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On the existence and uniqueness of non-negative solutions of a certain non-linear convolution equation

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Abstract. We consider some properties of non-negative solutions u of the one-dimensional non-linear equation

$$u^2 = K*u,$$

where the kernel K is a non-decreasing function vanishing on the half-line $(-\infty, 0)$ and having a jump at the origin.

We give some theorems concerning the existence and uniqueness of non-negative solutions whose support is bounded from the left.

In this paper we shall study the non-linear integral equation occurring in the mathematical theory of the infiltration of a fluid from a cylindrical reservoir into an isotropic homogeneous porous medium. It is known (see [1], [6]) that under some simplifying assumptions of the hydrogeological kind the free surface of the percolating fluid is described by Boussinesq's non-linear differential equation. The research concerning its approximative solutions leads in the case of radial symmetry to the one-dimensional integral equation

$$u^2(x) = \int_r^x K(x-\tau)u(\tau)\,d\tau \quad \text{for } x \in [0,1], \tag{1}$$

where K is given and the unknown function u denotes the height of the percolating fluid above the horizontal impervious base, multiplied by a positive factor. Thus from a physical point of view only non-negative solutions of (1) are interesting. Equation (1) was studied in [4] with a linear function K and in [2] with $K(x) = e^{2Ax}(1+x\ln A)$, $A > 1$. In the sequel we shall consider non-negative solutions of (1) under fairly general assumptions concerning the kernel function K.

1. Properties of non-negative solutions. We shall consider equation (1) for all $x \in R$. We suppose that the kernel K satisfies the following conditions:

(a) K is a non-decreasing function defined on R,

(b) $K(x) = 0$ for $x < 0$,

(c) the limit $\lim_{x\to 0^+} K(x) = g$ is a positive number.

If $u(x) = 0$ for $x < 0$, then equation (1) can be written as

$$u^2 = K*u. \tag{1'}$$

Henceforth we shall study equation (1').

Let M_+ be the set of all measurable functions on R whose support is bounded from the left.

LEMMA 1. *Let f and g be functions from M_+. If f is a non-negative non-decreasing function and g is a non-negative locally bounded function, then $f*g$ is a continuous function on R.*

Proof. Let the supports of f and g be bounded from the left by r. We can write

$$(f*g)(x) = \int_r^{x-r} f(x-\tau)g(\tau)\,d\tau \qquad \text{for all } x \in R.$$

We suppose that a sequence x_n converges to x. For sufficiently large x_0 we get

$$\int_r^{x_n-r} f(x_n-\tau)g(\tau)\,d\tau = \int_r^{x_0} f(x_n-\tau)g(\tau)\,d\tau \qquad \text{for all } x_n.$$

The function f is continuous almost everywhere, because it is monotone. Hence the sequence $f(x_n-\tau)g(\tau)$ converges to $f(x-\tau)g(\tau)$ for almost all $\tau \in [r, x_0]$. Since f is non-decreasing and

$$g(\tau) \leqslant L \qquad \text{for } \tau \in [r, x_0],$$

we have

$$f(x_n-\tau)g(\tau) \leqslant Lf(x_0) \qquad \text{for all } \tau \in [r, x_0] \text{ and all } x_n.$$

Using Lebesgue's theorem, we get

$$\lim_{n\to\infty} \int_r^{x_0} f(x_n-\tau)g(\tau)\,d\tau = \int_r^{x_0} f(x-\tau)g(\tau)\,d\tau$$

and

$$\int_r^{x_0} f(x-\tau)g(\tau)\,d\tau = \int_r^{x-r} f(x-\tau)g(\tau)\,d\tau.$$

Therefore $\lim_{n\to\infty}(f*g)(x_n) = (f*g)(x)$ for any sequence convergent to x. This implies that $f*g$ is continuous on R.

LEMMA 2. *If $u \in M_+$ is a non-negative solution of* (1′), *then u is a non-decreasing function.*

Proof. If $x_1 < x_2$, then

$$u^2(x_2)-u^2(x_1) = \int_{-\infty}^{x_1} [K(x_2-\tau)-K(x_1-\tau)]u(\tau)\,d\tau + \int_{x_1}^{x_2} K(x_2-\tau)u(\tau)\,d\tau$$

and

$$\int_{-\infty}^{x_1} [K(x_2-\tau)-K(x_1-\tau)]u(\tau)\,d\tau + \int_{x_1}^{x_2} K(x_2-\tau)u(\tau)\,d\tau \geqslant 0.$$

This implies that $u(x_2) \geqslant u(x_1)$.

THEOREM 1. *If u is a non-negative solution of* (1′) *such that $u \in M_+$, then u is a continuous function on R.*

Proof. Since by Lemma 2 u is a non-decreasing function, it is locally bounded. Hence by Lemma 1 we infer that $K*u$ is a continuous function. Therefore u^2 is continuous. Since u is non-negative, we infer that $u = \sqrt{u^2}$ is continuous.

For a real number r let Q_r be the set of all continuous functions f on R such that $f(x) = 0$ for $x \leqslant r$ and $f(x) > 0$ for $x > r$. We shall write $Q = \bigcup_{r\in R} Q_r$.

THEOREM 2. *If u is a non-negative solution of* (1′) *such that $u \in M_+$, then $u \in Q_r$ for some real number r.*

Proof. We infer by Lemma 2 that the support of u is $[r, +\infty)$, where r is some real number. Since by Theorem 1 u is continuous, we find that $u \in Q_r$.

THEOREM 3. *If u is a solution of* (1′) *such that $u \in Q_0$, then*

$$\tfrac{1}{2}gx \leqslant u(x) \leqslant \int_0^x K(\tau)\,d\tau \quad \textit{for } x \geqslant 0. \tag{2}$$

Proof. By Lemma 2 u is a non-decreasing function. Therefore

$$u^2(x) \leqslant u(x)\int_0^x K(x-\tau)\,d\tau.$$

After the substitution $s = x-\tau$ we have

$$u^2(x) \leqslant u(x)\int_0^x K(s)\,ds,$$

which implies the right-hand side of inequality (2).

The function u is differentiable almost everywhere, because it is monotone. Let u be differentiable in $x > 0$ and let h_n be a positive sequence convergent to zero. Then

$$[u^2(x)]' = \lim_{n\to\infty} \frac{1}{h_n}\Big[\int_0^{x+h_n} K(x+h_n-\tau)u(\tau)d\tau - \int_0^x K(x-\tau)u(\tau)d\tau\Big].$$

We have

$$\frac{1}{h_n}\Big[\int_0^{x+h_n} K(x+h_n-\tau)u(\tau)d\tau - \int_0^x K(x-\tau)u(\tau)d\tau\Big]$$

$$= \int_0^x \frac{1}{h_n}[K(x+h_n-\tau)-K(x-\tau)]u(\tau)d\tau + \frac{1}{h_n}\int_x^{x+h_n} K(x+h_n-\tau)u(\tau)d\tau.$$

From conditions (a) and (c) we get

$$\int_0^x \frac{1}{h_n}[K(x+h_n-\tau)-K(x-\tau)]u(\tau)d\tau + \frac{1}{h_n}\int_x^{x+h_n} K(x+h_n-\tau)u(\tau)d\tau$$

$$\geqslant g\frac{1}{h_n}\int_x^{x+h_n} u(\tau)d\tau.$$

Hence we obtain

$$[u^2(x)]' \geqslant g\cdot \lim_{n\to\infty}\frac{1}{h_n}\int_x^{x+h_n} u(\tau)d\tau.$$

Since u is continuous in x, we have (see [5])

$$\lim_{n\to\infty}\frac{1}{h_n}\int_x^{x+h_n} u(\tau)d\tau = u(x).$$

Therefore the last inequality can be written as

$$2u(x)u'(x) \geqslant g\cdot u(x),$$

from which we get

$$u'(x) \geqslant \frac{1}{2}g \quad \text{for a.a. } x > 0.$$

The inequality $u(x) \geqslant \int_0^x u'(\tau)d\tau$ is true for a non-decreasing function u

such that $u(0) = 0$ (see [5]). Then we have

$$u(x) \geqslant \int_0^x \tfrac{1}{2} g \, d\tau, \qquad \text{where } \int_0^x \tfrac{1}{2} g \, d\tau = \tfrac{1}{2} g x.$$

We get the left-hand side of (2).

2. Existence and uniqueness of the non-negative solution. For a function $f \in Q$ we denote by $T(f)$ the function

$$T(f) = (K * f)^{\frac{1}{2}}. \tag{3}$$

LEMMA 3. *If $f_1(x) \leqslant f_2(x)$ for all $x \leqslant b$, then $T(f_1)(x) \leqslant T(f_2)(x)$ for all $x \leqslant b$.*

Proof. From conditions (a) and (c) we infer that T is monotone.

We denote by P_K^b the set of all functions from Q_0 reduced to $(-\infty, b]$ and satisfying (2) on $[0, b]$. Let P_K be the subset of functions from Q_0 satisfying (2) on $[0, +\infty)$.

LEMMA 4. *The operator T transforms P_K^b into P_K^b.*

Proof. Let

$$F(x) = \begin{cases} \tfrac{1}{2} g(x) & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0, \end{cases}$$

and

$$G(x) = \begin{cases} \int_0^x K(\tau) \, d\tau & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Using conditions (a)–(c), we get

$$T(F)(x) \geqslant F(x) \quad \text{and} \quad T(G)(x) \leqslant G(x),$$

from which, by the definition of P_K^b and Lemma 3, we obtain

$$F(x) \leqslant T(\varphi)(x) \leqslant G(x) \quad \text{for } \varphi \in P_K^b.$$

From Lemma 1 we infer that $T(\varphi)$ is a continuous function. The lemma is proved.

For every $b > 0$ there exists a number $c > 0$ such that $K(c)/g < 2$ and $c < b$. For any $f_1, f_2 \in P_K^b$ the number

$$\sup_{0 < \tau \leqslant b} \frac{|f_1(\tau) - f_2(\tau)|}{p(\tau)}, \qquad \text{where } p(\tau) = e^{\beta\tau} \int_0^\tau [K(\sigma) - \tfrac{1}{2} g] \, d\sigma$$

$$\text{and } \beta = \frac{1}{g} \sup_{c \leqslant \tau \leqslant b} \frac{|K(\tau) - g|}{\tau},$$

is positive and finite. We can define the function

$$\varrho_b(f_1, f_2) = \sup_{0<\tau\leqslant b} \frac{|f_1(\tau)-f_2(\tau)|}{p(\tau)} \quad \text{for all } f_1, f_2 \in P_K^b.$$

LEMMA 5. *The function ϱ_b defines a metric in P_K^b and P_K^b is a complete space.*

Proof. It is easy to see that ϱ_b has properties of a metric. Now we show that P_K^b is complete. Let $f_n \in P_K^b$ be a Cauchy sequence. Thus for any $\varepsilon > 0$ there exists an N_ε such that

$$\frac{|f_m(\tau)-f_n(\tau)|}{p(\tau)} \leqslant \varepsilon \quad \text{for } n, m \geqslant N_\varepsilon \text{ and } \tau \in (0, b].$$

We have

$$\frac{|f_m(\tau)-f_n(\tau)|}{p(\tau)} \geqslant \frac{1}{M}|f_m(\tau)-f_n(\tau)| \quad \text{for } \tau \in (0, b],$$

because

$$p(\tau) \leqslant M = e^{\beta b}\int_0^b [K(\sigma)-\tfrac{1}{2}g]\,d\sigma \quad \text{for } \tau \in (0, b].$$

Hence

$$|f_m(\tau)-f_n(\tau)| \leqslant \varepsilon M \quad \text{for } n, m \geqslant N_\varepsilon \quad \text{and } \tau \in (0, b],$$

which implies that there exists a function f such that $f(\tau) = \lim_{n\to\infty} f_n(\tau)$ for $\tau \in (0, b]$. The function f is an element of P_K^b. If $m \to \infty$, we obtain

$$\frac{|f(\tau)-f_n(\tau)|}{p(\tau)} \leqslant \varepsilon \quad \text{for } n \geqslant N_\varepsilon \text{ and } \tau \in (0, b].$$

This implies that $\varrho_b(f, f_n) \leqslant \varepsilon$ for $n \geqslant N_\varepsilon$. We conclude that P_K^b is complete.

LEMMA 6. *For every $\tau \geqslant 0$*

$$\tfrac{1}{2}\tau\int_0^\tau [K(s)-\tfrac{1}{2}g]\,ds \geqslant \int_0^\tau\int_0^s [K(\sigma)-\tfrac{1}{2}g]\,d\sigma\,ds. \tag{4}$$

Proof. Let

$$l(\tau) = \tfrac{1}{2}\tau\int_0^\tau [K(s)-\tfrac{1}{2}g]\,ds - \int_0^\tau\int_0^s [K(\sigma)-\tfrac{1}{2}g]\,d\sigma\,ds.$$

The function l is absolutely continuous. Thus l' exists almost everywhere and $l(\tau) = \int_0^\tau l'(s)\,ds$ (see [5]). We have

$$l'(\tau) = \tfrac{1}{2}\Big(\tau[K(\tau)-\tfrac{1}{2}g] - \int_0^\tau [K(s)-\tfrac{1}{2}g]\,ds\Big) \quad \text{for a.a. } \tau \geqslant 0.$$

From (a) we get

$$\int_0^\tau [K(s)-\tfrac{1}{2}g]\,ds \leqslant \tau[K(\tau)-\tfrac{1}{2}g],$$

which implies that $l'(\tau) \geqslant 0$. Hence $l(\tau) \geqslant 0$ for $\tau \geqslant 0$. We infer that inequality (4) is true.

LEMMA 7. *For every* $\tau \in [0, b]$

$$K(\tau)e^{-\beta\tau} \leqslant K(c). \tag{5}$$

Proof. Since K is a non-decreasing function and β is a non-negative number, we have

$$K(\tau)e^{-\beta\tau} \leqslant K(\tau) \leqslant K(c) \quad \text{for } \tau \in [0, c].$$

We can write

$$K(\tau) = g + \tau\,\frac{K(\tau)-g}{\tau},$$

from which we obtain

$$K(\tau) g + \tau \sup_{c\leqslant\tau\leqslant b} \frac{K(\tau)-g}{\tau} \quad \text{for } \tau \in [c, b],$$

and, by the definition of β, we get

$$K(\tau) \leqslant g(1+\tau\beta) \quad \text{for } \tau \in [c, b].$$

Since the inequality

$$1+x \leqslant e^x$$

is true for all $x \in R$, we get

$$K(\tau) \leqslant g e^{\beta\tau} \quad \text{for } \tau \in [c, b],$$

from which it follows that

$$K(\tau)e^{-\beta\tau} \leqslant K(c) \quad \text{for } \tau \in [c, b].$$

This implies that inequality (5) is true for all $\tau \in [0, b]$.

LEMMA 8. *For* $f_1, f_2 \in P_K^b$

$$\varrho_b\big(T(f_2),\, T(f_1)\big) \leqslant \frac{K(c)}{2g}\,\varrho_b(f_2, f_1). \tag{6}$$

Proof. For $f_1, f_2 \in P_K^b$ and $\tau \in (0, b]$ we can write

$$|T(f_2)(\tau)-T(f_1)(\tau)| = \left|\frac{(K*f_2)(\tau)-(K*f_1)(\tau)}{(K*f_2)^{\frac{1}{2}}(\tau)+(K*f_1)^{\frac{1}{2}}(\tau)}\right|.$$

Using Lemma 4, we obtain

$$\left|\frac{(K*f_2)(\tau)-(K*f_1)(\tau)}{(K*f_2)^{1/2}(\tau)+(K*f_1)^{1/2}(\tau)}\right| \leqslant \left|\frac{(K*[f_2-f_1])(\tau)}{g\tau}\right|.$$

Since the inequolity

$$|f_2(\tau)-f_1(\tau)| \leqslant p(\tau)\varrho_b(f_2, f_1),$$

is true for all $f_1, f_2 \in P_K^b$ and $\tau \geqslant 0$, we have

$$\left|\frac{(K*[f_2-f_1])(\tau)}{g\tau}\right| \leqslant \frac{(K*p)(\tau)}{g\tau}\varrho_b(f_2, f_1) \quad \text{for } \tau > 0.$$

We obtain

$$|T(f_2)(\tau)-T(f_1)(\tau)| \leqslant \frac{(K*p)(\tau)}{g\tau}\varrho_b(f_2, f_1)$$

From the definition of p we get

$$(K*p)(\tau) = e^{\beta\tau}\int_0^\tau K(s)e^{-\beta s}\int_0^{\tau-s}\left(K(\sigma)-\tfrac{1}{2}g\right)d\sigma ds.$$

We have, by Lemma 7,

$$(K*p)(\tau) \leqslant e^{\beta\tau}K(c)\int_0^\tau\int_0^s [K(\sigma)-\tfrac{1}{2}g]d\sigma ds.$$

Using inequality (4), we obtain

$$(K*p)(\tau) \leqslant e^{\beta\tau}K(c)\tfrac{1}{2}\tau\int_0^\tau [K(s)-\tfrac{1}{2}g]ds.$$

The last inequality can be written as

$$(K*p)(\tau) \leqslant \tfrac{1}{2}K(c)\tau p(\tau),$$

from which we get

$$|T(f_2)(\tau)-T(f_1)(\tau)| \leqslant \frac{K(c)}{2g}p(\tau)\varrho_b(f_2, f_1) \quad \text{for } \tau \in (0, b].$$

This implies that inequality (6) is true.

THEOREM 4. *Equation* (1′) *has a unique solution in the set* Q_0 *satisfying inequality* (2).

Proof. Since for every $b > 0$ there exists a number $c > 0$ such that $K(c)/2g < 1$, by Lemma 8 the operator T is a contraction on the complete metric space P_K^b. Using Banach's theorem (see [3]), we infer that T has one and only one fixed point in P_K^b for every $b > 0$. This implies that T has one and only one fixed point in P_K.

COROLLARY. *For every real number d equation* (1′) *has a unique solution u in the set Q_d.*

The proof may be obtained by translation.

LEMMA 9. *The function*

$$r(x) = \int_0^x K(x-\tau)\int_0^\tau K(s)\,ds\,d\tau$$

is differentiable for a.a. $x \geqslant 0$ and

$$r'(x) = \int_0^x K(x-\tau)K(\tau)\,d\tau \qquad a.e.$$

Proof. We can write

$$\frac{r(x+h)-r(x)}{h} = \int_0^x K(\tau)\left(\frac{1}{h}\int_{x-\tau}^{x+h-\tau} K(s)\,ds\right)d\tau + \frac{1}{h}\int_x^{x+h} K(\tau)\int_0^{x+h-\tau} K(s)\,ds\,d\tau.$$

For $(h) < h_0$ we have

$$\left|K(\tau)\left(\frac{1}{h}\int_{x-\tau}^{x+h-\tau} K(s)\,ds\right)\right| \leqslant K(x)K(x+h_0) \qquad \text{for } \tau \in [0,x].$$

Then, by Lebesgue's theorem, we get

$$\lim_{h\to 0}\int_0^x K(\tau)\left(\frac{1}{h}\int_{x-\tau}^{x+h-\tau} K(s)\,ds\right)d\tau = \int_0^x K(\tau)K(x-\tau)\,d\tau.$$

Since K is continuous almost everywhere, we have

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} K(\tau)\int_0^{x+h-\tau} K(s)\,ds\,d\tau = 0 \qquad \text{for a.a. } x \geqslant 0.$$

We obtain

$$\lim_{h\to 0}\frac{r(x+h)-r(x)}{h} = \int_0^x K(\tau)K(x-\tau)\,d\tau \qquad \text{for a.a. } x \geqslant 0.$$

LEMMA 10. *If*

$$q(x) = \tfrac{1}{2}\int_0^x K(\tau)\,d\tau,$$

then $q \in P_K$ and $T(q)(x) \leqslant q(x)$ for all x.

Proof. We have

$$q(x) \leqslant \int_0^x K(\tau)\,d\tau.$$

Since, by conditions (a) and (c),

$$\tfrac{1}{2}\int_0^x K(\tau)d\tau - \tfrac{1}{2}gx = \tfrac{1}{2}\int_0^x [K(\tau)-g]d\tau$$

is a non-negative number, we have

$$q(x) \geqslant \tfrac{1}{2}gx \quad \text{for } x \geqslant 0.$$

We get $q \in P_K$. Let

$$w(x) = q^2(x) - [T(q)]^2(x).$$

The function $w(x)$ can be written as

$$w(x) = \tfrac{1}{4}\Big(\int_0^x K(\tau)d\tau\Big)^2 - \tfrac{1}{2}\int_0^x K(x-\tau)\int_0^\tau K(s)\,ds\,d\tau.$$

Since w is absolutely continuous and $w(0) = 0$, we have $w(x) = \int_0^x w'(\tau)d\tau$. By Lemma 9

$$w'(\tau) = \tfrac{1}{2}K(\tau)\int_0^\tau K(s)ds - \tfrac{1}{2}\int_0^\tau K(\tau-s)K(s)ds \quad \text{a.e.}$$

From (a) we obtain $w'(\tau) \geqslant 0$. This implies that $w(x) \geqslant 0$. The lemma s proved.

THEOREM 5. *If u is a solution of* (1′) *in the set* Q_0, *then*

$$u(x) \leqslant \tfrac{1}{2}\int_0^x K(\tau)d\tau \quad \textit{for all } x. \tag{7}$$

Proof. Since by Lemma 10 for every $b > 0$

$$q(x) \in P_K^b \quad \text{and} \quad T(q)(x) \leqslant q(x),$$

we have

$$T^{n+1}(q)(x) \leqslant T^n(q)(x)$$

and

$$T^n(q) \in P_K^b \quad \text{for every natural } n.$$

Hence $\lim_{n\to\infty} T^n(q)(x)$ exists for every $x \in [0, b]$. Let

$$\tilde{u}(x) = \lim_{n\to\infty} T^n(q)(x).$$

Since

$$T^{n+1}(q)(x) = (T * T^n)(q)(x),$$

we obtain by Lebesgue's theorem

$$\tilde{u}(x) = T(\hat{u})(x).$$

Therefore $\tilde{u}$ is a solution of (1′) in the set Q_0. From Theorem 4 we infer that equation (1′) has a unique solution u in Q_0. This implies that $\tilde{u} = u$. Since $u(x) \leqslant q(x)$ for all x, the theorem is proved.

COROLLARY. *If u is a solution of (1′) belonging to Q_0, then $u'(0+) = \frac{1}{2}g$.*

3. Regularity of solutions.

THEOREM 6. *Let $K^{(n)}(x)$ be a continuous function for $x \geqslant 0$. If u is a solution of (1′) in the set Q_0, then $u^{(n)}(x)$ is a continuous function for $x > 0$.*

Proof For $x > 0$ we have

$$[u^2(x)]' = K(0)u(x) + \int_0^x K'(x-\tau)u(\tau)d\tau.$$

Since u is continuous and $u(x) > 0$ for $x > 0$, we infer that

$$u'(x) = \frac{[u^2(x)]'}{2u(x)}$$

is continuous for $x > 0$.

We suppose that $u^{(k)}(x)$ is continuous for $x > 0$ and $k < n$. This implies that

$$[u^2(x)]^{(k+1)} = \sum_{i=0}^{k} K^{(i)}(0)u^{(k-i-1)}(x) + \int_0^x K^{(k+1)}(x-\tau)u(\tau)d\tau$$

is continuous for $x > 0$.

Using Leibnitz's formula, we have

$$u^{(k+1)}(x) = \frac{[u^2(x)]^{(k+1)} - \sum_{i=1}^{k}\binom{k+1}{i}u^{(i)}(x)u^{(k+1-i)}(x)}{2u(x)} \quad \text{for } x > 0,$$

which implies that $u^{(k+1)}(x)$ is continuous for $x > 0$ and $k < n$. The theorem is proved.

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