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ANOTHER PROOF THAT A CHAIN OF NON-EMPTY H-CLOSED SUBSPACES HAS A NON-EMPTY INTERSECTION

BY

J. MIODUSZEWSKI (KATOWICE)

This theorem, which may be found in Katětov's paper [4], was restated by Fomin and Iliadis in [2], Note 5 on p. 67, where it was shown that it implies in an easy way a theorem of Stone [6] asserting that if in a Hausdorff space each closed subspace is H-closed, then the space itself is compact (in fact, then each chain of non-empty closed subspaces has a non-empty intersection, and this is sufficient for the compactness in virtue of a criterion due to Alexandroff and Urysohn [1]). The aim of this note is to give a new proof of this theorem. Its proof in [2] depends on a rather detailed construction of extremally disconnected resolutions for arbitrary Hausdorff spaces (due originally to Iliadis [3]) and, in consequence (through constructions of ultrafilters), depends on the axiom of choice.

The proof given here depends on a rather natural lemma, proved in [5] by this author and L. Rudolf, that each closed subspace F of a Hausdorff space X is *properly embedded* in X which means that if $x \in X$, then for each neighbourhood V of x in X there exists a neighbourhood V' of x in X such that $\mathrm{Int}_A(F \cap \mathrm{Cl}_X V') \subset \mathrm{Cl}_X(V \cap F)$. This implies, in particular, that

(1)' *If $x \notin F$, then there exists an H, open in X, such that $H \cap F$ is dense in F and $x \notin \mathrm{Cl}_X H$ (to get H take $X - \mathrm{Cl}_X V'$).*

Before the proof of the theorem note that it may be assumed that all elements of the chain lie in one element, hence, in particular, that all elements of the chain are subspaces of an H-closed space X.

P r o o f o f t h e t h e o r e m. Let L be a chain consisting of H-closed subspaces of an H-closed space X. For each $F \in L$ let $H(F)$ be the family of all open subsets H of X such that

(2) *$H \cap F$ is dense in F.*

The union $S = \bigcup \{H(F): F \in L\}$ is a centered family. Indeed, if $H_1 \in H(F_1)$, $H_2 \in H(F_2), \ldots, H_n \in H(F_n)$, and $F_1 \supset F_2 \supset \ldots \supset F_n$ are members of the chain L, then $H_n \cap F_{n-1} \supset H_n \cap F_n \neq \emptyset$ and $H_n \cap F_{n-1}$ is a non-empty open subset of F_{n-1}, whence $H_n \cap F_{n-1} \cap H_{n-1} \neq \emptyset$ by (2). Consequently, $H_n \cap H_{n-1} \cap F_{n-2} \neq \emptyset$ and this is a non-empty open subset of F_{n-2}, whence, by (2), $H_n \cap H_{n-1} \cap F_{n-2} \cap H_{n-2} \neq \emptyset$ and, finally, $H_n \cap H_{n-1} \cap \ldots \cap F_1 \cap H_1 \neq \emptyset$ and $H_n \cap H_{n-1} \cap \ldots \cap H_1 \neq \emptyset$. Since each member of the family S is open and X is H-closed, we get $\bigcap \{\mathrm{Cl}\, H : H \in S\} \neq \emptyset$. But if $x \in \bigcap \{\mathrm{Cl}\, H : H \in S\}$, then $x \in \cap L$, because $x \notin F$ and $F \in L$ implies, in virtue of (1), the existence of $H \in H(F) \subset S$ such that $x \notin \mathrm{Cl}\, H$. Thus $\cap L \neq \emptyset$.

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REFERENCES

[1] P. S. Alexandroff et P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. K. Akademie Amsterdam, Deel XIV, Nr 1 (1929), p. 1 - 96.

[2] С. Илиадис и С. Фомин, *Метод центрированных систем в теории топологических пространств*, Успехи Математических Наук 21 (1966) p. 47 - 76.

[3] С. Илиадис, *Абсолюты хаусдорфовых пространств*, Доклады Академии Наук СССР 149 (1963), p. 22 - 25.

[4] M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis matem. fys. 69 (1940), p. 36 - 49.

[5] J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Dissertationes Mathematicae 66 (1969).

[6] M. H. Stone, *Application of Boolean algebra to topology*, Transactions of the American Mathematical Society 41 (1937), p. 375 - 481.

SILESIAN UNIVERSITY, KATOWICE

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