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NON-ISOMORPHIC STEINER TRIPLES WITH SUBSYSTEMS

BY

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Any family consisting of 3-element subsets of a v-element set, with the property that each pair of the set is contained in one and only one triple of the family, is called a *system of Steiner triples* in the v-element set and will be denoted by $B(3,1,v)$. As has been known since long [4], a system of Steiner triples does exist if and only if

$$v \equiv 1 \text{ or } 3 \pmod 6.$$

Let $N(v)$ be the number of all non-isomorphic systems of Steiner triples in a v-element set and let $N_k(v)$ be the number of all those non-isomorphic systems of Steiner triples in the v-element set which contain some $B(3,1,k)$.

The aim of the present paper is to give a lower estimate for $N_{6i+1}(v)$ and $N_{6i+3}(v)$ with respect to v.

In the proof we use the construction of Hanani [1] and its modification by Pukanow [2], [3]. Recall some definitions and theorems.

Let m be a positive integer and let $\tau_0, \tau_1, \ldots, \tau_{m-1}$ be mutually disjoint sets consisting of $t \geqslant m-1$ elements each. A system of t^2 m-tuples such that each m-tuple has exactly one element in common with each set τ_i and any two m-tuples have at most one element in common will be denoted by $T(m,t)$. The set of numbers t for which there exists at least one system $T(m,t)$ will be denoted by $T(m)$. We shall also use the notation $T_e(m,t)$ $(0 \leqslant e \leqslant t)$ instead of $T(m,t)$ to indicate that among the m-tuples belonging to $T(m,t)$ there are at least e disjoint sets of t mutually disjoint m-tuples. The set of numbers t for which systems $T_e(m,t)$ exist will be denoted by $T_e(m)$.

Let $t = p_1^{\alpha_1}\cdot \ldots \cdot p_n^{\alpha_n}$, where p_i are distinct primes and α_i are positive integers. The following assertions have been known since long (see [1]):

(A) *If* $p_i^{\alpha_i} \geqslant m$ *for* $i = 1, \ldots, n$, *then* $t \in T_t(m)$.

(B) *If* $p_i^{\alpha_i} \geqslant m-1$ *for* $i = 1, \ldots, n$, *then* $t \in T_{\cdot}(m)$.

Let $w_1, \ldots, w_m$ be mutually disjoint sets such that each of $w_1, \ldots$ $\ldots, w_{m-1}$ consists of t elements and w_m consists of $t-q$ elements. A system of $t\cdot(t-q)$ m-tuples and $t\cdot q$ $(m-1)$-tuples will be called a *semi-T-system*, denoted by $T(m, t, q)$, if

1° each m-tuple and each $(m-1)$-tuple has exactly one element in common with each of the sets w_i;

2° every pair consisting of two $(m-1)$-tuples, or one $(m-1)$-tuple and one m-tuple, or two m-tuples has at most one element in common.

The set of all numbers t for which there exists at least one $T(m, t, q)$ will be denoted by $T(m, t)$.

The following proposition is known (see [2], Theorem 8).

(C) *If there exists a system $T(m, t)$ and if $q \leqslant t$, then there exists a semi-T-system $T(m, t, q)$.*

Let E be a v-element set, let $K = \{k_i\}_{i=1,\ldots,n}$ be a finite set of integers such that $3 \leqslant k_i \leqslant v$ for $i = 1, \ldots, n$, and let λ be a positive integer. Each system of subsets of E such that the number of elements in each of them belongs to K, and each pair of elements of E is contained in exactly λ subsets of the system will be denoted by $B(K, \lambda, v)$. Elements of $B(K, \lambda, v)$ are called *blocks*. The set of numbers v for which there exists at least one $B(K, \lambda, v)$ will be denoted by $B(K, \lambda)$. If $K = \{k\}$, we write $B(k, \lambda, v)$ and $B(k, \lambda)$, and so $B(3, 1, v)$ is a system of Steiner triples.

THEOREM 1. *Let $K_1 = \{3, 4\}$. If $u \neq 6$ and $u \equiv 0$ or* 1 (mod 3), *then $u \in B(K_1, 1)$.*

Proof. We first consider the case

$$(*) \qquad u = 24, 28, 40, 42, 46, 48, 52, 58, 60, 64 \quad \text{or} \quad u \geqslant 66.$$

In that case we are able to construct semi-T-systems $T(m, t, q)$ for $m = 4$, $t = (u+q)/4$, and the values of q shown in Table 1. In fact, these values are chosen to satisfy $t \equiv 1 \pmod 6$. Consequently, there exist systems $T(4, t)$ in view of (A). It follows from Table 1 that, for $u \geqslant 66$ and all q, we have $u \geqslant 3q$ and so $q \leqslant t$. For smaller u in $(*)$ the same can be checked by taking the corresponding q. Hence, using (C) for $m = 4$, we see that there exists a semi-T-system $T_u = T(4, t, q)$ $(t = (u+q)/4)$.

Table 1

u (mod 24)	q	u (mod 24)	q	u (mod 24)	q	u (mod 24)	q
0	4	6	22	12	16	18	10
1	3	7	21	13	15	19	9
3	1	9	19	15	13	21	7
4	0	10	18	16	12	22	6

Since $t \equiv 1 \pmod 6$, for $i = 1, 2, 3$ there exists a system $B(3, 1, t)$ in w_i. We denote it by B_i.

If $q \equiv 0 \pmod 2$, then $t-q \equiv 1$ or $3 \pmod 6$ and we can construct $B_4 = B(3, 1, t-q)$ in w_4.

Putting

$$B = T_u \cup \bigcup_{i=1}^{4} B_i, \tag{**}$$

we see that B is a $B(K_1, 1, u)$, and so the assertion of Theorem 1 follows.

If $q \equiv 1 \pmod 2$, then $t-q \equiv 0$ or $4 \pmod 6$ and we have to consider four cases.

(1) $t-q \equiv 4 \pmod{12}$.

By virtue of [1], we may construct $B_4 = B(4, 1, t-q)$ in w_4 and use (**).

(2) $t-q \equiv 0 \pmod{12}$.

We adjoin to w_4 one auxiliary element and in the $(t-q+1)$-element set we construct $B(4, 1, t-q+1)$. Then we remove the adjoined element from the quadruples in which it appears and we get $B_4 = B(K_1, 1, t-q)$, which has $(t-q)/3$ triples and $[(t-q)(t-q-3)]/12$ quadruples. Now (**) gives the result.

(3) $t-q \equiv 10 \pmod{12}$.

We adjoin to w_4 three auxiliary elements and in the $(t-q+3)$-element set we construct $B(4, 1, t-q+3)$ in such a way that the three adjoined elements are in one quadruple. Thus any other quadruple has at most one adjoined element. Removing the quadruple that contains all three adjoined elements and the adjoined elements from the quadruples in which they appear single, we get $B_4 = (K_1, 1, t-q)$ which has $t-q-1$ triples and $[(t-q-1)(t-q-6)]/12$ quadruples. We use again (**).

(4) $t-q \equiv 6 \pmod{12}$.

In this case we remove some three elements a_1, a_2, a_3 from the set in which B_4 should be constructed. In the remaining set there exists a system of Steiner triples $B_0 = B(3, 1, t-q-3)$ satisfying Kirkman's condition [4]. Since $t-q > 6$ (in view of (*) and Table 1), B_0 splits into more than three groups according to this condition. Let C_1, C_2, and C_3 be any three of them. We adjoin a_i to every triple in C_i, thus getting $B_4 = B(K_1, 1, t-q)$. Again (**) gives the result.

It remains to consider $u < 66$ distinct from the values listed in (*). For $u \equiv 1$ or $3 \pmod 6$, $u < 66$, we construct $B(3, 1, u)$.

For $u = 16$ there exists $B(4, 1, 16)$.

For $u = 18$ we construct $B(3, 1, u-3)$ satisfying Kirkman's condition and we proceed as in case (4) for u instead of $t-q$, thus getting $B(K_1, 1, 18)$.

In all cases which follow, the existence of the corresponding T-systems or semi-T-systems is guaranteed by assertion (B).

If $u = 10$, we construct $T_{10} = T(4, 3, 2)$ and check that $\bigcup_{i=1}^{4} w_i \cup T_{10}$ is a $B(K_1, 1, 10)$.

If $u = 12$, we take $T_{12} = T_0(3, 4)$ and check that $\bigcup_{i=1}^{3} \tau_i \cup T_{12}$ is a $B(K_1, 1, 12)$.

If $u = 22$, we find $T_{22} = T(4, 7, 6)$ and in every w_i $(i = 1, 2, 3)$ we take a B_i which is a $B(3, 1, 7)$. Then $\bigcup_{i=1}^{3} B_i \cup T_{22}$ is a $B(K_1, 1, 22)$.

If $u = 30$, we construct $T_{30} = T(4, 9, 6)$ and in every w_i $(i = 1, 2, 3)$ we take a B_i which is a $B(3, 1, 9)$. We then put $\bigcup_{i=1}^{3} B_i \cup T_{30}$ to obtain a $B(K_1, 1, 30)$.

If $u = 34$, we take $T_{34} = T(4, 9, 2)$ and in w_1, w_2, w_3 we find $B(3, 1, 9)$-systems B_1, B_2, B_3, respectively, whereas in w_4 we find $B_4 = B(3, 1, 7)$. Then $\bigcup_{i=1}^{4} B_i \cup T_{34}$ is a $B(K_1, 1, 34)$.

If $u = 36$, we find $T_{36} = T(4, 9)$ and in every τ_i $(i = 1, 2, 3, 4)$ we construct a B_i which is a $B(3, 1, 9)$. Then $\bigcup_{i=1}^{4} B_i \cup T_{36}$ is a $B(K_1, 1, 36)$.

If $u = 54$, we find $T_{54} = T(4, 15, 6)$ and in w_1, w_2, w_3 we find $B(3, 1, 15)$-systems B_1, B_2, B_3, respectively, whereas in w_4 we find $B_4 = B(3, 1, 9)$. Then $\bigcup_{i=1}^{4} B_i \cup T_{54}$ is a $B(K_1, 1, u)$.

Remark 1. Constructions used in the proof of Theorem 1 allow us to evaluate precisely the number of triples and quadruples in $B(K_1, 1, u)$. It is evident that we may also take values for t and q other than those used in that proof without breaking conditions $u \geqslant 3q$ and $t \geqslant m$. We may put $t = (u+q_i)/4$, $q_i = 24i+q_0$, where q_0 is q taken from Table 1 according to u, and $0 \leqslant i \leqslant (u-3q)/72$.

COROLLARY 1. *For u sufficiently large there are $[u/72]$ non-isomorphic systems of blocks $B(K_1, 1, u)$.*

Proof. Given u, we can repeat all the described constructions for q_i instead of q. For different q_i's the resulting systems $B(K_1, 1, u)$ will contain different numbers of triples, and so different numbers of quadruples. In this way we get the conclusion.

For a given natural n let

$$K_n = \{3, 4, 3n, 3n+1\} \quad \text{and} \quad N = (p_1 \cdot \ldots \cdot p_k) \cdot (3n+1),$$

where $p_1, \ldots, p_k$ are all primes less than $3n$.

THEOREM 2. *For each n, if $u \equiv 0$ or 1 (mod 3), and $u > 3nN$, then there exists a system of blocks $B(K_n, 1, u)$ in which blocks consisting of $3n$ and $3n+1$ elements do occur.*

Proof. Let n be fixed and let u satisfy the assumption. For $m = 3n+1$, $t = (u+q)/m$, and q being the least positive integer such that

$$\frac{u+q}{m} \equiv 1 \pmod{p_1 \cdot \ldots \cdot p_k} \quad \text{and} \quad t-q \neq 6,$$

we are able to construct a semi-T-system $T(m, t, q)$; denote it by T_u. In fact, since

$$t \equiv 1 \pmod{p_1 \cdot \ldots \cdot p_k},$$

there exists a system $T(3n+1, t)$ in view of (A). It follows from $u > 3nN$ that $u > (m-1)\cdot q$, whence the required T_u exists by virtue of (C). In Table 2 we give values of q that correspond to $u \equiv 0$ or $1 \pmod 3$ in the case where $n = 2$, $m = 7$, $N = 210$.

Table 2

u (mod 210)	q	u (mod 210)	q	u (mod 210)	q	u (mod 210)	q	u (mod 210)	q
0	7	45	4	90	1	135	40	180	37
1	6	46	3	91	42	136	39	181	36
3	4	48	1	93	40	138	37	183	34
4	3	49	42	94	39	139	36	184	33
6	1	51	40	96	37	141	34	186	31
7	42	52	39	97	36	142	33	187	30
9	40	54	37	99	34	144	37	189	28
10	39	55	36	100	33	145	30	190	27
12	37	57	34	102	31	147	28	192	25
13	36	58	33	103	30	148	27	193	24
15	34	60	31	105	28	150	25	195	22
16	33	61	30	106	27	151	24	196	21
18	31	63	28	108	25	153	22	198	19
19	30	64	27	109	24	154	21	199	18
21	28	66	25	111	22	156	19	201	16
22	27	67	24	112	21	157	18	202	15
24	25	69	22	114	19	159	16	204	13
25	24	70	21	115	18	160	15	205	12
27	22	72	19	117	16	162	13	207	10
28	21	73	18	118	15	163	12	208	9
31	18	75	16	120	13	165	10		
33	16	76	15	121	12	166	9		
34	15	78	13	123	10	168	7		
36	13	79	12	124	9	169	6		
37	12	81	10	126	7	171	4		
39	10	82	9	127	6	172	3		
40	9	84	7	129	4	174	1		
42	7	85	6	130	3	174	42		
43	6	87	4	132	1	177	40		
		88	3	133	84	178	39		

Since $t \equiv 1 \pmod 6$, we may construct a $B_i = B(3, 1, t)$ in every w_i for $i = 1, \ldots, 3n = m-1$.

It is easily seen that $t-q \equiv 0$ or $1 \pmod 3$. Hence,

(i) if m and u are both even or both odd, then $t-q \equiv 1$ or $3 \pmod 6$ and we may construct a $B_m = B(3, 1, t-q)$ in w_m;

(ii) if m is odd and u is even, or conversely, then $t-q \equiv 0$ or $4 \pmod 6$.

Thus, since $t-q \neq 6$, we can apply Theorem 1 to construct a $B_m = B(K_1, 1, t-q)$ in w_m. In both cases, (i) and (ii), we state that $T_u \cup \bigcup_{i=1}^{m} B_i$ is a $B(K_n, 1, u)$ and that it contains blocks of $m-1$ or m elements since each T_u consists of such blocks only.

But it is easily seen that $u > 3nN$ implies $q < t$, whence $B(K_n, 1, u)$ contains blocks of $3n$ and blocks of $3n+1$ elements as well.

Remark 2. *If $n = 2$, Theorem 2 is valid for $u \geqslant 505$ instead of $u > 3nN = 1260$.*

We omit the proof.

Remark 3. We have to be more careful in the sequel when constructing systems $B(K_n, 1, u)$ in the proof above.

Let $S \subset U$ be any sets and let $B(3, 1, k)$ and $B(K_n, 1, u)$ be constructed in S and U, respectively. If for every block $\alpha \in B(3, 1, k)$ there exists $\beta \in B(K_n, 1, u)$ such that $\alpha \subset \beta$, then $B(3, 1, k)$ is said to be a *3-subsystem* of $B(K_n, 1, u)$. We may assert that every $B(K_n, 1, u) = T_u \cup \bigcup B_i$ contains two disjoint blocks β_1 and β_2 belonging to T_u and such that in no 3-subsystem of $B(K_n, 1, u)$ there is a triple $\{x, y, z\}$ common with β_1 or β_2. This can be done in the following way. If there exists a triple

$$\{x, y, z\} \in B(3, 1, s) \cap \beta_1,$$

where $B(3, 1, s)$ is a 3-subsystem of $B(K_n, 1, u)$, then we can renumber elements of w_1 in a way that if

$$w_1 \cap \beta_1 = x \in \{x, y, z\},$$

then we replace x by a certain x_1 such that

$$x \notin \{x_1, y, z\} \in B(3, 1, s).$$

This renumbering concerns only the system $B(3, 1, t)$ constructed in w_1, whereas all T-blocks remain unchanged. We may do the same for β_2. Details completing the proof can be found in [3].

A system of blocks $B(3, 1, u)$ is said to be *prime* if it has no subsystems. A system of blocks $B(3, 1, u)$ is said to be *1-prime* if it has no subsystem $B(3, 1, d)$, where $d \equiv 1 \pmod 6$.

Now we construct $B(3, 1, 2u+1)$ by applying the method of Hanani ([1], Theorem 5.5). Let, namely,

$$E_1 = \{1, \ldots, u\} \quad \text{and} \quad E_2 = \{u+1, \ldots, 2u\}.$$

In E_1 we construct $B(K_n, 1, u)$. Then we shift it for u, thus obtaining a $B(K_n, 1, u)$ in E_2. For every block

$$\{x_1, \ldots, x_k\} \in B(K_n, 1, u) \qquad (k \in K_n)$$

we can construct a system of Steiner triples $B(3, 1, 2k+1)$ in the set

$$\{x_1, \ldots, x_k, x_1+u, \ldots, x_k+u, 2u+1\}$$

in a way such that the union of all these systems is a system of Steiner triples in $\{1, \ldots, 2u, 2u+1\}$. We have

$$B(3, 1, 2u+1) = \bigcup B(3, 1, 2k+1), \tag{1}$$

where the union is taken over all blocks in $B(K_n, 1, u)$. Then, as is easily seen, we deduce

(α) If a triple in some $B(3, 1, 2k+1)$ contains the element $2u+1$, then it must be of the form $\{x_i, x_i+u, 2u+1\}$. Hence every triple not containing $2u+1$ is of the form $\{y_1, y_2, y_3\}$, where $|y_i - y_j| \neq u$ for $i, j = 1, 2, 3$.

For our purposes we must consider the summands in (1) to further conditions:

(β) Each $B(3, 1, 2k+1)$ is prime or 1-prime; if $k \equiv 0 \pmod 3$, then it is prime.

(γ) If a triple $\{x_i, x_j, x_k\}$ belongs to $B(K_n, 1, u)$, then it belongs to the corresponding $B(3, 1, 7)$.

Condition (β) can be satisfied according to [7], Lemma 1 and Remark 2, whereas (γ) can be required without breaking (β), since every system of Steiner triples in a 7-element set is prime.

LEMMA 1. *Any subsystem of $B(3, 1, 2u+1)$ not containing $2u+1$ is a 3-subsystem of $B(K_n, 1, u)$.*

Proof. We define a set of isomorphisms of $B(K_n, 1, u)$ in the following way. We choose any r elements $x_1, \ldots, x_r$ from $E = \{1, \ldots, u\}$ and replace every x_j $(1 \leqslant j \leqslant r)$ by $x_j + u$. Denote by B^i $(1 \leqslant i \leqslant 2^u)$ the resulting systems, isomorphic to $B(K_n, 1, u)$. Let $T = \{x_1, \ldots, x_k\}$ be a block belonging to some B^i. Then we set $\bar{x}_i = x_i + u$ if $x_i \leqslant u$ and $\bar{x}_i = x_i - u$ if $x_i > u$, and put $\bar{T} = \{\bar{x}_1, \ldots, \bar{x}_k\}$. Hence every summand in (1) is a system of Steiner triples in the set $T \cup \bar{T} \cup \{2u+1\}$. If S is a subsystem of $B(3, 1, 2u+1)$ constructed in a set not containing $2u+1$ (such a subsystem may exist or not), then we infer from (α) that every triple in S is a subset of a block in some B^i. Thus Lemma 1 is proved.

LEMMA 2. *There is exactly one element b in E such that any triple from $B(3, 1, 2u+1)$ not containing b generates together with b a subsystem which is prime or 1-prime.*

Proof. Now, $b = 2u+1$. If $\{x, y, z\} \in B(3,1,2u+1)$, $x, y, z \neq 2u+1$, then there exists exactly one system $B(3, 1, 2k_0+1)$ such that $\{x, y, z\} \in B(3,1,2k_0+1)$, where $B(3,1,2k_0+1)$ is a summand of (1) and a subsystem of $B(3,1,2u+1)$ $(k \in K_n)$. If this $B(3,1,2k_0+1)$ is prime, then it is the subsystem generated in $B(3,1,2u+1)$ by the set $\{x, y, z, 2u+1\}$ and the assertion follows. If $B(3,1,2k_0+1)$ is 1-prime, then this set generates either the whole of $B(3,1,2k_0+1)$ or a subsystem thereof. In both cases the generated subsystem is 1-prime, and so the assertion is satisfied.

Now, we show that, for every $x \in E$, if $x \neq 2u+1$, then there exists a triple $\{p, r, s\} \in B(3,1,2u+1)$ such that in $B(3,1,2u+1)$ there is neither a prime nor 1-prime subsystem containing triples with x and the triple $\{p, r, s\}$.

Let β_1 and β_2 be two blocks described in Remark 3. Denote by β_1^i and β_2^i their images under the mappings defined in the proof of Lemma 1. Let $B_0 = B(3, 1, 2k+1)$ be a system of triples in the set $\beta_1 \cup \bar{\beta}_1 \cup \{2u+1\}$. If $x \in \beta_1 \cup \bar{\beta}_1$, then we may choose an arbitrary triple $\{p, r, s\}$ in B_0 not containing $2u+1$. To see this we first have to prove that $\{p, r, s\}$ does not belong to any 3-subsystem of $B(K_n, 1, u)$. In fact, among the isomorphisms just mentioned, there is one such that $\{p, r, s\} \notin \beta_1$. But then it cannot belong to any 3-subsystem of the corresponding B^i. If, for some j, $\{p, r, s\} \notin \beta_1^j$, then $p \notin \beta_1^j$, say. But then p does not enter into any block in B^j, and so it is trivial that $\{p, r, s\}$ does not belong to any subsystem of B^j. Thus the latter is true for every B^i, especially for the original system $B(K_n, 1, u)$. According to Lemma 1, $\{p, r, s\}$ does not belong to any subsystem of $B(3,1,2u+1)$ in which $2u+1$ does not occur. Hence any subsystem to which the triple $\{p, r, s\}$ belongs must contain a block with the element $2u+1$. Observe that B_0 is prime on account of Remark 1 in [7]. Hence any subsystem in which there occur elements p, r, s contains the whole B_0 as a proper subsystem, and so it is not prime. It also cannot be 1-prime, since $k \equiv 0 \pmod 3$. If $x \in \beta_1 \cup \bar{\beta}_1$, then $x \notin \beta_2 \cup \bar{\beta}_2$ and the proof runs as before.

LEMMA 3. *Let K consist of numbers $k_1, \ldots, k_r$ such that every system $B(3,1,2k_j+1)$ $(1 \leqslant j \leqslant r)$ is a summand of* (1). *Then every subsystem of $B(3, 1, 2u+1)$ in* (1) *constructed in a set of $2k+1$ elements $(k \in K)$ and containing $2u+1$ is identical with one of the summands $B(3,1,2k+1)$.*

Proof. There must be a triple t in S not containing $2u+1$. Such a triple is of the form $\{x_i, x_j, x_s+u\}$ or $\{x_i, x_j+u, x_s+u\}$. Since the pair x_i, x_j belongs to exactly one element of $B(K_n, 1, u)$, t belongs to exactly one summand $B(3,1,2k+1)$, B_0 say. But t together with the element $2u+1$ generates the whole of $S = B_0$, as well as the whole B_0, since both these systems are prime.