text{as} \quad \lambda \to 0$$

for each function $\varphi \in D$.

Integrating $n-1$ times by parts and taking into account (3), we obtain

$$(7)\quad \int_{-\infty}^{\infty} \frac{F(x_0+\lambda x)}{\lambda^n} \varphi^{(n-1)}(x)\,dx = (-1)^{n-1} \int_{-\infty}^{\infty} \frac{f(x_0+\lambda x)-f(x_0)}{\lambda} \varphi(x)\,dx\,.$$

Similarly we obtain

$$(8)\quad \int_{-\infty}^{\infty} \frac{C}{n!} x^n \varphi^{(n-1)}(x)\,dx = (-1)^{n-1} \int_{-\infty}^{\infty} Cx\varphi(x)\,dx\,.$$

By (6), (7), (8) we conclude that

$$\int_{-\infty}^{\infty} \frac{f(x_0+\lambda x)-f(x_0)}{\lambda} \varphi(x)\,dx \to \int_{-\infty}^{\infty} Cx\varphi(x)\,dx \text{ as } \lambda \to 0 \text{ for each function } \varphi \in D$$

is the necessary condition for the existence of the distributional derivative C of a continuous function f at a point x_0. Q. E. D.

THEOREM. *Let* $0 < |a| < 1$ *and let the functions* $g_\nu(x)$ $(\nu = 1, 2, \ldots)$ *satisfy in the interval* $(-\infty, \infty)$ *the following conditions*:

$$(1)\quad |g_\nu(x)| \leqslant 1\,,$$

(2) $|g_\nu(\bar{x}) - g_\nu(\bar{\bar{x}})| \leqslant M|\bar{x}-\bar{\bar{x}}|^\alpha$ *for certain* M *and* α *such that* $M > 0$, $0 < \alpha < 1$

and

(3) *the oscillation of the functions $g_\nu(x)$ in any interval of length d is greater than 2ω, where ω is a constant satisfying the condition $0 < \omega < 1$.*

Let us take into account the function

$$f(x) = \sum_{\nu=1}^{\infty} a^\nu g_\nu(b^\nu x) .$$

It is defined and continuous in $(-\infty, \infty)$ for an arbitrary value of the constant b.

If a and b satisfy the conditions

$$0 < |a| < \frac{1}{1+\dfrac{2|a|}{\omega}}, \tag{4}$$

$$|b| > \left[\frac{\dfrac{Md^a}{\omega|a|}-1}{1-|a|-\dfrac{2|a|}{\omega}} + \frac{1}{|a|}\right]^{1/a}, \tag{5}$$

then $f(x)$ satisfies Hölder's condition with an exponent a^* such that $|a||b|^{a^*} < 1$, and it has neither the usual, nor the approximative, nor the distributional derivative at any point.

Proof. It follows from (2) and (3) that $M \geqslant 2\omega/d^a$, and thus it follows from (4) and (5) that $|b| > 1$ and $|a||b|^a > 1$. Since $|a| < 1$ and $|b| > 1$, one can choose a^* such that $0 < a^* < a$ and $|a||b|^{a^*} < 1$.

We shall prove that the function $f(x)$ satisfies Hölder's condition with the exponent a^* and with the constant $M^* = \tilde{M}\sum_{\nu=1}^{\infty}(|a||b|^{a^*})^\nu$, where $\tilde{M} = \max\{M, 2\}$.

In fact, if $|\bar{x}-\bar{\bar{x}}| \leqslant 1$, we have on the basis of (2)

$$|g_\nu(\bar{x}) - g_\nu(\bar{\bar{x}})| \leqslant M|\bar{x}-\bar{\bar{x}}|^{a^*} .$$

By (1)

$$|g_\nu(\bar{x}) - g_\nu(\bar{\bar{x}})| \leqslant 2;$$

thus, if $|\bar{x}-\bar{\bar{x}}| > 1$, the following estimation is true:

$$|g_\nu(\bar{x}) - g_\nu(\bar{\bar{x}})| \leqslant 2|\bar{x}-\bar{\bar{x}}|^{a^*} .$$

Accepting $\tilde{M} = \max\{M, 2\}$ we have

$$|g_\nu(\bar{x}) - g_\nu(\bar{\bar{x}})| \leqslant \tilde{M}|\bar{x}-\bar{\bar{x}}|^{a^*}$$

for arbitrary values $\bar{x}$, $\bar{\bar{x}}$.

Hence
$$|g_\nu(b^\nu \bar{x}) - g_\nu(b^\nu \bar{\bar{x}})| \leqslant \tilde{M}(|b|^\nu)^{a^*} |\bar{x} - \bar{\bar{x}}|^{a^*}$$
and therefore
$$|f(\bar{x}) - f(\bar{\bar{x}})| \leqslant \sum_{\nu=1}^{\infty} |a|^\nu |g_\nu(b^\nu \bar{x}) - g_\nu(b^\nu \bar{\bar{x}})| \leqslant \tilde{M} \cdot \sum_{\nu=1}^{\infty} (|a| \, |b|^{a^*})^\nu |\bar{x} - \bar{\bar{x}}|^{a^*} .$$

Conditions (1) and (2) of the Theorem coincide with conditions (1) and (2) of Lemma I.

For the proof of the non-existence of the usual derivative it suffices to show that it follows from conditions (2) and (3) of the Theorem that there exist constants γ, d', d'' for which there exists a sequence $\{h_n\}$ satisfying assumptions (5) and (6) of Lemma I.

We shall prove more: It is possible to choose the values of the constants γ, d', d'' so that inside the intervals $Q_n = \left\langle \frac{d'}{|b|^n}, \frac{d''}{|b|^n} \right\rangle$ there are intervals P_n such that $\frac{|P_n|}{d''/|b|^n} \geqslant K$, where $K = \text{const}$, $0 < K < 1$, and thus that every sequence $\{h_n\}$ which satisfies the condition $h_n \in P_n$ satisfies condition (6) of Lemma I as well, i.e.
$$\left|g_n\big(b^n(x_0 + h_n)\big) - g_n(b^n x_0)\right| > \gamma .$$
(The choice of intervals P_n depends in general on the point x_0, but the constant K does not.)

Condition (5) of Lemma I is satisfied for such sequences $\{h_n\}$ automatically as $P_n \subset Q_n$.

By virtue of condition (3) of the Theorem there is an $h_n > 0$ such that
$$|b^n(x_0 + h_n) - b^n x_0| = |b|^n h_n \leqslant d \quad \text{or} \quad 0 < h_n \leqslant \frac{d}{|b|^n}$$
and
$$\left|g_n\big(b^n(x_0 + h_n)\big) - g_n(b^n x_0)\right| > \omega .$$
Hence, by virtue of condition (2) we have $\omega < M^a h_n |b|^{an}$ or
$$h_n > \left(\frac{\omega}{M}\right)^{1/a} \frac{1}{|b|^n} .$$

Let us take an arbitrary constant $\varrho > 1$ and write
$$h_n^{**} = \max\left\{h_n\colon \left|g_n\big(b^n(x_0 + h_n)\big) - g_n(b^n x_0)\right| = \omega,\ h_n \leqslant d/|b|^n\right\} ,$$
$$h_n^{*} = \max\left\{h_n\colon \left|g_n\big(b^n(x_0 + h_n)\big) - g_n(b^n x_0)\right| = \omega/\varrho,\ h_n < h_n^{**}\right\} .$$
Since the functions $g_\nu(x)$ are continuous (on the basis of (2)), such h_n^{**} and h_n^{*} certainly exist.

If $h_n \in \langle h_n^*, h_n^{**}\rangle$, we have the inequality

$$(7)\qquad |g_n(b^n(x_0+h_n))-g_n(b^n x_0)| > \omega/\varrho .$$

It follows from condition (2) that

$$\omega-\frac{\omega}{\varrho} = |g_n(b^n(x_0+h_n^*))-g_n(b^n(x_0+h_n^{**}))| \leqslant M|b|^{an}|h_n^*-h_n^{**}|^a$$

or

$$(8)\qquad h_n^{**}-h_n^* \geqslant \left[\frac{\omega}{M}\left(1-\frac{1}{\varrho}\right)\right]^{1/a}\frac{1}{|b|^n}.$$

Similarly we obtain

$$(9)\qquad h_n^* \geqslant \left(\frac{\omega}{\varrho M}\right)^{1/a}\frac{1}{|b|^n}.$$

It follows from the definition h_n^{**} that

$$(10)\qquad h_n^{**} \leqslant \frac{d}{|b|^n}.$$

It follows from (7), (8), (9) and (10) that if we accept

$$d' = \left(\frac{\omega}{\varrho M}\right)^{1/a}, \qquad d'' = d, \qquad \gamma = \frac{\omega}{\varrho}, \qquad P_n = \langle h_n^*, h_n^{**}\rangle,$$

then

$$P_n \subset Q_n = \left\langle\frac{d'}{|b|^n}, \frac{d''}{|b|^n}\right\rangle$$

and

$$(11)\qquad \frac{|P_n|\,|b|^n}{d''} \geqslant K,$$

where

$$K = \frac{\left[\frac{\omega}{M}\left(1-\frac{1}{\varrho}\right)\right]^{1/a}}{d}.$$

(Obviously $0 < K < 1$, as $\varrho > 1$ and $\omega < Md^a$.)

Now, it follows from Lemma I that if

$$(12)\qquad 0<|a|<\frac{1}{1+\frac{2\varrho}{\omega}}, \qquad |b| > \left[\frac{\frac{Md^a\varrho}{\omega|a|}-1}{1-|a|-\frac{2\varrho|a|}{\omega}}+\frac{1}{|a|}\right]^{1/a}$$

and if $\{h_n\}$ is an arbitrary sequence such that $h_n \in P_n$, we have

$$\left|\frac{f(x_0+h_n)-f(x_0)}{h_n}\right| > W_n \,, \tag{13}$$

where

$$W_n = W_n\left(|a|, |b|, M, a, \frac{\omega}{\varrho}, \left(\frac{\omega}{\varrho M}\right)^{1/a}, d\right)$$

and

$$W_n \to \infty \quad \text{as} \quad n \to \infty \,.$$

Since $0 < h_n \leqslant d/|b|^n$ and $|b| > 1$, we have $h_n \to 0$ as $n \to \infty$. Hence it follows that if a and b satisfy (12), the approximative derivative of the function $f(x)$ does not exist.

Let us notice now that on the basis of (10) and (8)

$$\frac{h_n^{**}}{h_n^{*}} = \frac{1}{1-\dfrac{h_n^{**}-h_n^{*}}{h_n^{**}}} \geqslant \frac{1}{1-\dfrac{(h_n^{**}-h_n^{*})\,|b|^n}{d}} \geqslant \frac{1}{1-K}\,,$$

i.e.

$$P_n' = \left\langle h_n^{*}, \frac{1}{1-K}\,h_n^{*}\right\rangle \subset P_n = \langle h_n^{*}, h_n^{**}\rangle$$

and therefore we may apply Lemma I for $h_n \in P_n'$.

By Lemma II the necessary condition for the existence of the finite distributional derivative of a continuous function f at a point x_0 is:

$$\int_{-\infty}^{\infty} \frac{f(x_0+\lambda_n x)-f(x_0)}{\lambda_n}\,\varphi(x)\,dx \to \int_{-\infty}^{\infty} Cx\varphi(x)\,dx$$

for each function $\varphi \in D$ and for every sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ as $n \to \infty$.

Let us take

$$\lambda_n = h_n^{*}$$

and

$$\varphi(x) = \varphi^{*}(x)$$

such that

$$\varphi^{*}(x) \in D\,,$$

$$\varphi^{*}(x)\begin{cases} > 0 & \text{for } x \in \left(1, \dfrac{1}{1-K}\right), \\ = 0 & \text{for } x \leqslant 1 \text{ and } x \geqslant \dfrac{1}{1-K}\,. \end{cases}$$

Let $x \in (1, 1/(1-K))$, i.e. $\lambda_n x \in P'_n$, and let the constants a and b satisfy conditions (12). In view of Lemma I we then have

$$\left|\frac{f(x_0+\lambda_n x)-f(x_0)}{\lambda_n}\right| > W_n x . \tag{13'}$$

(W_n is the same as in (13).)

One can assume without loss of generality that

$$\frac{f(x_0+\lambda_n x)-f(x_0)}{\lambda_n} > W_n x \quad \text{for} \quad x \in \left(1, \frac{1}{1-K}\right). \tag{14}$$

(In fact, it follows from the continuity of the function $f(x)$ and from inequality (13′) that

$$\operatorname{sgn} \frac{f(x_0+\lambda_n x)-f(x_0)}{\lambda_n} = \operatorname{sgn} \frac{f(x_0+\lambda_n)-f(x_0)}{\lambda_n}$$

for $x \in \left(1, \frac{1}{1-K}\right)$.

If inequality (14) is not satisfied, it suffices to choose an arbitrary sequence of components with a constant signum from the sequence

$$\left\{\frac{f(x_0+\lambda_n x)-f(x_0)}{\lambda_n}\right\},$$

and if they are negative, it is necessary to consider the inequality

$$\frac{f(x_0+\lambda_{n_\nu} x)-f(x_0)}{\lambda_{n_\nu}} < -W_{n_\nu}$$

instead of (14).)

It follows from (14) and from the definition of the function $\varphi^*(x)$ that

$$\int_{-\infty}^{\infty} \frac{f(x_0+\lambda_n x)-f(x_0)}{\lambda_n} \varphi^*(x)\, dx > W_n \int_{-\infty}^{\infty} x\varphi^*(x)\, dx . \tag{15}$$

Since $\lambda_n = h_n^*$, $\lambda_n \to 0$ as $n \to \infty$. Moreover, $W_n \to \infty$ as $n \to \infty$ and $\int_{-\infty}^{\infty} x\varphi^*(x) dx > 0$. Therefore it follows from (15) that

$$\int_{-\infty}^{\infty} \frac{f(x_0+\lambda_n x)-f(x_0)}{\lambda_n} \varphi^*(x)\, dx \to \infty \quad \text{as} \quad n \to \infty .$$

Thus the necessary condition for the existence of a finite distributional derivative is not satisfied.

Let us notice that conditions (12), which guarantee that the function $f(x)$ has neither the approximative nor the distributional derivative, differ from conditions (4) and (5) in the dependence on an arbitrary constant $\varrho > 1$ only. However, it is obvious from conditions (4) and (5) that it is possible to choose ϱ so near 1 that conditions (12) are also satisfied. Thus the fulfilment of conditions (4) and (5) for the constants a and b secures the non-existence of the approximative and the distributional derivatives. Q. E. D.

References

[1] S. Łojasiewicz, *Sur la valeur et la limite d'une distribution en un point*, Studia Math. 15 (1958), pp. 1-36.

[2] Z. Zieleźny, *Über die Mengen der regulären und singulären Punkte einer Distribution*, Studia Math. 19 (1960), pp. 27-52.

Reçu par la Rédaction le 10. 7. 1962