DYNAMICAL SYSTEMS AND ERGODIC THEORY
BANACH CENTER PUBLICATIONS, VOLUME 23
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1989

DENSE ORBITS OF HOROSPHERICAL FLOWS

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Given a dynamical system, namely a group T acting on a locally compact second countable space X, it is of interest to know whether there exist dense orbits and, if so, whether we could describe in some way the set of points whose orbits are dense. The question of existence is answered affirmatively, using ergodic theory, if there is a measure on X, with full support, which is invariant (or 'quasi-invariant') and ergodic under the action of T. Though in this situation one knows that orbits of almost all points are dense, in general it is hard to say anything regarding which points have dense orbits.

For certain specific classes of dynamical systems there are adequate answers in this respect. We refer the reader to [11] for some details in this regard. We recall here the classical result due to G. A. Hedlund [20] asserting that for the horocycle flow associated to a surface of constant negative curvature and finite Riemannian area, every orbit is either dense or periodic; if the surface is compact then there are only the dense orbits, while if it is noncompact there are as many one-parameter families (indexed in a certain canonical way) of periodic points as the number of cusps of the surface.

There is a natural generalization of the horocycle flows called '*the horospherical flows*', acting on homogeneous spaces of finite invariant volume: Let G be a connected Lie group and let C be a closed subgroup such that G/C admits a finite G-invariant measure. A subgroup U of G is said to be *horospherical* if there exists $g \in G$ such that

$$U = \{u \in G \mid g^j u g^{-j} \to e \quad \text{as } j \to \infty\},$$

where e is the identity element. The action of a horospherical subgroup of G on a homogeneous space G/C as above is called a *horospherical flow*.

Generalizing the result of Hedlund for the case of compact surfaces (and also a strengthening of that result by H. Furstenberg [18]), R. Bowen [3] and R. Ellis and W. Perrizo [15], proved, by different methods, that if C is a *cocompact* discrete subgroup of G and U is a horospherical subgroup corresponding to an element g such that $\mathrm{Ad}\,g$ is a semisimple linear transformation and the action of g on G/C is weak mixing then the U-action on G/C is strictly ergodic (that is, the G-invariant probability measure on G/C is the only U-invariant probability measure on G/C) and consequently all U-orbits on G/C are dense. Thanks to certain recent observations of D. Witte [28] it truns out that the same conclusion can be upheld so long as G/C is compact and U acts ergodically (cf. Corollary 5.6).

When G/C is noncompact all orbits of horospherical flows cannot be dense. In fact we noted earlier that the horocycle flow, which is itself a special case of a horospherical flow with $G = \mathrm{SL}(2, \boldsymbol{R})$, admits periodic orbits. Generalizing the situation, in [12] the present author showed that if G is a reductive Lie group (in particular, if it is semisimple) and C is a lattice (that is, a discrete subgroup such that the quotient G/C admits a finite invariant measure) then the closure of any orbit of a horospherical subgroup U is always a homogeneous space with finite invariant measure; that is, for any $x \in G/C$ there exists a closed subgroup H containing U such that $\overline{Ux} = Hx$ and Hx admits a finite H-invariant measure.

The main point of this paper is to show that for any connected Lie group G, closed subgroup C such that G/C admits a finite G-invariant measure and a horospherical subgroup U acting ergodically on G/C, the U-orbit of $x \in G/C$ is dense if and only if the U-orbit of the image of x in the largest semisimple factor (namely $G/\overline{RC}$, where R is the radical of G) is dense (cf. Theorem 5.1). This together with the results in [12] constitutes, in a certain sense, a complete description of dense orbits of ergodic horospherical flows. This leads to the following general observation about the orbits of ergodic horospherical flows: Any orbit of such a flow is either dense or is contained in a submanifold M of codimension at least 2, embedded as a closed subset of G/C, and further, there is a closed subgroup H of G such that M is an orbit of H with a finite H-invariant measure (cf. Corollaries 5.3 and 5.5).

We also show (cf. Theorem 5.1) that for an ergodic horospherical flow, defined by U on G/C as above, the G-invariant probability measure on G/C is the only U-invariant measure that factors to the G-invariant measure on the largest semisimple factor $G/\overline{RC}$.

A horospherical subgroup consists entirely of unipotent elements; that is, it is a unipotent subgroup. More generally it is expected that the orbit closure for the action of any unipotent subgroup is a homogeneous space with finite invariant measure and that such measures in fact account for all ergodic invariant measures (cf. [10] Conjectures I and II formulated for one-parameter subgroups). We therefore carry out some of the results in the generality of actions of unipotent subgroups. In particular, we note that for actions of such subgroups there is an invariant partition into homogeneous subspaces on each of which the action is ergodic (cf. Theorem 3.2). Thus, though in general studying the ergodic case may not lead to adequate understanding in the nonergodic case, for actions of unipotent subgroups that is indeed the case.

Along the way, in § 2, we consider, given an ergodic action of certain groups, the set of elements of the group which act ergodically, and make certain observations which may be of independent interest.

§ 1. Preliminaries

Let G be a connected Lie group. For the most part we deal with homogeneous spaces G/C, where C is a closed subgroup such that G/C admits a finite G-invariant (Borel) measure. If, further, C is discrete then C is called a *lattice* in G.

Let Ad denote the adjoint representation of G over its Lie algebra. An element $u \in G$ is said to be *unipotent* if $\mathrm{Ad}\, u$ is a unipotent linear transformation (that is, 1 is the only eigenvalue). A subgroup U of G is said to be *unipotent* if all its elements are unipotent. A subgroup U of G is said to be *horospherical* if there exists $g \in G$ such that

$$U = \{u \in G | \; g^j u g^{-j} \to e \text{ as } j \to \infty\},$$

where e is the identity element. Let g be the element as above. The Lie algebra $\mathfrak{G}$ of G can then be decomposed into $\mathrm{Ad}\, g$-invariant subspaces as

$$\mathfrak{G} = \mathfrak{G}^- + \mathfrak{G}^0 + \mathfrak{G}^+ \text{ (direct sum)}$$

such that if λ is any (complex) eigenvalue of the restriction of $\mathrm{Ad}\, g$ to $\mathfrak{G}^-$, $\mathfrak{G}^0$ or $\mathfrak{G}^+$ respectively, then $|\lambda| < 1$, $|\lambda| = 1$ or $|\lambda| > 1$ respectively. It is easy to see (cf. [6] § 1) that $\mathfrak{G}^-$, $\mathfrak{G}^0$ and $\mathfrak{G}^+$ are Lie subalgebras of $\mathfrak{G}$ and that U is the connected Lie subgroup with $\mathfrak{G}^-$ as the corresponding Lie subalgebra. It is also easy to see that $[\mathfrak{G}^0, \mathfrak{G}^-] \subset \mathfrak{G}^-$ and $[\mathfrak{G}^0, \mathfrak{G}^+] \subset \mathfrak{G}^+$, which in particular implies that U is normalized by any connected normal Lie subgroup M such that all eigenvalues of the restriction of $\mathrm{Ad}\, g$ to the Lie subalgebra corresponding to M are of absolute value 1 (this will be used in § 5). It can also be deduced from commutation relations, similar to the above, for generalized eigenspaces of $\mathrm{Ad}\, g$ (cf. [6] § 1) that any horospherical subgroup is unipotent.

In the following sections we would be interested in dense orbits of actions of unipotent subgroups in general, and of horospherical subgroups in particular, on G/C as above; the latter are called horospherical flows. It is well known that if X is a compact space equipped with a probability measure μ such that $\mu(\Omega) > 0$ for all nonempty open subsets and S is a solvable group acting on X such that μ is the only S-invariant measure on X then all orbits of S are dense in X. In general however the spaces that we consider would be noncompact and the invariant measures would not be unique. The following generalization of the above fact would be useful in the sequel.

1.1. Proposition. *Let X and Y be locally compact second countable spaces and $\varphi: X \to Y$ be a proper surjective (continuous) map. Let μ be a probability measure on X such that $\mu(\Omega) > 0$ for all nonempty open subsets of X. Let S be a solvable group acting on X and Y in such a way that φ is equivariant (that is, $\varphi(sx) = s\varphi(x)$ for all $s \in S$ and $x \in X$). Suppose that the set of S-invariant probability measures λ on X such that $\varphi(\lambda) = \varphi(\mu)$, consists of μ alone. Then for $x \in X$ the orbit Sx is dense in X if and only if $S\varphi(x)$ is dense in Y.*

Proof. If Sx is dense in X then $S\varphi(x) = \varphi(Sx)$ is clearly dense in Y. Now let $x \in X$ be such that $S\varphi(x)$ is dense and let $Z = \overline{Sx}$. It is easy to see that the set of probability measures ν on X such that $\nu(X-Z) = 0$ and $\varphi(\nu) = \varphi(\mu)$ is a nonempty compact convex subset of the space of measures on X equipped with the weak* topology on measures. Also the set is invariant under the S-action induced by the S-action on X (that is, $s\nu(E) = \nu(s^{-1}E)$ for all $s \in S$, Borel sets E and measures ν). Hence by the Kakutani–Markov theorem there is a fixed point, which means that there exists a S-invariant probability measure ν such that $\nu(X-Z) = 0$ and $\varphi(\nu) = \varphi(\mu)$. The condition in the hypothesis then forces that $\nu = \mu$. Thus $\mu(X-Z) = 0$ which, in turn implies that $Z = X$ or, equivalently, that Sx is dense in X.

§ 2. Ergodic transformations from ergodic actions

In this section we prove some general results which will be applied in the later sections. Throughout the section we let $(X, \mathcal{M})$ be a standard Borel space and μ a nonatomic probability measure on $(X, \mathcal{M})$. The actions of (locally compact topological) groups H on $(X, \mathcal{M}, \mu)$ are assumed to preserve $\mathcal{M}$ and μ and be continuous in the sense that the map $h \to \int \varphi(hx)\psi(x)\,d\mu(x)$ is continuous for all $\varphi, \psi \in L^2(X, \mu)$. It may be recalled that such an action yields a continuous unitary representation π of H over the Hilbert space $L^2(X, \mu)$ defined by $\pi(h)f(x) = f(h^{-1}x)$ for all $f \in L^2(X, \mu)$ and $x \in X$. Further, the action is ergodic if and only if there is no nonconstant function f in $L^2(X, \mu)$ such that $\pi(h)f = f$ for all $h \in H$ (cf. [5] or [26] for instance).

2.1. Proposition. *Let $V = \boldsymbol{R}^n$, $n \geqslant 1$ and let π be a continuous unitary representation of V over a separable Hilbert space $\mathcal{H}$. Then there exist* *countably many proper (vector) subspaces* $\{V_i\}_1^\infty$ *of* V *such that if* $w \in V$ *is an element not contained in* V_i *for any* i *and* $\xi \in \mathscr{H}$ *is such that* $\pi(tw)\xi = \xi$ *for all* $t \in \boldsymbol{R}$ *then* $\pi(v)\xi = \xi$ *for all* $v \in V$. *In particular, given an ergodic measure-preserving V-action on* $(X, \mathscr{M}, \mu)$ *there exist countably many proper subspaces* $\{V_i\}_1^\infty$ *of* V *such that for any* w *not contained in* V_i *for any* i *the action of the one-parameter subgroup* $\{tw |\ t \in \boldsymbol{R}\}$, *obtained by restriction, is ergodic.*

Proof. By restricting to the orthocomplement of the set of vectors fixed by $\pi(v)$ for all v, we may assume that $\mathscr{H}$ contains no non-zero vector fixed by $\pi(v)$ for all $v \in V$. Let V^* be the dual vector space of V. We recall that for any subset S of V, ann S denotes the subset

$$\{\chi \in V^* |\ \chi(v) = 0 \text{ for all } v \in S\}.$$

To the representation π there corresponds a (unique) projection-valued measure P defined on the Borel subsets of V^* such that

$$\langle \pi(v)\varphi, \psi \rangle = \int \exp 2\pi i\chi(v)\, d\langle P(\chi)\varphi, \psi\rangle$$

for all $\varphi, \psi \in \mathscr{H}$, where $\langle\ ,\ \rangle$ denotes the inner product in $\mathscr{H}$. It is straightforward to deduce from this that for $v \in V$ there exists $\xi \in \mathscr{H}$, such that $\xi \neq 0$ and $\pi(tv)\xi = \xi$ for all $t \in \boldsymbol{R}$ if and only if $P(\text{ann}\{v\})$ is a nonzero projection. Let $\mathscr{F}$ be the family of subspaces W^* of V^* such that $P(W^*)$ is a nonzero projection and let $\mathscr{F}_m$ be the set of minimal elements of $\mathscr{F}$. We note that in view of dimension considerations any element of $\mathscr{F}$ contains a minimal element. We note also that since $\mathscr{H}$ does not contain any nonzero vector fixed by $\pi(v)$ for all $v \in V$, the zero subspace is not contained in $\mathscr{F}$. For any two distinct subspaces W_1^* and W_2^* of $\mathscr{F}_m$ we have $P(W_1^* \cap W_2^*) = 0$; since P is a projection-valued measure, this implies that the projections $P(W_1^*)$ and $P(W_2^*)$ are orthogonal. Since $\mathscr{H}$ is separable any family of pairwise orthogonal projections must be countable. Thus $\mathscr{F}_m$ is countable. Say $\mathscr{F}_m = \{V_1^*, V_2^*, \ldots\}$. Let V_i, $i \geqslant 1$, be the subspace of V dual to V_i^*. Since $V_i^* \neq \{0\}$ for any i, each V_i is a proper subspace. Now if w is an element of V not contained in V_i for any i then ann $\{w\}$ does not contain V_1^* for any i and consequently $P(\text{ann}\{w\}) = 0$; as noted before this implies that there is no nonzero vector fixed by $\pi(tw)$ for all $t \in \boldsymbol{R}$.

The second part may be deduced by applying the above to the representation of V on $L^2(X, \mu)$ corresponding to the given action.

2.2. Proposition. *Let* U *be a connected nilpotent Lie group. Consider an ergodic measure-preserving action of* U *on* $(X, \mathscr{M}, \mu)$. *Then for almost all* $u \in U$, *with respect to the Haar measure on* U, *the action of* u *on* $(X, \mathscr{M}, \mu)$ *is ergodic.*

Proof. We first note that the set of $u \in U$ for which the action on $(X, \mathscr{M}, \mu)$ is ergodic is in fact a Borel subset. This may be deduced from the fact that the set of all ergodic transformations of a Lebesgue space is a G_δ set in the weak topology (cf. [19] pp. 78–80) and the assumption that the U-action on $(X, \mathcal{M}, \mu)$ is continuous.

We note also that there is no loss of generality in assuming U to be simply connected. Then the exponential map $\exp: \mathfrak{U} \to U$, where $\mathfrak{U}$ is the Lie algebra of U, is an analytic isomorphism. We only need to prove that for almost all $\xi \in \mathfrak{U}$ the action of $\exp \xi$ on $(X, \mathcal{M}, \mu)$ is ergodic. It is easy to see that if $\{\varphi_t\}_{t \in \boldsymbol{R}}$ is a μ-preserving ergodic flow then the action of φ_t is ergodic for all but countably many t; we only need to avoid t which are rational multiples of the periods of the characters on $\boldsymbol{R}$ for which there exist nonconstant eigenfunctions. Hence to prove the proposition it is enough to prove that for almost all $\xi \in \mathfrak{U}$ the action of the one-parameter subgroup $\{\exp t\xi \mid t \in \boldsymbol{R}\}$ is ergodic.

We shall also need the property of nilpotent Lie groups U as above that unless U is one dimensional $U/[U, U]$ (where $[U, U]$ denotes the commutator subgroup) is of dimension at least 2. This may be argued as follows: Suppose $U/[U, U]$ is of dimension $\leqslant 1$. Then there exists a one-parameter subgroup H such that $U = H[U, U]$. But then we get $[U, U] = [H[U, U], H[U, U]] \subset [U, [U, U]]$, since $[H, H]$ is the trivial subgroup; U being nilpotent, that inclusion is possible only if $[U, U]$ is trivial, which in turn implies $U = H$, so that it is one dimensional.

We now complete the proof by induction on the dimension of U. For low dimensions the assertion is evident. Now let U be a simply-connected nilpotent Lie group of dimension $\geqslant 2$ acting μ-preservingly and ergodically on $(X, \mathcal{M}, \mu)$. Let $U_1 = [U, U]$ and let $\mathfrak{U}_1$ be the Lie subalgebra corresponding to U_1 (viz. $\mathfrak{U}_1 = [\mathfrak{U}, \mathfrak{U}]$). Consider the unitary representation say π of U over $L^2(X, \mu)$ corresponding to the U-action. Consider the subspace

$$\mathscr{H} = \{f \in L^2(X, \mu) \mid \pi(u)f = f \text{ for all } u \in U_1\}.$$

Then $\mathscr{H}$ is a closed subspace of $L^2(X, \mu)$ invariant under $\pi(u)$ for all $u \in U$. We thus get a representation π' of U/U_1 over $\mathscr{H}$ defined by $\pi'(uU_1)f = \pi(u)f$ for all $f \in \mathscr{H}$. Since U/U_1 is isomorphic to $\boldsymbol{R}^n$ for some n, by Proposition 2.1 there is a subset Y of $\mathfrak{U}/\mathfrak{U}_1$ such that $\mathfrak{U}/\mathfrak{U}_1 - Y$ is a countable union of proper subspaces and for any $\theta \in Y$, there is no nonconstant function in $\mathscr{H}$ invariant under $\{\pi'(\exp t\theta) \mid t \in \boldsymbol{R}\}$. For any $\theta \in \mathfrak{U}/\mathfrak{U}_1$ let U_θ be the subgroup of U containing $U_1 = [U, U]$ and such that $U_\theta/U_1 = \{\exp t\theta \mid t \in \boldsymbol{R}\}$. Clearly for any $\theta \in Y$ the action of U_θ on $(X, \mathcal{M}, \mu)$ is ergodic, as there are no nonconstant U_θ-invariant functions in $L^2(X, \mu)$. Since U is of dimension at least 2, as noted above U/U_1 is also of dimension at least 2 and hence for any $\theta \in \mathfrak{U}/\mathfrak{U}_1$, U_θ is a simply connected nilpotent Lie group of dimension *less* than the dimension of U. Thus we see that for almost all $\theta \in \mathfrak{U}/\mathfrak{U}_1$ (viz. for $\theta \in Y$) the action of U_θ is ergodic and for these θ, by the induction hypothesis, the action of $\{\exp t\xi | \ t \in \boldsymbol{R}\}$ is ergodic for almost all $\xi \in \mathfrak{U}_\theta$, the Lie algebra corresponding to U_θ. Hence by Fubini theorem we get that for almost all $\xi \in \mathfrak{U}$ the action of $\{\exp t\xi | \ t \in \boldsymbol{R}\}$ on X is ergodic with respect to μ.

2.3. *Remark*. It may be noted that the statement analogous to Proposition 2.2 is not true for a general solvable group. In fact, for certain solvable groups there exist transitive actions with finite invariant measure such that no element of the group acts ergodically (cf. [1] for details). The inductive argument in the proof of Proposition 2.2 breaks down because for a solvable group G, $G/[G, G]$ could be one dimensional.

The analogous statement does not held for actions of semisimple groups either. For instance, for the action of $SL(2, \boldsymbol{R})$ on $SL(2, \boldsymbol{R})/\Gamma$, where Γ is a lattice in $SL(2, \boldsymbol{R})$ (e.g. $\Gamma = SL(2, \boldsymbol{Z})$) the action of no elliptic element can be ergodic; since these form an open subset of $SL(2, \boldsymbol{R})$, the set of elements acting ergodically cannot have full measure. There, however, exists an open set of elements of any noncompact simple Lie group acting ergodically whenever the group acts ergodically and preserving a finite measure (cf. [22]).

§ 3. Partition into closed invariant sets

To begin with we note the following.

3.1. THEOREM. *Let G be a connected Lie group and C be a closed subgroup such that G/C admits a G-invariant probability measure say μ. Let U be a connected unipotent Lie subgroup of G. Then the following conditions are equivalent.*

(i) *The U-action on G/C is ergodic with respect to μ.*

(ii) *The U-action on G/C is topologically transitive (that is, there exists $xC \in G/C$ such that the U-orbit UxC/C is dense in G/C).*

(iii) *If V is the smallest normal subgroup of G containing U then VC is dense in G.*

Proof. The implication (i) $\Rightarrow$ (ii) is standard (cf. [26], Theorem 5.5 for example) and the implication (ii) $\Rightarrow$ (iii) is obvious (because if x is as in (ii) then $x^{-1}Ux$ is contained in V). The implication (iii) $\Rightarrow$ (i) can be deduced from the Mautner–Moore phenomenon, viz. Theorem 1.1 of [23], as follows. Let E be a U-invariant Borel subset and let χ be the characteristic function of E. Let π be the unitary representation of G on $L^2(G/C, \mu)$ associated to the left action. Then by Theorem 1.1 of [23] χ is invariant (as an element of $L^2(G/C, \mu)$ under $\pi(g)$ for all $g \in N$, where N is the smallest closed normal subgroup of G such that for every one-parameter subgroup U_1 of U, $U_1 N/N$

is an Ad-compact subgroup (cf. [23] for terminology). Since U is a unipotent subgroup $U_1 N/N$ being Ad-compact implies that it is contained in the center of G/N. In particular, UN/N is normal in G/N and hence UN is normal in G. Since V is a normal subgroup containing U, the definition of N shows that V contains N. Hence UN is contained in V. Since V is the smallest closed normal subgroup containing U and UN is normal we conclude that $V = \overline{UN}$. Since χ is invariant under $\pi(U)$ and $\pi(N)$ and π is a continuous unitary representation, we get that χ is invariant under $\pi(g)$ for all $g \in V$. But since V is a normal subgroup such that VC is dense in G, any V-invariant function in $L^2(G/C, \mu)$ is, in fact, constant μ almost everywhere (cf. [8] Lemma 9.1 for an idea of the proof of this). Thus χ is constant a.e. and, therefore, either $\mu(E) = 0$ or $\mu(G/C - E) = 0$. This shows that the U-action on G/C is ergodic.

For results on ergodicity of actions of more general subgroups the reader may consult [4].

The next theorem shows that in studying the orbit behaviour for actions of unipotent subgroups, there is no loss of generality, in a certain sense, in assuming the action to be ergodic.

3.2. THEOREM. *Let G be a connected Lie group and C be a closed subgroup such that G/C admits a G-invariant probability measure. Let U be a connected unipotent Lie subgroup of G. Then there is a partition ξ of G/C into closed subsets such that for each element E of ξ there exists a closed connected subgroup H containing U such that $E = Hx$ for some (and hence any) $x \in E$, E admits a (unique) H-invariant probability measure say μ_E and the action of U on E is ergodic with respect to μ_E.*

Proof. Let V be the smallest closed normal subgroup of G containing U and let η be the partition $\{g\overline{VC} \mid g \in G\}$. Since $g\overline{VC} = \overline{VgC}$, each element of η is U-invariant. Since G/C admits a G-invariant probability measure, say μ, it follows from the Fubini–Weil formula (cf. [27], Chapter II § 9) for invariant measures on homogeneous spaces that there exists a $\overline{VC}$-invariant probability measure λ on $\overline{VC}/C$ such that

$$\int f(x)\, d\mu(x) = \int \left(\int f(gh)\, d\lambda(h) \right) d\bar{\mu}(g\overline{VC}) \tag{3.3}$$

for all continuous functions f with compact support on G/C, where $\bar{\mu}$ denotes the quotient measure on $G/\overline{VC}$.

If $\overline{VC} = G$ then by Theorem 3.1 the action of U on G/C is ergodic, and hence we may choose ξ to be the trivial partition (consisting of G/C alone). Now suppose that $\overline{VC}$ is a proper subgroup and let $G_1 = (\overline{VC})^0$, the connected component of the identity in $\overline{VC}$. Then $\overline{VC} = G_1 C$, and hence $\overline{VC}/C$ is canonically equivalent to $G_1/G_1 \cap C$ equipped with a (unique) G_1-

variant probability measure. Similarly for any $g \in G$, $g\overline{VC}/C$ is canonically equivalent to $gG_1g^{-1}/g(G_1 \cap C)g^{-1}$. Since G_1 is of dimension less than that of G, by an obvious induction hypothesis we may assume that for any $\gamma = g\overline{VC}/C$, $g \in G$, there exists a partition ξ'_γ of $gG_1g^{-1}/g(G_1 \cap C)g^{-1}$ into closed subsets satisfying the conclusion of the theorem for gG_1g^{-1} and $g(G_1 \cap C)g^{-1}$ in the place of G and C respectively. It is easy to see that under the equivalence of $g\overline{VC}/C$ with $gG_1g^{-1}/g(G_1 \cap C)g^{-1}$ the partition ξ'_γ of the latter corresponds to a partition ξ_γ of the former into closed subsets satisfying the following conditions: for any element E of ξ_γ there exists a closed connected subgroup H of gG_1g^{-1}, containing U such that $E = Hx$ for any $x \in E$, E admits a unique H-invariant probability measure and the U-action on E is ergodic with respect to the H-invariant measure. Thus if ξ is the partition of G/C refining η and such that for each $\gamma = g\overline{VC}/C$, the restriction of ξ to γ is ξ_γ, then ξ evidently satisfies the requirements of the theorem.

3.4. *Remark.* It may be noted that the decomposition as in Theorem 3.2 is different from the 'ergodic decomposition' in the usual sense. The latter notion is defined only upto sets of measure zero and in fact, in the present case, it is not difficult to see that η as in the proof of the theorem is an ergodic decomposition a.e. The partition ξ is finer. On the other hand, it is not clear whether ξ is countably separated. It may however be of interest to compare it with the decomposition as in [25].

§ 4. Commuting extensions

In this section we prove a result about the behaviour of orbit closures under extensions by normal subgroups which commute with the unipotent subgroup in question. We prove the following.

4.1. THEOREM. *Let G be a connected Lie group and let M be a closed connected normal subgroup of G. Let C be a lattice in G such that MC is closed and MC/C is compact. Let U be a connected unipotent subgroup of G contained in the centraliser of M. Let $x \in G$ and suppose that there exists a closed subgroup L of G such that $\overline{UxMC} = LxC$ and LxC/C admits a finite L-invariant measure. Then there exists a closed subgroup H of G such that $\overline{UxC} = HxC$ and HxC/C admits a finite H-invariant measure. Further, if the U-action on G/C is ergodic with respect to the G-invariant probability measure, then any measure on G/C, which is U-invariant and factors to a G-invariant measure on G/MC (under the canonical quotient map), is itself G-invariant. In particular, if $x \in G$ is such that $UxMC$ is dense in G then UxC is dense in G (and hence the U-orbit UxC/C of xC is dense in G/C).*

Proof. We first note that it is enough to prove the conclusion for $x = e$, the identity element; the general case could then be deduced by considering xCx^{-1} in the place of C.

Since LC/C is canonically equivalent to $L/L \cap C$, the latter admits a finite L-invariant measure. Hence by Theorem 3.2 there exists a closed connected subgroup H of L containing U such that $H(L \cap C)$ is closed, $H(L \cap C)/(L \cap C)$ admits a H-invariant probability measure and the action of U on $H(L \cap C)/(L \cap C)$ is ergodic with respect to that measure. We note also that if $L = G$, and the action of U on G/C admits one dense orbit then H has to be G.

Let η: $G/C \to G/MC$ be the canonical quotient map. Since MC/C is compact, η is a proper map. Hence HMC is a closed subset; since it contains UMC, we get that $HMC = LC$. Since HMC/MC is canonically equivalent to $H/H \cap MC$ and UMC is dense in HMC, we conclude that $U(H \cap MC)$ is dense in H.

Observe that since C is discrete M is an open subgroup of MC (this is the only place where we use the assumption that C is a lattice and not any closed subgroup such that the homogeneous space admits a finite invariant measure). Hence $H \cap M$ is an open subgroup of $H \cap MC$. In turn, $(H \cap M)(H \cap C)$ is also an open subgroup of $H \cap MC$. In particular $(H \cap M)(H \cap C)$ is a closed subgroup of G and further, since $H/H \cap C$ admits a finite H-invariant measure, $(H \cap M)(H \cap C)$ is of finite index in $H \cap MC$. Let $x_1, x_2, \ldots, x_r$, where $r \geqslant 1$, be such that

$$H \cap MC = \bigcup_{i=1}^{r} (H \cap M)(H \cap C)x_i.$$

Then

$$H = \overline{U(H \cap MC)} = \bigcup_{i=1}^{r} \overline{U(H \cap M)(H \cap C)}\, x_i,$$

which implies that $\overline{U(H \cap M)(H \cap C)}$ must have an interior point in H. Since the U-action on

$$H/H \cap C \ \ (\simeq H(L \cap C)/(L \cap C))$$

is ergodic (and hence admits a dense U-orbit) this further implies that $U(H \cap M)(H \cap C)$ is dense in H.

Now consider the quotient map

$$\varphi\colon\ H/H \cap C \to H/(H \cap M)(H \cap C).$$

We claim that φ is a proper map. To prove this it is enough to show that the quotient map

$$H/H \cap C \to H/H \cap MC$$

is a proper map or, in turn, that the quotient map of HC/C onto HMC/MC is a proper map. But the latter indeed is proper since it is the restriction of the quotient map η of G/C onto G/MC.

We shall now show that if λ is any U-invariant measure on $H/H \cap C$ such that $\varphi(\lambda)$ is H-invariant, then λ is H-invariant. The idea of the proof is reminiscent of H. Furstenberg's proof of unique ergodicity of certain extensions (cf. [17]). One way of proving the above assertion, motivated by Furstenberg's argument would be to note that U is a nilpotent Lie group and hence there are versions of the individual ergodic theorem for measure-preserving U-actions, with respect to suitable averaging sequences (cf. [2] and [16] for details), so that we can define analogues of generic points and then repeat Furstenberg's argument. However, these versions of the individual ergodic theorem are technically more involved and presumably not so well known. Here we shall give an argument using Proposition 2.2, so that we need only the usual notion of generic points (cf. [14]).

Let μ be the H-invariant probability measure on $H/H \cap C$. Recall that μ is ergodic with respect to the U-action. Let λ be any U-invariant ergodic probability measure on $H/H \cap C$ such that $\varphi(\lambda)$ is H-invariant. By Proposition 2.2 there exists $u \in U$ such that both λ and μ are ergodic with respect to the action of u. Let $\mathscr{G}(\lambda)$ and $\mathscr{G}(\mu)$ be the sets of generic points of the u-action on $H/H \cap C$, with respect to the measures λ and μ respectively. Then $\lambda(\mathscr{G}(\lambda)) = \mu(\mathscr{G}(\mu)) = 1$ (cf. [14], Proposition 5.9). Since $\varphi(\lambda) = \varphi(\mu)$, viz. the unique H-invariant probability measure on $H/(H \cap M)(H \cap C)$, the last condition yields, in particular, that there exist $x \in \mathscr{G}(\lambda)$ and $y \in \mathscr{G}(\mu)$ such that $\varphi(x) = \varphi(y)$; thus, in particular there exists $m \in H \cap M$ such that $x = my$. But since u commutes with m and μ is m-invariant a routine argument (as in [17]) implies that for any $z \in \mathscr{G}(\mu)$, $mz \in \mathscr{G}(\mu)$. Thus we get that $x \in \mathscr{G}(\mu)$, which is impossible unless $\lambda = \mu$. Thus μ is the only ergodic U-invariant probability measure on $H/H \cap C$ which projects under φ to the H-invariant measure. By ergodic decomposition, this implies that μ is the only U-invariant probability measure whose projection under φ is H-invariant. By Proposition 1.1 this implies that $U(H \cap C)$ is dense in H. Under the canonical equivalence of HC/C and $H/H \cap C$ this means that UC is dense in HC so that $\overline{UC} = HC$ as asserted by the theorem. The equivalence as above also implies that HC/C admits a finite H-invariant measure.

Now let D be any subgroup of G satisfying the conditions of Theorem 4.1 in the place of C, and suppose that the action of U, as above, on G/D is ergodic. Then there exists $g \in G$ such that UgD/D is dense in G/D or, equivalently, UgD is dense in G. Put $C = gDg^{-1}$. Then C satisfies the conditions of the theorem and further, UC is dense in G. As noted above, in this case H as in the theorem must be G. Hence, by what we proved above, any U-invariant measure on G/C which factors to a G-invariant measure on G/MC is itself G-invariant. The corresponding assertion for G/D evidently follows from this, which shows that the assertion in the theorem about invariant measures in the case of ergodic actions holds. The last part then follows from Proposition 1.1.

§ 5. Orbits of horospherical flows

We now deduce the following theorem, which may be considered the main result of this paper, on orbits and invariant measures of horospherical flows (cf. § 1 for definition).

5.1. THEOREM. *Let G be a connected Lie group and let C be a closed subgroup of G such that G/C admits a (unique) G-invariant probability measure, say μ. Let U be a horospherical subgroup of G acting ergodically on G/C (with respect to μ). Let R be the radical of G and let η: $G/C \to G/\overline{RC}$ be the canonical quotient map. Then the following conditions are satisfied:*

(i) *if λ is a U-invariant probability measure on G/C such that $\eta(\lambda)$ is G-invariant then $\lambda = \mu$ and*

(ii) *if $x \in G$ is such that the U-orbit of $x\overline{RC}$ is dense in $G/\overline{RC}$ then the U-orbit UxC/C of xC is dense in G/C.*

It may be noted here that since G/R is semisimple, the results of [12] give satisfactory conditions under which the U-orbit of $x\overline{RC}$ is dense in $G/\overline{RC}$ and similarly, when UR/R is a maximal horospherical subgroup in G/R the results of [10] give sufficient conditions for $\eta(\lambda)$ to be G-invariant. It should be noted that in general there exist other invariant measures. We refer the reader to the survey [11] for details.

To begin with the proof, we note that there is no loss of generality in assuming that C does not contain any nontrivial normal subgroup of G. Since U is a unipotent subgroup of G (cf. § 1) acting ergodically on G/C, thanks to the observations of D. Witte, that assumption entails that the following conditions hold:

(a) C is discrete (namely, a lattice in G);
(b) R is nilpotent and
(c) $R \cap C$ is a (cocompact) lattice in R.

Witte makes the observations when there is a single unipotent element acting ergodically on G/C (cf. [28] Proposition 2.6). However, his arguments would go through for any unipotent subgroup. In any case, the subgroup U as in the hypothesis is a connected nilpotent Lie group, and hence by Proposition 2.2 it contains an element acting ergodically on G/C; thus we may apply Witte's observations in the original form and conclude the above assertions.

For proving the theorem, apart from the results of earlier sections we need the following theorem from [10] which it would be convenient to recall separately.

5.2. THEOREM (cf. [10] Theorem 4.1). *Let G be a connected Lie group and Γ be a lattice in G. Let U be a connected Lie subgroup of G and let V be the smallest normal subgroup of G containing U. Suppose that $(\overline{V\Gamma})^0$ is a nilpotent subgroup. Let φ: $G/\Gamma \to G/\overline{V\Gamma}$ be the canonical quotient map and let λ be any U-invariant measure on G/Γ such that $\varphi(\lambda)$ is G-invariant. Then λ is G-invariant.*

Proof of Theorem 5.1. Let V be the smallest closed normal subgroup of G containing $U \cap R$ and let $W = (\overline{VC})^0$. Then W is a closed connected subgroup normalised by C. Let Ad denote the adjoint representation of G over its Lie algebra. By Borel's density theorem the Zariski closure of $\{\mathrm{Ad}\,x \mid x \in C\}$ contains the nilradical and also all simple noncompact Lie subgroups of G (cf. [9]). Since R is the nilradical of G, we conclude that $\{\mathrm{Ad}\,x \mid x \in C\}$ contains $\{\mathrm{Ad}\,g \mid g \in G_1\}$, where G_1 is the smallest closed normal subgroup of G such that G/G_1 is a compact semisimple Lie group. In particular it follows that G_1 normalises W (cf. [24] Chapter 8 for similar arguments). Since U has to be contained in G_1 and the action of U on G/C is ergodic, it follows that $G_1 C$ is dense in G. Since G_1 and C both normalise W, the preceding assertion implies that W is a normal subgroup of G.

Now let $G' = G/W$, $C' = CW/W = \overline{VC}/W$ and $U' = UW/W$. We note that C' is a lattice in G' and U' is a horospherical subgroup acting ergodically on G'/C'. Note that by condition (c) above, $W \subset R$. Let $R' = R/W$. Then R' is the radical of G', $R'C' = RC/W$ is closed and $R' \cap C'$ is a lattice in R'. We would now like to apply Theorem 4.1 to the U'-action on G'/C'. For this purpose we show that U' is contained in the centraliser of R'. Observe that $U' \cap R'$ is the trivial subgroup, since $U \cap R$ is contained in W. Let $g \in G'$ be such that U' is the horospherical subgroup corresponding to g. Since G' admits a lattice, it is unimodular and hence $|\det \mathrm{Ad}\,g| = 1$, where Ad is the adjoint representation of G' and 'det' stands for the determinant. Since a similar assertion must also hold for gR'/R' in the semisimple group G'/R', it follows that the determinant of the restriction of $\mathrm{Ad}\,g$ to the Lie subalgebra say $\mathfrak{R}'$ of R' must also be of absolute value 1. Since $U' \cap R'$ is the trivial subgroup, no eigenvalue of the restriction of $\mathrm{Ad}\,g$ to $\mathfrak{R}'$ is of absolute value less than 1 (cf. § 1). The preceding assertion therefore implies that all eigenvalues of the restriction of $\mathrm{Ad}\,g$ to $\mathfrak{R}'$ are of absolute value 1. This implies, in particular that R' normalises U' (cf. § 1). Now if $r \in R'$ and $u \in U'$ then $rur^{-1}u^{-1} = (rur^{-1})u^{-1} = r(ur^{-1}u^{-1}) \in U' \cap R'$, the trivial subgroup. Hence U' is contained in the centraliser of R'. Thus the conditions of Theorem 4.1 are satisfied for G', C', U' and R' in the place of G, C, U and M respectively. Also the U'-action on G/C is ergodic. Therefore by that theorem any U'-invariant measure ν on G'/C' which factors to the G'-invariant measure on $G'/R'C'$ under the canonical quotient map, is itself G'-invariant.

Now let λ be any U-invariant measure on G/C which factors to a G-invariant measure on G/RC, under the canonical quotient map. Let

$$\varphi: G/C \to G'/C' \simeq G/WC$$

be the quotient map onto G/WC. The preceding deduction from Theorem 4.1 then implies that $\varphi(\lambda)$ is G-invariant. Since $W = (\overline{V\Gamma})^0$, where V is the smallest closed normal subgroup of G containing $U \cap R$, applying Theorem 5.2 (for $U \cap R$ in the place of U) we conclude that λ is G-invariant. Thus assertion (i) of Theorem 5.1 holds. Assertion (ii) follows from assertion (i) and Proposition 1.1.

5.3. COROLLARY. *Let G be a connected Lie group and C be a closed subgroup such that G/C admits a finite G-invariant measure. Let U be a horospherical subgroup of G and let $x \in G$. Then either*

(i) *the U-orbit of xC in G/C is dense in G/C or*

(ii) *there exists a closed connected subgroup H containing U such that HxC/C is a proper closed subset and admits a finite H-invariant measure.*

If the U-action is ergodic and (i) *does not hold then H as in* (ii) *contains the radical of G.*

Proof. If the action of U on G/C is not ergodic then the assertion follows from Theorem 3.2. Now suppose that the U-action is ergodic. There is also no loss of generality in assuming that C contains no nontrivial normal subgroup of G and, as in the proof of Theorem 5.1, this assumption leads to conditions (a), (b) and (c) as listed there. Now G/R is a connected semisimple Lie group and by conditions (a) and (c), RC/C is a cocompact lattice in G/R. Further, UR/R is a horospherical subgroup acting ergodically on G/RC. Hence by Theorem A of [12] either $UxRC$ is dense in G or there exists a closed subgroup H containing U and R such that $HxRC$ is a proper closed subset and $HxRC/RC$ admits a finite H-invariant measure. In the former case by Theorem 5.1 UxC/C is dense in G/C. In the latter case we see that $HxRC = HRxC = HxC$ is a proper closed subset. Also in view of condition (c) the quotient map of G/C into G/RC is a proper map. Since $HxRC/RC$ admits a finite H-invariant measure the preceding observation yields that HxC/C admits a finite H-invariant measure. This shows that, in the latter case, assertion (ii) holds, except for the connectedness condition on H, which can be readily incorporated, since U and R are connected, by replacing H by H^0.

5.4. *Remark.* It is tempting to argue further by induction within the subgroup H as in Corollary 5.3 above and conclude that $\overline{UxC/C}$ must be a homogeneous space of a suitable closed subgroup. However, in general U may not be a horospherical subgroup in H. In [12], a similar difficulty was overcome using the theory of algebraic groups. However there is no direct parallel to that in the general case.

5.5. COROLLARY. *Let the notation be as in Corollary* 5.3 *and suppose further that the* U*-action on* G/C *is ergodic. Then for any* $x \in G$ *the* U*-orbit* UxC/C *of* xC *is either dense in* G/C *or it is contained in a closed submanifold of* G/C *of codimension at least* 2.

Proof. Let $x \in G$ be such that UxC/C is not dense in G/C. Then by Corollary 5.3 there exists a closed connected subgroup H containing U and the radical R such that HxC/C is a proper closed subset and admits a finite H-invariant measure. Recall also that, the U-action being ergodic, C is discrete (condition (a) as above). Hence HxC/C is a submanifold whose codimension in G/C equals the codimension of H in G. Suppose if possible, that H is of codimension 1 in G. Let N be the largest closed connected normal subgroup of G contained in H. We note that N contains R. Now G/N is a semisimple Lie group admitting a subgroup H/N of codimension 1 which does not contain any nontrivial closed normal subgroup of G/N. This implies that G/N is locally isomorphic to $\mathrm{SL}(2, \boldsymbol{R})$ (cf. [21] for a more general result). Let $W = \overline{NC}$. Clearly $(G/N)/(W/N)$ admits a G/N-invariant probability measure. Since G/N is a simple Lie group, by Borel's density theorem this implies that either $W/N = G/N$ or W/N is discrete. Observe that $xW = x\overline{NC} = \overline{NxC}$ is contained in HxC which is a proper closed subset of G and hence $W/N \neq G/N$. Hence W/N is discrete; namely, a lattice in G/N. Any connected 2-dimensional subgroup of $\mathrm{SL}(2, \boldsymbol{R})$ is conjugate to the subgroup

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \; a, b \in \boldsymbol{R},\ a > 0 \right\}.$$

We deduce that any such subgroup contains a semisimple element g such that $\mathrm{Ad}\, g$ has an eigenvalue of absolute value other than 1, and also the horospherical subgroup corresponding to g. Since G/N is locally isomorphic to $\mathrm{SL}(2, \boldsymbol{R})$ and H/N is a two dimensional subgroup it is easy to deduce from the above observation that H/N contains an element $\bar{h}$ such that $\mathrm{Ad}\, \bar{h}$ has an eigenvalue of absolute value other than 1, and also the corresponding horospherical subgroup. By Proposition 1.5 of [13] this implies that all orbits of H/N on $(G/N)/(W/N)$ are dense; in other words all orbits of H on G/W are dense. But observe that $HxW = Hx\overline{NC} \subset \overline{HxNC} = \overline{HNxC} = \overline{HxC} = HxC$ so that the H-orbit of xW is not dense in G/W. Thus we arrive at a contradiction which shows that H must be of codimension at least 2.

Finally, we note the following.

5.6. COROLLARY (cf. [3] and [15]). *Let G be a connected Lie group and let C be a closed subgroup such that G/C is compact and admits a G-invariant probability measure, say μ. Let U be a horospherical subgroup of G acting ergodically on G/C (with respect to μ) or equivalently, such that there exists a dense U-orbit. Then the U-action on G/C is strictly ergodic; that is, μ is the only U-invariant probability measure on G/C. Consequently the action is minimal; that is, all the orbits are dense.*

This result was proved by R. Bowen [3] and later by a different method by R. Ellis and W. Perrizo [15] under certain additional restrictions (viz. C *discrete*, U the horospherical subgroup defined by a *semisimple* element g whose action on G/C is *weak mixing*). One could deduce the above, using Theorem 5.1, from the special case of their result for semisimple Lie groups, where it is fairly apparent that the additional conditions are automatically satisfied (with a suitable choice for g). On the other hand, one could also deduce it directly from their result using D. Witte's observations (a), (b) and (c) as in the proof of Theorem 5.1 to show that their conditions are satisfied. We omit the details.

Acknowledgement. I thank the Sonderforschungsbereich – 170, Geometrie und Analysis, Göttingen and especially Professor M. Denker for their hospitality during the Autumn of 1985 and the Summer of 1986.

I also thank the Stefan Banach International Mathematical Centre, Warsaw, and Professor F. Przytycki in particular, for inviting me to participate in the Semester on Dynamical Systems and Ergodic Theory in the Autumn of 1986.

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