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Extremal pairs of Aharonov

by J. ŚLADKOWSKA (Gliwice)

Abstract. Let f, g be functions holomorphic and univalent in the unit disc $U = \{z: |z| < 1\}$, $f(0) = 0$, $g(0) = 0$, and such that $f(z)\, g(\zeta) \neq 1$ for any $z, \zeta \in U$. (f, g) is called a *pair of Aharonov*.

In this paper we obtain a system of differential-functional equations for an extremal pair and we give some general properties of pairs which are solutions of that system.

Introduction. Let f, g be functions holomorphic and univalent in the unit disc $U = \{z: |z| < 1\}$, of the form

$$(1) \qquad f(z) = b_{1f} z + a_{2f} z^2 + \ldots, \qquad g(z) = b_{1g} z + a_{2g} z^2 + \ldots$$

and such that

$$(2) \qquad f(z)\, g(\zeta) \neq 1 \quad \text{for any } z, \zeta \in U.$$

The functions $f(z)$ and $h(z) = 1/g(z)$ are said to have *disjoint sets* of values since, by (2), $f(U) \cap h(U) = \emptyset$. Pairs (f, g) with properties (1) and (2) have been investigated by several authors, D. Aharonov [1], [2], among them, and recently by J. A. Hummel [7], who called them *univalent pairs of Aharonov*; following Hummel, we will denote by $\mathscr{A}$ the set of all univalent pairs of Aharonov.

Any pair $(f, 1/g)$, where $(f, g) \in \mathscr{A}$, represents a special case of a system of n functions meromorphic and univalent in U, normalized by prescribing their values at zero, with disjoint sets of values; as it turned out lately, investigation of several well-known families of univalent functions such as, for instance, bounded functions, those of Bieberbach–Eilenberg, of Grunsky–Shah, of Gelfer, and the like, resolves itself to a considerable extent into investigating such pairs. This is accomplished in such a way that to each of the functions of the above-mentioned classes one assigns a function which constitutes, together with the former, a pair of Aharonov.

Systems of functions with disjoint sets of values have been dealt with for a long time. Those investigations were initiated by Russian mathematicians. The first of them was M. A. Lavrentev who estimated, as early as 1934, the product of the mapping radii for a pair of functions with disjoint sets of values by means of a variational method invented by him [16]. The next one was G. M. Golousin who gave in [4] variational formulae of the type of interior variation of Schiffer–Golousin–Spencer–Schaeffer for a system of n functions with disjoint sets of values and, with use of them, estimated the product of the mapping radii for $n = 3$. The investigations were continued by N. A. Lebedev [10]–[15] and Yu. A. Alenicyn, and later on, by L. L. Gromova [5], [6] and J. A. Aleksandrov and M. N. Nikulšina [3]. N. A. Lebedev [11], among others, applied the variational formulae of Golousin to pairs of Aharonov, to examine the set of values of the functional $f(z)g(\zeta)$, where $z, \zeta \in U$ are arbitrary fixed points, as (f, g) range over $\mathscr{A}$. In the late 50's also mathematicians from the western countries, J. A. Jenkins [8], [9], D. Aharonov, J. A. Hummel, to list a few, started to deal with pairs of functions with disjoint sets of values and succeeded in grasping what it is that the families of bounded functions, of Bieberbach–Eilenberg functions and of Grunsky–Shah functions have in common. All of them except Jenkins who availed himself also of the method of extremal metric [8], and Aleksandrov who found a parametric representation of the type of Loewner [3], applied the area method which leads to an inequality of the type of Grunsky–Nehari's (e.g. Hummel [7]), which in turn leads, through a suitable choice of parameters, to estimating a great many functionals defined on the set $\mathscr{A}$. What is more, the area method enables one in numerous cases actually to find all extremal functions, and thereby to prove the accuracy of the estimates obtained. The disadvantage of this method lies in that with its help one can investigate functionals of a specific form only, namely those for which the part of the Gâteaux differential in which the functions f and g occur explicitly, is a perfect square, while for instance the above-mentioned functional $f(z)g(\zeta)$, estimated by Lebedev, does not passess this property. Although variational formulae have been repeatedly applied to estimating different functionals in the set of pairs of Aharonov as well as in the set of systems of functions with disjoint sets of values, the authoress has not encountered in the literature any differential-functional equations for an extremal pair, derived from these formulae and analogous to the differential-functional equation of Schiffer–Schaeffer–Spencer–Charzyński which is derived from the interior variation of a univalent function. This paper just aims at filling this gap and at establishing certain general properties of the resulting system of equations, and also of extremal pairs which are solutions to that system(1).

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(1) During the preparation this paper the authoress did not know the paper of J. A. Hummel and M. N. Schiffer, *Variations methods for Bieberbach–Eilenberg functions and for pairs*, Ann. Acad. Sci. Fennicae, Ser. A.I. Mathematica, Vol. 3 (1977), p. 3–42, which contains some of the results presented here.

1. Variations of a pair of Aharonov. Let $\mathcal{F}_0$ denote a set whose elements are any pairs of functions holomorphic in the disc U and vanishing at zero. Obviously, $\mathcal{A} \subset \mathcal{F}_0$, $\mathcal{F}_0$ is a linear space over the field of complex numbers if the operations are defined as usual: if $(f_1, g_1), (f_2, g_2) \in \mathcal{F}_0$ and $\lambda \in C$, then

$$(f_1, g_1)+(f_2, g_2) = (f_1+f_2, g_1+g_2), \quad \lambda(f_1, g_1) = (\lambda f_1, \lambda g_1).$$

We give the space $\mathcal{F}_0$ topological structure by assuming that a sequence of pairs (f_n, g_n) converges to (f, g) if and only if $f_n \to f$ and $g_n \to g$ almost uniformly in U. In this way we have made $\mathcal{F}_0$ into a linear-topological space. Let now $\Phi = \Phi((f, g))$ be a real functional defined at least on $\mathcal{A}$. By an extremal pair of Aharonov in the family $\mathcal{A}$, with respect to the functional Φ, we will mean a pair $(f, g) \in \mathcal{A}$ such that for every pair $(f^*, g^*) \in \mathcal{A}$ either the inequality $\Phi((f^*, g^*)) \leqslant \Phi(f, g)$, or $\Phi(f^*, g^*) \geqslant \Phi(f, g)$ is satisfied. (Accordingly, (f, g) is called a *maximal pair* or a *minimal pair.*)

Let $(f, g) \in \mathcal{A}$. A family of pairs $\{(f_\lambda, g_\lambda), 0 \leqslant \lambda \leqslant \lambda_0\}$ is called a *variation of* (f, g) in $\mathcal{A}$ if $(f_\lambda, g_\lambda) \in \mathcal{A}$, $0 \leqslant \lambda \leqslant \lambda_0$, $(f_0, g_0) = (f, g)$ and the limits

$$\dot{f}_0(z) = \frac{\partial}{\partial\lambda} f_\lambda(z)|_{\lambda=0} = \lim_{\lambda\to 0+} \frac{f_\lambda(z)-f(z)}{\lambda}, \tag{3}$$

$$\dot{g}_0(z) = \frac{\partial}{\partial\lambda} g_\lambda(z)|_{\lambda=0} = \lim_{\lambda\to 0+} \frac{g_\lambda(z)-g(z)}{\lambda} \tag{3'}$$

exist in the sense of the almost uniform convergence in U.

It follows from (3) and (3′) that

$$f_\lambda(z) = f(z)+\lambda\dot{f}_0(z)+o(\lambda), \quad g_\lambda(z) = g(z)+\lambda\dot{g}_0(z)+o(\lambda), \tag{3''}$$

where $o(\lambda)/\lambda \to 0$ almost uniformly in U. Basing on Golousin's method of constructing variations of functions of the class S [4] and making use of Loewner's variation [17], p. 185, Lebedev constructed examples of variations of a pair $(f, g) \in \mathcal{A}$, [12]. And so, the following families are variations:

(α) *In the case where the set* $C\setminus(f(U)\cup h(U))$ *has interior points and* w *is an interior point, there exist variations* $\{(f_\lambda^{(1)}, g_\lambda^{(1)})\}$ *and* $\{(f_\lambda^{(2)}, g_\lambda^{(2)})\}$ *(with* $\lambda > 0$ *sufficiently small) such that*

$$f_\lambda^{(1)}(z) = f(z)+\lambda a\frac{f(z)}{f(z)-w}, \quad g_\lambda^{(1)}(z) = g(z)-\lambda a\frac{g^2(z)}{1-wg(z)}+o(\lambda), \tag{4}$$

$$f_\lambda^{(2)}(z) = f(z)-\lambda a\frac{f^2(z)}{1-wf(z)}+o(\lambda), \quad g_\lambda^{(2)}(z) = g(z)+\lambda a\frac{g(z)}{g(z)-w}, \tag{4'}$$

where $a \in C$ *is arbitrary and* $o(\lambda)/\lambda \to 0+$ *almost uniformly in* U *as* $\lambda \to 0+$.

Proof. Consider the functions

$$f_\lambda(z) = f(z)+\lambda a\frac{f(z)}{f(z)-w}, \quad h_\lambda(z) = h(z)+\lambda a\frac{h(z)}{h(z)-w},$$

where $h(z) = [g(z)]^{-1}$ and w is an interior point of the set $C \setminus (f(U) \cup h(U))$; notice that, for a sufficiently small λ and for arbitrary complex a, these are functions with disjoint sets of values. Indeed, if $z, z_1 \in U$, then

$$f_\lambda(z) - h_\lambda(z_1) = (f(z) - h(z_1)) \left(1 - \lambda a \frac{w}{(f(z)-w)(h(z_1)-w)}\right).$$

Since w is an exterior point for the domains $f(U)$ and $h(U)$, there exists a $\varrho > 0$ such that $|f(z) - w| \geqslant \varrho$ and $|h(z_1) - w| \geqslant \varrho$, whence it follows that

$$\left|\frac{w}{(f(z)-w)(h(z_1)-w)}\right| \leqslant \frac{|w|}{\varrho^2},$$

and consequently,

$$|f_\lambda(z) - h_\lambda(z_1)| \geqslant |f(z) - h(z_1)| (1 - \lambda |a| |w|/\varrho^2) > 0$$

for $\lambda < \varrho^2/|a||w|$; thus we have, for such λ, $f_\lambda(z) \neq h_\lambda(z_1)$ with any $z, z_1 \in U$. Putting

$$f_\lambda^{(1)}(z) = f_\lambda(z),$$

$$g_\lambda^{(1)}(z) = \frac{1}{h_\lambda(z)} = \frac{1}{h(z)} \frac{1}{1 + \lambda a \dfrac{1}{h(z) - w}} = g(z) - \lambda a \frac{g^2(z)}{1 - w g(z)} + o(\lambda),$$

where $o(\lambda)/\lambda \to 0$ as $\lambda \to 0+$ almost uniformly in U, we find with no difficulty that the pairs of Aharonov $(f_\lambda^{(1)}, g_\lambda^{(1)})$ constitute a variation of the form (4).

(β) *There exist variations* $\{(f_\lambda^{(3)}, g_\lambda^{(3)})\}$ *and* $\{(f_\lambda^{(4)}, g_\lambda^{(4)})$ *(with* $\lambda > 0$ *sufficiently small) such that*

$$(5) \quad f_\lambda^{(3)}(z) = f(z) + \lambda a \frac{f(z)}{f(z) - f(\zeta)} - \lambda a \left(\frac{f(\zeta)}{\zeta f'(\zeta)^2}\right) \frac{z f'(z)}{z - \zeta} + \lambda \bar{a} \overline{\left(\frac{f(\zeta)}{\zeta f'(\zeta)^2}\right)} \frac{z^2 f'(z)}{1 - \bar{\zeta} z} + o(\lambda),$$

$$g_\lambda^{(3)}(z) = g(z) - \lambda a \frac{g^2(z)}{1 - f(\zeta) g(z)} + o(\lambda),$$

$$(5') \quad f_\lambda^{(4)}(z) = f(z) - \lambda a \frac{f^2(z)}{1 - g(\zeta) f(z)} + o(\lambda),$$

$$g_\lambda^{(4)}(z) = g(z) + \lambda a \frac{g(z)}{g(z) - g(\zeta)} - \lambda a \left(\frac{g(\zeta)}{\zeta g'(\zeta)^2}\right) \frac{z g'(z)}{z - \zeta} + \lambda \bar{a} \overline{\left(\frac{g(\zeta)}{\zeta g'(\zeta)^2}\right)} \frac{z^2 g'(z)}{1 - \bar{\zeta} z} + o(\lambda),$$

where $\zeta \in U$, $a \in C$ *are arbitrary and* $o(\lambda)/\lambda \to 0$ *almost uniformly in* U *as* $\lambda \to 0+$.

Proof. Let $\zeta \in U$ be arbitrary. Consider the annulus $R = \{r < |z| < 1\}$, $|\zeta| < r < 1$. In this annulus the function

$$F_\lambda(z) = f(z) + \lambda a \frac{f(z)}{f(z) - f(\zeta)}$$

is a function holomorphic and, for λ sufficiently small, univalent in R. Along with $F_\lambda(z)$ let us consider the function

$$h_\lambda(z) = h(z) + \lambda a \frac{h(z)}{h(z) - f(\zeta)}$$

in the disc U. $h_\lambda(z)$ is a function holomorphic in U since $h(z) \neq f(\zeta)$ and, for sufficiently small λ, univalent. The domains $F_\lambda(R)$ and $h_\lambda(U)$ are disjoint if λ is small enough. Clearly, for any $z \in R$ and $z_1 \in U$,

$$F_\lambda(z) - h_\lambda(z_1) = \big(f(z) - h(z_1)\big)\left(1 - \lambda a \frac{f(\zeta)}{\big(f(z) - f(\zeta)\big)\big(h(z_1) - f(\zeta)\big)}\right).$$

But $f(\zeta)$ is an exterior point of either of the domains $f(R)$ and $h(U)$; therefore there exists a $\varrho > 0$ such that $|f(z) - f(\zeta)| \geqslant \varrho$ and $|h(z_1) - f(\zeta)| \geqslant \varrho$ for every $z \in R$ and $z_1 \in U$, whence it follows that

$$\left|\frac{f(\zeta)}{\big(f(z) - f(\zeta)\big)\big(h(z_1) - f(\zeta)\big)}\right| \leqslant \frac{|f(\zeta)|}{\varrho^2}$$

for every $z \in R$ and every $z_1 \in U$. Thus

$$|F_\lambda(z) - h_\lambda(z_1)| \geqslant |f(z) - h(z_1)|\left(1 - \lambda |a| \frac{|f(\zeta)|}{\varrho^2}\right) > 0 \quad \text{for } 0 < \lambda < \frac{\varrho^2}{|a|\,|f(\zeta)|},$$

whence $F_\lambda(z) \neq h_\lambda(z_1)$ for any $z \in R$ and $z_1 \in U$.

Let now Φ_λ denote the union of $F_\lambda(R)$ and the bounded component of the set $C \backslash F_\lambda(R)$. From the Golousin theorem [17], p. 186, it follows after easy calculation that the function $f_\lambda(z)$, $f(0) = 0$, univalent in U and mapping U onto Φ_λ, is of the form

$$f_\lambda(z) = f(z) + \lambda a \frac{f(z)}{f(z) - f(\zeta)} - \lambda a \left(\frac{f(\zeta)}{\zeta f'(\zeta)^2}\right) \frac{z f'(z)}{z - \zeta} + \\ + \lambda \bar{a} \left(\frac{\overline{f(\zeta)}}{\bar{\zeta} \overline{f'(\zeta)^2}}\right) \frac{z^2 f'(z)}{1 - \bar{\zeta} z} + o(\lambda),$$

where $o(\lambda)/\lambda \to 0$ as $\lambda \to 0+$ almost uniformly in U. Moreover, as it follows from the above-described construction of the functions $f_\lambda(z)$, and $h_\lambda(z)$ the domains $f_\lambda(U)$ and $h_\lambda(U)$ are disjoint, and hence the functions $f_\lambda^{(3)}(z) = f_\lambda(z)$ and $g_\lambda^{(3)}(z) = [h_\lambda(z)]^{-1} = g(z) - \lambda a \dfrac{g^2(z)}{1 - g(z) f(\zeta)} + o(\lambda)$ constitute a pair of Aharonov.

(γ) *There exist variations* $\{(f_\lambda^{(5)}, g_\lambda^{(5)})\}$ *and* $\{(f_\lambda^{(6)}, g_\lambda^{(6)})\}$ *(with* $\lambda > 0$ *sufficiently small) such that*

$$f_\lambda^{(5)}(z) = f(z) - \lambda z f'(z)\frac{\zeta+z}{\zeta-z} + o(\lambda), \quad g_\lambda^{(5)}(z) = g(z), \tag{6}$$

$$f_\lambda^{(6)}(z) = f(z), \quad g_\lambda^{(6)}(z) = g(z) - zg'(z)\frac{\zeta+z}{\zeta-z} + o(\lambda), \tag{6'}$$

where $\zeta \in \partial U$ *is arbitrary and* $o(\lambda)/\lambda \to 0$ *almost uniformly in* U *as* $\lambda \to 0+$.

We shall give here yet another evident example of a variation, which arises by rotation of the disc U about its centre.

(δ) *There exist variations* $\{(f_\lambda^{(7)}, g_\lambda^{(7)})\}$ *and* $\{(f_\lambda^{(8)}, g_\lambda^{(8)})\}$ *(with* $\lambda > 0$ *sufficiently small) such that*

$$f_\lambda^{(7)}(z) = f(z) \pm i\lambda z f'(z) + o(\lambda), \quad g_\lambda^{(7)}(z) = g(z), \tag{7}$$

$$f_\lambda^{(8)}(z) = f(z), \quad g_\lambda^{(8)}(z) = g(z) \pm i\lambda z g'(z) + o(\lambda), \tag{7'}$$

where $o(\lambda)/\lambda \to 0$ *almost uniformly in* U *as* $\lambda \to 0+$.

2. Variations of functionals in the space of pairs. Let $\mathscr{F} \subset \mathscr{F}_0$ be an arbitrary subset of the space $\mathscr{F}_0$, and let Φ be a real functional defined in $\mathscr{F}$. We say that Φ has a *complex derivative in the sense of Gâteaux* at the point $(f, g) \in \mathscr{F}$ with respect to $\mathscr{F}$ if there exists a linear continuous functional $L_{(f,g)}$, defined in $\mathscr{F}_0$, such that

$$\Phi\big((f^*, g^*)\big) = \Phi\big((f, g)\big) + \lambda \ \mathrm{re}\ L_{(f,g)}\big((h, k)\big) + o(\lambda), \tag{8}$$

where $(f^*, g^*) \in \mathscr{F}$, $\lambda > 0$ and $(f^*, g^*) = (f, g) + \lambda(h, k) + o(\lambda)$, where $(h, k) \in \mathscr{F}_0$ and $o(\lambda)/\lambda \to 0$ as $\lambda \to 0+$, almost uniformly in $U \times U$. From (8) and (3″) the following theorem results.

THEOREM 1. *Let* Φ *be a real functional defined in the set* $\mathscr{A}$ *and suppose that for a pair* $(f, g) \in \mathscr{A}$ *the functional* Φ *has a derivative in the sense of Gâteaux with respect to the set* $\mathscr{A}$. *If* (f, g) *is a maximal pair for the functional* Φ *in the set* $\mathscr{A}$, *then for every variation* $\{(f_\lambda, g_\lambda)\}$ *of the pair* (f, g) *in* $\mathscr{A}$ *there holds the inequality*

$$\mathrm{re}\ L_{(f,g)}\big((\dot{f}_0, \dot{g}_0)\big) \leqslant 0 \tag{9}$$

where $\dot{f}_0, \dot{g}_0$ *are defined by formulae* (3) *and* (3′).

Proof. This theorem is an immediate consequence of the representation (8).

Writing down condition (9) for the variations defined in (α), (β), (γ), (δ), we obtain the following four properties which are shared by maximal pairs (f, g) in the family $\mathscr{A}$ with respect to the functional Φ having at the point (f, g) a complex derivative $L_{(f,g)}$ in the sense of Gâteaux.

PROPERTY I. *For any* $w \in \operatorname{int} C \setminus (f(U) \cup h(U))$, *the equalities*

$$(10)\qquad \begin{gathered} L_{(f,g)}\left(\left(\frac{f(z)}{f(z)-w}, \frac{-g^2(z)}{1-wg(z)}\right)\right) = 0, \\ L_{(f,g)}\left(\left(\frac{-f^2(z)}{1-wf(z)}, \frac{g(z)}{g(z)-w}\right)\right) = 0; \end{gathered}$$

hold; *z is here an apparent variable, and the left-hand sides of* (10) *are functions of the variable w.*

Proof. This property follows immediately from (3), (3′) (3″), (9) and the arbitrariness of a complex.

PROPERTY II. *For an arbitrary* $\zeta \in U$, *the equalities*

$$(11)\qquad \begin{gathered} L_{(f,g)}\left(\left(\frac{f(z)}{f(z)-f(\zeta)} - \left(\frac{f(\zeta)}{\zeta f'(\zeta)^2}\right)\frac{zf'(z)}{z-\zeta}, \frac{g^2(z)}{1-f(\zeta)g(z)}\right)\right) + \\ + \bar{L}_{(f,g)}\left(\left(\left(\overline{\frac{f(\zeta)}{\zeta f'(\zeta)^2}}\right)\frac{z^2 f'(z)}{1-\bar{\zeta} z}, 0\right)\right) = 0, \\ L_{(f,g)}\left(\left(\frac{-f^2(z)}{1-g(\zeta)f(z)}, \frac{g(z)}{g(z)-g(\zeta)} - \left(\frac{g(\zeta)}{\zeta g'(\zeta)^2}\right)\frac{zg'(z)}{z-\zeta}\right)\right) + \\ + \bar{L}_{(f,g)}\left(\left(0, \left(\overline{\frac{g(\zeta)}{\zeta g'(\zeta)^2}}\right)\frac{z^2 g'(z)}{1-\bar{\zeta} z}\right)\right) = 0 \end{gathered}$$

hold; *z is here an apparent variable, and the left-hand sides of* (11) *are functions of the variable* ζ, $\bar{L}_{(f,g)}((h, k)) = \overline{L_{(f,g)}((h, k))}$.

Proof. This property follows immediately from (4), (4′), (3″), (9) and the arbitrariness of a complex.

PROPERTY III. *For any* $\zeta \in \partial U$, *the inequalities*

$$(12)\qquad \begin{gathered} \operatorname{re} L_{(f,g)}\left(\left(-zf'(z)\frac{\zeta+z}{\zeta-z}, 0\right)\right) \leqslant 0, \\ \operatorname{re} L_{(f,g)}\left(\left(0, -zg'(z)\frac{\zeta+z}{\zeta-z}\right)\right) \leqslant 0, \end{gathered}$$

hold; *z is here an apparent variable, and the left-hand sides of* (12) *are functions of the variable* ζ,

Proof. This property follows at once from (5), (5′), (3″) and (9).

PROPERTY IV. *There hold equalities*

$$(13)\qquad \operatorname{im} L_{(f,g)}((zf'(z), 0)) = 0, \quad \operatorname{im} L_{(f,g)}((0, zg'(z))) = 0.$$

Proof. This property follows at once from (6), (6′), (3″) and (9).

Taking account of the representation of continuous linear functionals in the set of analytic functions on U (see [18], p. 34), we observe that there exist functions $\varphi_1(z)$ and $\varphi_2(z)$ analytic in $C\setminus\{z\colon |z| < r\}$, $0 < r < 1$, vanishing at ∞, and circles $C_1 = \{z\colon |z| = r_1\}$, $C_2 = \{z\colon |z| = r_2\}$, $r < r_1, r_2 < 1$, such that for every $(h, k) \in \mathscr{F}_0$

$$\begin{aligned} L_{(f,g)}((h, k)) &= L_{(f,g)}((h, 0)) + L_{(f,g)}((0, k)) \\ &= \frac{1}{2\pi i}\int_{C_1} h(z)\varphi_1(z)\,dz + \frac{1}{2\pi i}\int_{C_2} k(z)\varphi_2(z)\,dz. \end{aligned}$$

We now apply this result to an extension of the functional $L_{(f,g)}$ from the class $\mathscr{F}_0$ to a continuous linear functional on the class of all pairs (h, k) of functions meromorphic in U, where h has no poles on C_1 and k has no poles on C_2. As a consequence, taking account of (13), we may write (11) in the form

$$(14)\qquad \begin{aligned} &\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 L_{(f,g)}\left(\left(\frac{f(\zeta)f(z)}{f(\zeta)-f(z)}, \frac{f(\zeta)g^2(z)}{1-f(\zeta)g(z)}\right)\right) \\ &\qquad = \tfrac{1}{2}L_{(f,g)}\left(\left(zf'(z)\frac{\zeta+z}{\zeta-z}, 0\right)\right) + \tfrac{1}{2}\bar{L}_{(f,g)}\left(\left(zf'(z)\frac{1+\bar{\xi}}{1-\bar{\xi}z}, 0\right)\right), \\ &\left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right)^2 L_{(f,g)}\left(\left(\frac{g(\zeta)f^2(z)}{1-g(\zeta)f(z)}, \frac{g(\zeta)g(z)}{g(\zeta)-g(z)}\right)\right) \\ &\qquad = \tfrac{1}{2}L_{(f,g)}\left(\left(0, zg'(z)\frac{\zeta+z}{\zeta-z}\right)\right) + \tfrac{1}{2}\bar{L}_{(f,g)}\left(\left(0, zg'(z)\frac{1+\bar{\xi}z}{1-\bar{\xi}z}\right)\right), \end{aligned}$$

where $\zeta \in U$, $\zeta \notin C_1 \cup C_2$.

Assuming

$$(14')\qquad P(w) = L_{(f,g)}\left(\left(\frac{f^2(z)}{w-f(z)}, \frac{wg^2(z)}{1-wg(z)}\right)\right),$$

we observe that system (14) can be given the following, more symmetric form:

$$(15)\qquad \begin{aligned} &\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 \left(P(f(\zeta)) + L_{(f,g)}((f(z), 0))\right) \\ &\qquad = \tfrac{1}{2}L_{(f,g)}\left(\left(zf'(z)\frac{\zeta+z}{\zeta-z}, 0\right)\right) + \tfrac{1}{2}\bar{L}_{(f,g)}\left(\left(zf'(z)\frac{1+\bar{\xi}z}{1-\bar{\xi}z}, 0\right)\right), \\ &\left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right)^2 \left(P\left(\frac{1}{g(\zeta)}\right) + L_{(f,g)}((0, g(z)))\right) \\ &\qquad = \tfrac{1}{2}L_{(f,g)}\left(\left(0, zg'(z)\frac{\zeta+z}{\zeta-z}\right)\right) + \tfrac{1}{2}\bar{L}_{(f,g)}\left(\left(0, zg'(z)\frac{1+\bar{\xi}z}{1-\bar{\xi}z}\right)\right). \end{aligned}$$

PROPERTY V. *The right-hand sides of* (15) *are non-negative on the circle* ∂U.

Proof. The property follows immediately from inequalities (12).

3. Functionals of finite order and their variations. Let z_μ, $\mu = 1,\ldots, M$, ζ_ν, $\nu = 1,\ldots, N$, be any points of the disc U, and l_μ, $\mu = 1,\ldots, M$, λ_ν, $\nu = 1,\ldots, N$, any integers $\geqslant 0$. $M = 0$ or $N = 0$ will indicate that there are no points z_μ or ζ_ν. Let $l_0 \geqslant 0$, $\lambda_0 \geqslant 0$ be any integers. Put

$$n = l_0+l_1+\ldots+l_M+\lambda_0+\lambda_1+\ldots+\lambda_N. \tag{16}$$

To any pair $(f, g)\in\mathscr{F}_0$, $f(z) = b_{1f}z+b_{2f}z^2+\ldots$, $g(z) = b_{1g}z+b_{2g}z^2+\ldots$, let us assign an n-dimensional vector $v((f,g))$ with the components

$$\begin{aligned} b_{kf},\quad k = 1,\ldots,l_0,\quad b_{kg},\quad k = 1,\ldots,\lambda_0,\quad f^{(k)}(z_\mu),\quad k = 0,\ldots,l_\mu-1,\\ \mu = 1,\ldots,M,\quad g^{(k)}(\zeta_\nu),\quad k = 0,\ldots,\lambda_\nu-1,\ \nu = 1,\ldots,N. \end{aligned} \tag{17}$$

$l_0 = 0$ or $\lambda_0 = 0$ indicate that in $v((f, g))$ the coefficients of the functions f or g do not appear as components. Let us take into consideration the set

$$V = \{v((f, g)) \colon (f, g)\in\mathscr{A}\} \subset C^n$$

and let $X(w)$ be a real-valued function of class $C^{(1)}$ in some open neighbourhood of the set V. Define in $\mathscr{A}$ a continuous functional Φ by

$$\begin{aligned} \Phi((f, g)) = X(v((f, g))) = X(b_{1f},\ldots,b_{l_0f}, b_{1g},\ldots,b_{\lambda_0g},\\ f(z_1),\ldots,f^{(l_1-1)}(z_1),\ldots,g(\zeta_1),\ldots,g^{(\lambda_1-1)}(\zeta_1),\ldots). \end{aligned} \tag{18}$$

Following Pommerencke [17], p. 184, we shall call functionals of this form functionals of finite order, and the number n defined by means of (16) – the order of Φ. The differential of the function $X(w)$ is given by the formula

$$dX = 2\ \mathrm{re}\ \{X_w dw\},\qquad X_w = \left(\frac{\partial X}{\partial w_1},\ldots,\frac{\partial X}{\partial w_n}\right),\qquad \text{where } \frac{\partial X}{\partial w_j} = \frac{1}{2}\left(\frac{\partial X}{\partial u_j}-i\frac{\partial X}{\partial v_j}\right),$$

$w_j = u_j+iv_j$. The vector v is a linear function of its components (17), and therefore, for any pair $(f, g)\in\mathscr{A}$, if $(f^*, g^*) = (f, g)+\lambda(h, k)+(o_1(\lambda), o_2(\lambda))$, where $(h, k)\in\mathscr{F}_0$ and $o_j(\lambda)/\lambda\to 0$ as $\lambda\to 0$ almost uniformly in U, we have

$$\Phi((f^*, g^*)) = \Phi((f, g))+\lambda\ \mathrm{re}\ L_{(f,g)}((h, k))+o(\lambda),$$

where

$$L_{(f,g)}((h, k)) = 2X_w(v((f, g)))v((h, k)) \tag{19}$$

and $o(\lambda)/\lambda\to 0$ as $\lambda\to 0$. Put

$$h(z) = c_1z+c_2z^2+\ldots,\qquad k(z) = d_1z+d_2z^2+\ldots;$$

then

$$(20)\quad L_{(f,g)}\big((h,k)\big) = \sum_{k=0}^{l_0-1} \gamma_{0k}\big((f,g)\big)c_{k+1} + \sum_{k=0}^{\lambda_0-1} \delta_{0k}\big((f,g)\big)d_{k+1} + \\ + \sum_{\mu=1}^{M}\sum_{k=0}^{1_\mu-1} \gamma_{\mu k}\big((f,g)\big)h^{(k)}(z_\mu) + \sum_{\nu=1}^{N}\sum_{k=0}^{\lambda_\nu-1} \delta_{\nu k}\big((f,g)\big)k^{(k)}(\zeta_\nu).$$

We shall now examine the form that the system of equations (15) takes on in our case. To begin with, we transform the left-hand sides of these equations by introducing Faber's polynomials. And thus, making use of the relation

$$\frac{1}{1-tf(z)} = 1 + \sum_{k=1}^{\infty} \tfrac{1}{k} t F'_{kf}(t) z^k,$$

where F_{kf} is the k th polynomial of Faber for the function $1/f$, and of an analogous relation for the function g, we get

$$(21)\quad \begin{aligned} \frac{f^2(z)}{w-f(z)} &= \sum_{k=1}^{\infty} \tfrac{1}{k}\big(F'_{kf}(1/w) - kb_{kf}\big)z^k, \\ \frac{wg^2(z)}{1-wg(z)} &= \sum_{k=1}^{\infty} \tfrac{1}{k}\big(F'_{kg}(w) - kb_{kg}\big)z^k, \end{aligned}$$

where

$$\begin{aligned} \tfrac{1}{k}F'_{kf}(1/w) &= b_{1f}^k w^{-(k-1)} + B_{k-2,f} w^{-(k-2)} + \ldots + b_{kf}, \\ \tfrac{1}{k}F'_{kg}(w) &= b_{1g}^k w^{k-1} + B_{k-2,g} w^{k-2} + \ldots + b_{kg}. \end{aligned}$$

Further, we find that

$$(22)\quad \begin{aligned} \left(\frac{d}{dz}\right)^k \frac{f^2(z)}{w-f(z)} &= \frac{k!\,f^2(z)[f'(z)]^k}{\big(w-f(z)\big)^{k+1}} + \sum_{j=1}^{k} \frac{\varphi_{kj}(z)}{\big(w-f(z)\big)^j}, & k = 1, 2, \ldots, \\ \left(\frac{d}{dz}\right)^k \frac{wg^2(z)}{1-wg(z)} &= \frac{k!\,g^2(z)[g'(z)]^k w^{k+1}}{\big(1-wg(z)\big)^{k+1}} + \sum_{j=1}^{k} \frac{\psi_{kj}(z)\,w^j}{\big(1-wg(z)\big)^j}, & k = 1, 2, \ldots, \end{aligned}$$

where the functions $\varphi_{kj}(z)$ and $\psi_{kj}(z)$ do not depend on w. Put $z_0 = \zeta_0 = 0$. Let k_μ, $\mu = 0, \ldots, M$, $\varkappa_\nu$, $\nu = 0, \ldots, N$, denote the greatest indices such that $\gamma_{\mu,k_\mu-1}\big((f,g)\big) \neq 0$ and $\delta_{\nu,\varkappa_\nu-1}\big((f,g)\big) \neq 0$ in (20). From (21) and (22), in view of the fact that $f(z_\mu) \neq 0$, $f'(z_\mu) \neq 0$, $g(\zeta_\nu) \neq 0$, $g'(\zeta_\nu) \neq 0$, $\mu \neq 0$, $\nu \neq 0$, we infer that the function $P(w)$ from (14') has in this case the form

$$(23)\quad P(w) = \sum_{k=0}^{k_0-1} \gamma_{0k} \frac{1}{k+1}\big(F'_{k+1,f}(1/w) - (k+1)b_{k+1,f}\big) + \\ + \sum_{\mu=1}^{M}\sum_{k=0}^{k_\mu-1} \gamma_{u,k}\left(\frac{k!\,[f(z_\mu)]^2[f'(z_\mu)]^k}{\big(w-f(z_\mu)\big)^{k+1}} + \sum_{j=1}^{k} \frac{\varphi_{kj}(z_\mu)}{\big(w-f(z_\mu)\big)^j}\right) +$$

$$+\sum_{k=0}^{\varkappa_0-1}\delta_{0k}\frac{1}{k+1}(F'_{k+1,g}(w)-(k+1)b_{k+1,g})+$$

$$+\sum_{\nu=1}^{N}\sum_{k=0}^{\varkappa_\nu-1}\delta_{\nu k}\left(\frac{k!\,[g(\zeta_\nu)]^2[g'(\zeta_\nu)]^k w^{k+1}}{(1-g(\zeta_\nu)w)^{k+1}}\right)+\sum_{j=1}^{k}\frac{\psi_{kj}(\zeta_\nu)w^j}{(1-wg(\zeta_\nu))^j},$$

or simply

$$P(w)=\sum_{\mu=0}^{M}\sum_{j=1}^{k'_\mu}\frac{\alpha_{\mu j}}{(w-f(z_\mu))^j}+\sum_{\nu=0}^{N}\sum_{j=1}^{\varkappa'_\nu}\frac{\beta_{\nu j}w^j}{(1-wg(\zeta_\nu))^j},\tag{24}$$

where $k'_0=k_0-1$, $k'_\mu=k_\mu$, $\mu>0$, $\varkappa'_0=\varkappa_0-1$, $\varkappa'_\nu=\varkappa_\nu$, $\nu>0$, $\alpha_{\mu k'_\mu}\neq 0$, $\mu=0,\dots,M$, $\beta_{\nu\varkappa'_\nu}\neq 0$, $\nu=0,\dots,N$. The terms in (23) with $\mu=0$ or $\mu>0$ are omitted when $l_0=0$ or $M=0$; the same concerns the terms with $\nu=0$ or $\nu>0$ when $\lambda_0=0$ or $N=0$; besides, we see that $P(w)$ is a rational function of degree $n'=k'_0+\dots+k'_M+\varkappa'_0+\dots+\varkappa'_N$, with obvious changes in the cases of $k'_0=0$, $\varkappa'_0=0$, $M=0$, $N=0$. If we now write the left-hand sides of equations (15) in the forms

$$\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 P_1(f(\zeta))\quad\text{and}\quad\left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right)^2 P_2(g(\zeta)),$$

then, on account of the relations

$$P_1(w)=P(w)+L_{(f,g)}((f,0))\quad\text{and}\quad P_2(w)=P(1/w)+L_{(f,g)}((0,g)),$$

P_1 and P_2 are also rational functions of degree n', as described above.

We shall now calculate the right-hand sides in (15). On account of the relations

$$zf'(z)\frac{\zeta+z}{\zeta-z}+\sum_{k=1}^{\infty}\left(kb_{kf}+2\sum_{j=1}^{k-1}(k-j)b_{k-j,f}\zeta^{-j}\right)z^k,$$

$$\left(\frac{d}{dz}\right)^k\left[zf'(z)\frac{\zeta+z}{\zeta-z}\right]=\frac{k!\,z^2f'(z)}{(\zeta-z)^{k+1}}+\sum_{j=0}^{k}\frac{\chi_{kj}(z)}{(\zeta-z)^j},$$

where $\chi_{kj}(z)$ does not depend on ζ, and of analogous relations written for g, on account of the fact that the right-hand sides of equations (15) are real-valued on the circle ∂U, we find (denoting those right-hand sides by Q_1 and Q_2, respectively) that they are of the form

$$Q_1(\zeta)=\sum_{\mu=0}^{M}\left(\sum_{j=1}^{k'_\mu}\left(\frac{a_{\mu j}}{(\zeta-z_\mu)^j}+\frac{\bar a_{\mu j}\zeta^j}{(1-\bar z_\mu\zeta)^j}\right)\right)+a,\qquad a_{\mu k'_\mu}\neq 0,\ \mu=0,\dots,M,$$

$$Q_2(\zeta)=\sum_{\nu=0}^{N}\left(\sum_{j=1}^{\varkappa'_\nu}\left(\frac{b_{\nu j}}{(\zeta-\zeta_\nu)^j}+\frac{\bar b_{\nu j}\zeta^j}{(1-\bar\xi_\nu\zeta)^j}\right)\right)+b,\qquad b_{\nu\varkappa'_\nu}\neq 0,\ \nu=0,\dots,N,$$

with a, b real; thus they are rational functions of degree $2n_1' = 2(k_0'+\ldots+k_M')$ and $2n_2' = 2(\varkappa_0'+\ldots+\varkappa_N')$ respectively, with obvious changes in the cases of $k_0' = 0$, $\varkappa_0' = 0$, $M = 0$, $N = 0$. Obviously, we have $n_1'+n_2' = n'$. Moreover, since the functions Q_1 and Q_2 are of constant sign on ∂U, it follows that all zeros of these functions, situated on the unit circle, must be of even order.

The results obtained above will be summarized in the following theorem.

THEOREM 2. *Let Φ be a functional defined on $\mathscr{A}$, of finite order n, of the form* (19). *Suppose that for a pair $(f, g) \in \mathscr{A}$, Φ has a complex derivative in the sense of Gâteaux $L_{(f,g)}$, of the form* (20). *Let (f, g) be an extremal pair for Φ; then*

$$\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 P_1(f(\zeta)) = Q_1(\zeta), \quad \left(\frac{\zeta g'(\zeta)}{g(\zeta)}\right)^2 P_2(g(\zeta)) = Q_2(\zeta), \tag{25}$$

where

$$P_1(w) = P(w)+L_{(f,g)}((f, 0)), \quad P_2(w) = P(1/w)+L_{(f,g)}((0, g)),$$

$$P(w) = L_{(f,g)}\left(\left(\frac{f^2(z)}{w-f(z)}, \frac{wg^2(z)}{1-wg(z)}\right)\right),$$

$$Q_1(\zeta) = \tfrac{1}{2} L_{(f,g)}\left(\left(zf'(z)\frac{\zeta+z}{\zeta-z}, 0\right)\right)+\tfrac{1}{2}\bar{L}_{(f,g)}\left(\left(zf'(z)\frac{1+\bar{\xi}z}{1-\bar{\xi}z}, 0\right)\right),$$

$$Q_2(\zeta) = \tfrac{1}{2} L_{(f,g)}\left(\left(0, zg'(z)\frac{\zeta+z}{\zeta-z}\right)\right)+\tfrac{1}{2}\bar{L}_{(f,g)}\left(\left(0, zg'(z)\frac{1+\bar{\xi}z}{1-\bar{\xi}z}\right)\right),$$

$P(w)$ being rational function of the form

$$P(w) = \sum_{\mu=0}^{M}\sum_{j=1}^{k_\mu'}\frac{\alpha_{\mu j}}{(w-f(z_\mu))^j}+\sum_{\nu=0}^{N}\sum_{j=1}^{\varkappa_\nu'}\frac{\beta_{\nu j}w^j}{(1-wg(\zeta_\nu))^j},$$

$$\alpha_{\mu k_\mu'} \neq 0,\ \mu = 0,\ldots,M,\ \beta_{\nu\varkappa_\nu'} \neq 0,\ \nu = 0,\ldots,N,$$

of order $n' = k_0'+k_1'+\ldots+k_M'+\varkappa_0'+\varkappa_1'+\ldots+\varkappa_N'$, where $k_0' = k_0-1$, $k_j' = k_j$, $j = 1,\ldots,M$, $\varkappa_0' = \varkappa_0-1$, $\varkappa_j' = \varkappa_j$, $j = 1,\ldots,N$, $k_j \leqslant l_j$, $j = 0,\ldots,M$, $\varkappa_j \leqslant \lambda_j$, $j = 0,\ldots,N$; here we assume $k_0' = 0$ or $\varkappa_0' = 0$ if $k_0 = 0$ or $\varkappa_0 = 0$ and $k_j' = 0$ or $\varkappa_j = 0$, $j > 0$, if $M = 0$ or $N = 0$; the functions $Q_1(\zeta)$ and $Q_2(\zeta)$ are rational functions of the form

$$\begin{aligned} Q_1(\zeta) &= \sum_{\mu=0}^{M} \pm \sum_{j=1}^{k_\mu'}\left(\frac{a_{\mu j}}{(\zeta-z_\mu)^j}+\frac{\bar{a}_{\mu j}\zeta^j}{(1-\bar{z}_\mu\zeta)^j}\right)\Bigg)+a, \\ Q_2(\zeta) &= \sum_{\nu=0}^{N}\left(\sum_{j=1}^{\varkappa_\nu'}\left(\frac{b_{\nu j}}{(\zeta-\zeta_\nu)^j}+\frac{\bar{b}_{\nu j}\zeta^j}{(1-\bar{\xi}_\nu\zeta)^j}\right)\right)+b, \end{aligned} \tag{26}$$

with a, b real, of orders $2n_1' = 2(k_0'+\ldots+k_M')$ and $2n_2' = 2(\varkappa_0'+\ldots+\varkappa_N')$, respectively. The functions $Q_1(\zeta)$ and $Q_2(\zeta)$ are non-negative on the circle ∂U.

Equations (25), similarly, to the case of functions of the class S, cf. [17], p. 192–193, may be used to gather information about extremal domains, i.e., domains $f(U)$ and $g(U)$, and consequently, about the disjoint domains $f(U)$ and $h(U)$ in the case where (f, g) cónstitute an extremal pair. The following theorem holds.

THEOREM 3. *Let Φ be a functional defined on $\mathscr{A}$, of finite order, having a complex derivative in the sense of Gâteaux for a pair (f, g). If this derivative is not identically zero, then $P_1(w) \not\equiv 0$, $P_2(w) \not\equiv 0$, and the set $C\setminus(f(U)\cup h(U))$ has no interior points.*

Proof. The assumption that the derivative $L_{(f,g)}((h, k))$ is non-zero is equivalent, in virtue of (20), to saying that $\gamma_{\mu k} \neq 0$ for some μ and some k, or $\delta_{\nu k} \neq 0$ for some ν and some k; hence and from (23), in view of the fact that $f(z_\mu) \neq 0$, $f'(z_\mu) \neq 0$, $g(\zeta_\nu) \neq 0$, $g'(\zeta_\nu) \neq 0$, $f(z_\mu)$ different from one another, $g(\zeta_\nu)$ different from one another, $f(z_\mu)g(\zeta_\nu) \neq 1$, it follows that $P(w) \not\equiv \text{const}$, and thus $P_1(w)$ and $P_2(w)$ are not identically zero. Presently we shall prove that the set $C\setminus(f(U)\cup h(U))$ has no interior points. Suppose, on the contrary, that this set contains some disc U_0. From Property I, (10) it follows, in particular, that

$$L_{(f,g)}\left(\left(\frac{f(z)}{w-f(z)}, \frac{g^2(z)}{1-wg(z)}\right)\right) = 0$$

for every $w \in U_0$. From the relation

$$P_1(w) = wL_{(f,g)}\left(\left(\frac{f(z)}{w-f(z)}, \frac{g^2(z)}{1-wg(z)}\right)\right)$$

we see that $P_1(w) = 0$ in U_0, and since $P_1(w)$ is a rational function, we have $P_1(w) = 0$ in C, despite of what was shown above.

4. Example. To illustrate the theorems given above, we shall now find an estimate for the following functional. Let

$$X(w) = \operatorname{re}\left\{\log\frac{w_2 w_4}{1-w_1 w_2}\right\}, \tag{27}$$

where $(w_1, w_2, w_3, w_4) \in C^4$ and $w_1 w_3 \neq 1$, $w_2 w_4 \neq 0$. Let

$$\Phi((f, g)) = \operatorname{re}\left\{\log\frac{f'(z_1)g'(z_2)}{1-f(z_1)g(z_2)}\right\}, \tag{27'}$$

where $z_1, z_2 \in U$ are arbitrary but fixed. The functional Φ is, obviously, defined in the set of Aharonov's pairs $\mathscr{A}$, since $f(z_1)g(z_2) \neq 1$ and $f'(z_1)g'(z_2) \neq 0$ for $(f, g) \in \mathscr{A}$. Φ also possesses a complex derivative in the sense of Gâteaux at each point $(f, g) \in \mathscr{A}$; according to (19), (27), this derivative is defined as the functional

$$(28) \qquad L_{(f,g)}((h, k)) = \frac{g'(z_2)h'(z_1)+f'(z_1)k'(z_2)}{f'(z_1)g'(z_2)} + 2\frac{g(z_2)h(z_1)+f(z_1)k(z_2)}{1-f(z_1)g(z_2)},$$

where $(h, k) \in \mathscr{F}_0$.

We now show that the functional Φ attains its maximal value in the family $\mathscr{A}$. To this end, we assign to each pair (f, g) the pair

$$\hat{f}(z) = b_{1f}^{-1} f(z), \qquad \hat{g}(z) = b_{1g}^{-1} g(z);$$

these are functions of the class S; however, they do not necessarily form a pair of Aharonov. But the functions

$$(29) \qquad \check{f}(z) = b_{1f} b_{1g} \hat{f}(z), \qquad \check{g}(z) = \hat{g}(z)$$

constitute a pair of Aharonov, and, as can be easily seen, the equality

$$(30) \qquad \Phi\big((\check{f}(z), \check{g}(z))\big) = \Phi\big((f, g)\big)$$

holds. Next, let us notes that the functional Φ is bounded from above. Indeed, making use of the estimate $|b_{1f} b_{1g}| \leqslant 4$, which follows immediately from the Koebe covering theorem and from the Schwarz lemma, and from two inequalities valid for the class S: $|f(z)| \leqslant \frac{z}{(1-|z|)^2}$, $|f'(z)| \leqslant \frac{1+|z|}{(1-|z|)^3}$, on account of (29) and (30) we obtain

$$\begin{aligned}\Phi\big((f, g)\big) &= \Phi\big((\check{f}, \check{g})\big) = \Phi\big((b_{1f} b_{1g}\hat{f}, \hat{g})\big) \\ &= \log |b_{1f} b_{1g} \hat{f}'(z_1)\hat{g}'(z_2)| - 2\log |1 - b_{1f} b_{1g}\hat{f}(z_1)\hat{g}(z_2)| \\ &\leqslant \log \frac{4(1+|z_1|)(1+|z_2|)}{(1-|z_1|)(1-|z_2|)\big|(1-|z_1|)^2(1-|z_2|)^2 - 4|z_1||z_2|\big|} < +\infty,\end{aligned}$$

whenever

$$(31) \qquad (1-|z_1|)^2(1-|z_2|)^2 - 4|z_1||z_2| \neq 0.$$

Let $M = \sup_{(f,g)\in\mathscr{A}} \Phi\big((f, g)\big)$. In virtue what was said above, $M < +\infty$. On the other hand, $M > -\infty$, since there exist pairs of Aharonov, e.g. the pair $(f, g) = (z, z)$, for which $\Phi\big((f, g)\big) > -\infty$. Of course, there exists a sequence $\{(f_n, g_n)\}$, $(f_n, g_n) \in \mathscr{A}$, such that $\Phi\big((f_n, g_n)\big) \to M$. The sequence $\{b_{1f_n} b_{1g_n}\}$, being bounded, contains a convergent subsequence; also the sequences $\{\hat{f}_n\}$ and $\{\hat{g}_n\}$, consisting of functions of the class S, also contain convergent subsequences. Thus we may just assume that the sequences $\{\check{f}_n\}$ and $\{\check{g}_n\}$ converge to the functions $\check{f}$ and $\check{g}$, respectively; $(\check{f}, \check{g}) \in \mathscr{A}$ or $\check{f} = 0$. If we had $\check{f} = 0$, then $\Phi(f_n, g_n) = \Phi\big((\check{f}_n, \check{g}_n)\big) \to -\infty$, which is impossible since $\Phi(f_n, g_n) \to M > -\infty$.

Hence we infer that under our assumptions the upper bound of the functional Φ in the family $\mathscr{A}$ is attained at the pair $(\tilde{f}, \tilde{g})$.

Let now (f, g) be an arbitrary maximal pair. In order to find a system of equations (25), we first check by an easy calculation using (14′) and (28), that

$$P(w)+L_{(f,g)}((f, 0)) = w^2\left(\frac{1}{w-f(z_1)}+\frac{g(z_2)}{1-wg(z_2)}\right)^2,$$
$$P(1/w)+L_{(f,g)}((0, g)) = \left(\frac{1}{1-wf(z_1)}+\frac{g(z_2)}{w-g(z_2)}\right)^2.$$

The right-hand sides of the desired equations are, by (26), of the form

$$Q_1(\zeta) = \frac{a_{11}}{\zeta-z_1}+\frac{\bar{a}_{11}\zeta}{1-\bar{z}_1\zeta}+\frac{a_{12}}{(\zeta-z_1)^2}+\frac{\bar{a}_{12}\zeta^2}{(1-\bar{z}_1\zeta)^2}+$$
$$+\frac{a_{21}}{\zeta-z_2}+\frac{\bar{a}_{21}\zeta}{1-\bar{z}_2\zeta}+\frac{a_{22}}{(\zeta-z_2)^2}+\frac{\bar{a}_{22}\zeta^2}{(1-\bar{z}_2\zeta)^2}+a,$$
$$Q_2(\zeta) = \frac{b_{11}}{\zeta-z_1}+\frac{\bar{b}_{11}\zeta}{1-\bar{z}_1\zeta}+\frac{b_{12}}{(\zeta-z_1)^2}+\frac{\bar{b}_{12}\zeta^2}{(1-\bar{z}_1\zeta)^2}+$$
$$+\frac{b_{21}}{\zeta-z_2}+\frac{\bar{b}_{21}\zeta}{1-\bar{z}_2\zeta}+\frac{b_{22}}{(\zeta-z_2)^2}+\frac{\bar{b}_{22}\zeta^2}{(1-\bar{z}_2\zeta)^2}+b,$$

where a and b are real. The functions Q_1 and Q_2 are non-negative on the circle ∂U. Consequently, relations (25) take the form

$$f'^2(\zeta)\left(\frac{1}{f(\zeta)-f(z_1)}+\frac{g(z_2)}{1-f(\zeta)g(z_2)}\right)^2 = \frac{Q_1(\zeta)}{\zeta^2},$$

(32)

$$g'^2(\zeta)\left(\frac{1}{g(\zeta)-g(z_2)}+\frac{f(z_1)}{1-g(\zeta)f(z_1)}\right)^2 = \frac{Q_2(\zeta)}{\zeta^2}.$$

These relations hold in the disc U, whence it follows that all roots and poles of the functions Q_1 and Q_2 lying in U are even-tuple and so are, on account of the symmetry of these functions with respect to the circle ∂U, roots and poles lying outside U. On the other hand, the roots of the functions Q_1 and Q_2 lying on the circle ∂U (if there are any) are also even-tuple, because these functions are positive-valued on the circle. Summing up, we see that the two functions Q_1 and Q_2 have only even-tuple roots and poles and therefore they are squares of some rational functions q_1 and q_2. Extracting a square root from both sides of equalities (32), we obtain equivalent equalities

$$f'(\zeta)\left(\frac{1}{f(\zeta)-f(z_1)}+\frac{g(z_2)}{1-f(\zeta)g(z_2)}\right) = \frac{q_1(\zeta)}{\zeta},$$

(33)

$$g'(\zeta)\left(\frac{1}{g(\zeta)-g(z_2)}+\frac{f(z_2)}{1-g(\zeta)f(z_1)}\right) = \frac{q_2(\zeta)}{\zeta},$$

where q_1 and q_2 are rational functions, real-valued on the circle ∂U. The left-hand sides of (33) are regular functions at the point $\zeta = 0$; therefore $q_1(\zeta)/\zeta$ and $q_2(\zeta)/\zeta$ are also regular at $\zeta = 0$ and so they are of the form

$$\frac{q_1(\zeta)}{\zeta} = \frac{A_{11}}{z_1(\zeta - z_1)} + \frac{\bar{A}_{11}}{1-\bar{z}_1\zeta} + \frac{A_{21}}{z_2(\zeta - z_2)} + \frac{\bar{A}_{21}}{1-\bar{z}_2\zeta},$$

$$\frac{q_2(\zeta)}{\zeta} = \frac{B_{11}}{z_1(\zeta - z_1)} + \frac{\bar{B}_{11}}{1-\bar{z}_1\zeta} + \frac{B_{21}}{z_2(\zeta - z_2)} + \frac{\bar{B}_{21}}{1-\bar{z}_2\zeta}.$$

Integrating equalities (33) in a neighbourhood of $\zeta = 0$, we get

$$\begin{aligned} &\log(f(\zeta) - f(z_1)) - \log(1 - g(z_2)f(\zeta)) \\ &\quad = \frac{A_{11}}{z_1}\log(\zeta - z_1) - \frac{\bar{A}_{11}}{\bar{z}_1}\log(1-\bar{z}_1\zeta) + \frac{A_{21}}{z_2}\log(\zeta - z_2) - \\ &\qquad - \frac{\bar{A}_{21}}{\bar{z}_2}\log(1-\bar{z}_2\zeta) + \log\alpha, \end{aligned} \tag{34}$$

$$\begin{aligned} &\log(g(\zeta) - g(z_2)) - \log(1 - f(z_1)g(\zeta)) \\ &\quad = \frac{B_{11}}{z_1}\log(\zeta - z_1) - \frac{\bar{B}_{11}}{\bar{z}_1}\log(1-\bar{z}_1\zeta) + \frac{B_{21}}{z_2}\log(\zeta - z_2) - \\ &\qquad - \frac{\bar{B}_{21}}{\bar{z}_2}\log(1-\bar{z}_2\zeta) + \log\beta. \end{aligned}$$

From further comparison of the left- and right-hand sides of equality (34) at follows that $A_{11}/z_1 = 1$, $A_{21} = 0$, $B_{21}/z_2 = 1$ and $B_{11} = 0$. Taking account of the above and dropping the logarithms in (34), we obtain

$$\frac{f(\zeta) - f(z_1)}{1 - g(z_2)f(\zeta)} = \alpha\frac{\zeta - z_1}{1-\bar{z}_1\zeta}, \quad \frac{g(\zeta) - g(z_2)}{1 - f(z_1)g(\zeta)} = \beta\frac{\zeta - z_2}{1-\bar{z}_2\zeta}. \tag{35}$$

for $\zeta \in U$. Putting further $\zeta = 0$ in (35), we arrive at the conclusion that $\alpha = f(z_1)/z_1$ and $\beta = g(z_2)/z_2$, and so equalities (35) will take the form

$$\frac{f(\zeta) - f(z_1)}{1 - g(z_2)f(\zeta)} = \frac{f(z_1)}{z_1}\cdot\frac{\zeta - z_1}{1-\bar{z}_1\zeta}, \quad \frac{g(\zeta) - g(z_2)}{1 - f(z_1)g(\zeta)} = \frac{g(z_2)}{z_2}\cdot\frac{\zeta - z_2}{1-\bar{z}_2\zeta}. \tag{36}$$

Moreover, we claim that

$$|f(z_1)g(z_2)| = |z_1 z_2|. \tag{37}$$

Indeed this follows from the corollary to Theorem 3; on the circle ∂U there exist two points ζ_1 and ζ_2 such that $f(\zeta_1) = h(\zeta_2) = 1/g(\zeta_2)$. Putting in the first equality of (36) $\zeta = \zeta_1$, in the other one $\zeta = \zeta_2$, and equating the left-hand sides of the relations thus obtained, we get

$$\frac{f(z_1)}{z_1}\cdot\frac{\zeta_1 - z_1}{1-\bar{z}_1\zeta_1} = \frac{z_2}{g(z_2)}\cdot\frac{1-\bar{z}_2\zeta_2}{\zeta_2 - z_2},$$

from which relation (37) follows.

Finally, let us calculate the value of the functional Φ at the maximal pair (f, g). With that end in view, let us pass to the limit with $\zeta \to z_1$ and $\zeta \to z_2$ in the first and the second equality (36), respectively; we obtain

$$\frac{f'(z_1)}{1-f(z_1)g(z_2)} = \frac{f(z_1)}{z_1}\cdot\frac{1}{1-|z_1|^2}, \qquad \frac{g'(z_2)}{1-f(z_1)g(z_2)} = \frac{g(z_2)}{z_2}\cdot\frac{1}{1-|z_2|^2};$$

hence, taking account on (37), we have

$$\frac{|f'(z_1)g'(z_2)|}{|1-f(z_1)g(z_2)|^2} = \frac{1}{(1-|z_1|^2)(1-|z_2|^2)}. \tag{38}$$

But the pair (f, g) is maximal for the functional Φ in the family $\mathscr{A}$; hence, by (38), the inequality

$$|f'(z_1)\cdot g'(z_2)| \leqslant \frac{|1-f(z_1)g(z_2)|^2}{(1-|z_1|^2)(1-|z_2|^2)} \tag{38'}$$

necessarity holds for each pair $(f, g) \in \mathscr{A}$.

This inequality was obtained by Jenkins [8], p. 206, by the method of quadratic differentials. Since f, g, f', g' are continuous, we now can drop the assumption that $(1-|z_1|)^2(1-|z_2|)^2-4|z_1||z_2| \neq 0$.

In the case where condition (31) is satisfied, the existence of a pair maximizing the functional Φ has been proved. Let us now reflect upon whether there exists a maximal pair in the case where this condition is not satisfied.

Consider the pair (f, g) of functions defined by the relations

$$\frac{f-az_1}{1-bz_2 f} = a\frac{z-z_1}{1-\bar{z}_1 z}, \qquad \frac{g-bz_2}{1-az_1 g} = b\frac{z-z_2}{1-\bar{z}_2 z}, \tag{39}$$

where $z_1, z_2 \in U$ are arbitrary, $a, b \in C$ are arbitrary, and $|ab| = 1$. Formulae (39) define f and g as functions holomorphic and univalent in the disc U, Indeed, the values $-1/bz_2$ and $-1/az_1$ with which the homographies on the left-hand sides of (39) assume infinite values are not in the set of values assumed by the homographies on the right-hand sides of (39) in the disc U, since we have, in view of $|ab| = 1$,

$$-\frac{1}{bz_2} \neq a\frac{z-z_1}{1-\bar{z}_1 z} \quad \text{and} \quad -\frac{1}{az_1} \neq \frac{z-z_2}{1-\bar{z}_2 z} \quad \text{for } z \in U.$$

Put $z = 0$ in (39); we get

$$\frac{f(0)-az_1}{1-bz_2 f(0)} = -az_1 \quad \text{and} \quad \frac{g(0)-bz_2}{1-az_1 g(0)} = -bz_2,$$

from which it follows that $f(0) = g(0) = 0$. It is still to be proved that the functions f and h have disjoint sets of values. Suppose, on the contrary, that for some $\zeta_1, \zeta_2 \in U$ we have $f(\zeta_1) = h(\zeta_2) = 1/g(\zeta_2)$. On account of (39), we would have

$$a\frac{\zeta_1 - z_1}{1-\bar{z}_1\zeta_1} = \frac{1}{b}\cdot\frac{1-\bar{z}_2\zeta_2}{\zeta_2 - z_2},$$

which is impossible in view of the fact that $|ab| = 1$.

Consequently, the functions satisfying (39) constitute a pair of Aharonov.

Let us eventually calculate the value of the functional Φ for functions defined by means of (39). Putting $z = z_1$ and $z = z_2$ in the first and in the second of relations (39), we see that $f(z_1) = az_1$, $g(z_2) = bz_2$. Dividing the first of relations (39) by $z - z_1$ and the second one by $z - z_2$ and passing to the limit with $z \to z_1$ and $z \to z_2$ we obtain

$$\frac{f'(z_1)}{1-f(z_1)g(z_2)} = \frac{a}{1-|z_1|^2}, \quad \frac{g'(z_2)}{1-f(z_1)g(z_2)} = \frac{b}{1-|z_2|^2};$$

on account of the condition $|ab| = 1$, for the pair (f, g) there appears the equality sign in inequality (38'). Consequently, inequality (38') is exact in the class $\mathscr{A}$ of the pairs of Aharonov for any points $z_1, z_2 \in U$.

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