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NON-CLOSED THIN SETS IN HARMONIC ANALYSIS

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This paper is concerned with a condition implying Sidonicity in some abelian groups. Let G be a locally compact non-discrete abelian group and Γ its dual. By G_d we denote the group G with discrete topology. $E \in \mathrm{Sid}(G_d)$ means that E is a Sidon set in G_d. For a countable non-void set $E \subset G$ with compact closure $\bar{E}$, let E' denote the set of limit points of E. For a compact K, in general, let $C(K)$ be the space of continuous functions on K, and $A(K)$ that of restricted Fourier transforms $\hat{g}|K$ ($g \in L_1(\Gamma)$) with the quotient norm. We put

$$C_0 = \{f \in C(\bar{E}) \colon f|E' = 0\},$$

$$A^0 = \{f \in A(G) \colon f|E' = 0\},$$

$$A_0 = \{f \in A(\bar{E}) \colon f|E' = 0\}.$$

So we have $A_0 = A^0/I$, where $I = \{f \in A^0 \colon f|E = 0\}$. The corresponding dual spaces are

$$C_0^* = M_d(E \setminus E')$$

(the space of (atomic) measures in $E \setminus E'$), and

$$A^{0*} = L_\infty(\Gamma)/J,$$

where

$$J = \Big\{\varphi \in L_\infty(\Gamma) \colon \int \varphi g = 0 \;\; \forall g \in L_1(\Gamma),\; \hat{g}|E' = 0\Big\},$$

$$A_0^* = \Big\{[\varphi]_J \colon \int \varphi g = 0 \;\; \forall g \in L_1(\Gamma),\; \hat{g}|E = 0\Big\},$$

and $[\varphi]_J$ means the class of $\varphi \in L_\infty(\Gamma)$ modulo the ideal J.

It follows easily from the regularity of $A(G)$ that A_0 is dense in C_0. Moreover, it can be shown that if $f \in C_0$ is such that

$$\sum_{t \in E \setminus E'} |f(t)| < \infty,$$

then $f \in A_0$. In fact, if $E \setminus E' = (t_j)$ $(1 \leqslant j < \infty)$, we choose, for each j, a function $k_j \in A(G)$ such that

$$k_j(t_j) = f(t_j), \quad k_j | \bar{E} \setminus \{t_j\} = 0 \quad \text{and} \quad \|k_j\|_A \leqslant |f(t_j)|,$$

which is possible whatever be f (see, e.g., Theorem 2.6.1 of [6]). Then, putting $k = \sum_j k_j$, we have

$$k(t) = f(t) \ \forall t \in \bar{E} \quad \text{and} \quad \|k\|_A < \infty.$$

Hence $f \in A_0$.

THEOREM 1. *If* $C_0 = A_0$, *then* $E \setminus E' \in \mathrm{Sid}(G_d)$.

Proof. Let π be the canonical imbedding $A_0 \to C_0$. Then, for $\mu \in M_d(E \setminus E')$ and $f \in A_0$, we have

$$\langle \pi^* \mu, f \rangle = \langle \mu, \pi f \rangle = \int_G f d\mu = \int_\Gamma g \hat{\mu}, \quad \text{where } g \in L_1(\Gamma),\ \hat{g} = f.$$

So

$$(1) \qquad \|\pi^* \mu\| = \sup_{\hat{g} \in A_0} \frac{1}{\|g\|_1} \left| \int_\Gamma g \hat{\mu} \right| \leqslant \sup_{\|g\|_1 = 1} \left| \int_\Gamma g \hat{\mu} \right| = \|\hat{\mu}\|_\infty \overset{\text{def}}{=} \|\mu\|_{PM}.$$

If $A_0 = C_0$, then the norms $\|\pi^* \mu\|$ and $\|\mu\|$ (the total variation) are equivalent and since, by (1), $\|\pi^* \mu\| \leqslant \|\mu\|_{PM}$, we have $\|\mu\|_{PM} > a \|\mu\|$ with a positive constant a. It means that every finite sum

$$P(x) = \sum_{t_k \in E \setminus E'} a_k (x, t_k) \qquad (x \in \Gamma)$$

is minorized in absolute value by $a \sum |a_k|$. Moreover, since Γ is dense in its Bohr compactification $\tilde{\Gamma}$, we have

$$\max_{x \in \tilde{\Gamma}} |P(x)| \geqslant a \sum |a_k|.$$

As $\tilde{\Gamma}$ is the dual of G_d, this estimation is equivalent to $E \setminus E' \in \mathrm{Sid}(G_d)$.

As an easy consequence of Theorem 1 (*via* Bohr compactification) one can obtain the following result:

Let Λ be a countable set in a discrete abelian group H and let the following hold:

(*) Every function from $c_0(\Lambda)$, i.e. every function defined and tending to 0 on Λ, can be extended to the Fourier transform of an atomic measure on the dual of H.

Then Λ is Sidon.

Obviously, the assumption cannot be satisfied unless Λ is weakly isolated, i.e. if it does not contain any limit point in Bohr topology. Further, this result is by no means unexpected. For $H = \mathbf{Z}$ we know even more for a long time, namely

(A) If $\Lambda \subset H$ is such that every $\varphi \in c_0(\Lambda)$ is a restriction of $\hat{\mu}$ for some (not necessarily atomic) measure on the dual on H, then Λ is Sidon.

The assumption $H = \mathbf{Z}$ appears to be inessential in view of a result of Curtis and Figà-Talamanca [1] (see also [3], p. 284) who proved that, for any locally compact abelian group X, the space $C_0(X)$ of continuous functions vanishing at infinity factorizes into $A(X)$ and $C_0(X)$: $C_0 = AC_0$. So, if $\varphi \in c_0(\Lambda)$, we take a $g \in A(H)$ such that $\varphi = g|\Lambda \cdot \psi$ with $\psi \in c_0(\Lambda)$. Then, according to the assumption of (A), ψ is the restriction of some Fourier-Stieltjes transform: $\psi = \hat{\nu}|\Lambda$. The measure $\varrho = \check{g} * \nu$ is absolutely continuous and $\varphi = \hat{\varrho}|\Lambda$. Since $\varphi \in c_0(\Lambda)$ is arbitrary, (A) is proved. In the case $H = \mathbf{Z}$ the known argument runs along the same lines with g — a suitable real even convex function from $c_0(\mathbf{Z})$. It is classical that such g is an element of $A(\mathbf{Z})$ ([8], p. 180).

The author does not know whether, for weakly isolated Sidon sets (then, may be, for all Sidon sets), condition (*) is satisfied (**P 941**). It would be so if Theorem 1 admitted a converse. We do not see any reason for such conjecture (1). We now prove

THEOREM 2. *If E' is not of synthesis and $\bar{E}$ is of synthesis, then $A_0 \neq C_0$.*

Proof. By assumption, there exists a pseudomeasure T with support in E' (briefly, $T \in PM(E')$) and an $f \in A_0$ such that $\langle T, f\rangle \neq 0$. But T is a functional on $A(\bar{E})$ because $\bar{E}$ has synthesis. Thus, if we had $A_0 = C_0$, the restriction $T_1 = T|A_0$ would be a non-zero measure ν on $E \setminus E'$. If $t_0 \in E\setminus E'$ is an atom of ν, we take a $g \in A(G)$ such that $g(t_0) = 1$ and $g(t) = 0$ on an open set including $\bar{E}\setminus\{t_0\} \supset E'$. Then $g|\bar{E} \in A_0$ and $\langle T_1, g\rangle = 0$, but $\int g d\nu \neq 0$ — a contradiction.

THEOREM 3. *$A_0 = C_0$ does not imply $A(\bar{E}) = C(\bar{E})$.*

Proof. In order to produce an isolated set E such that $C_0 = A_0$ but $C(\bar{E}) \neq A(\bar{E})$ we take E — a countable Sidon set in $\mathbf{T}_d$ (an independent set will do) such that

(i) E' is countable without being a Helson set, for example $E' = \{0, 1, \ldots, 1/n, \ldots\}$ (see [4], p. 32), and

(ii) E and E' are independent in the sense that subgroups of $\mathbf{T}$ generated by E and E' are disjoint except for 0.

Let $T \in PM(\bar{E})$. Then the Fourier transform $\hat{T}$ is an almost periodic function on Γ ([4], p. 49). So $\hat{T}(\cdot)$ is represented by a Fourier series

$$\sum_{t_n \in \bar{E}} a_n(t_n, \cdot).$$

(1) Added in proof. It fails in fact (an example by Y. Meyer).

By (ii), the partial series

$$\sum_{t_n \in E} a_n(t_n, \cdot) \qquad \text{and} \qquad \sum_{t_n \in E'} a_n(t_n, \cdot)$$

represent also some almost periodic functions φ_1 and φ_2. Since E is Sidon in T_d, the first series is absolutely convergent, and so $\varphi_1 = \hat{\mu}$ with $\mu \in M(E)$ (i.e., μ — a measure in E). Thus we have $T = \mu + T_1$ with $\hat{T}_1 = \varphi_2$ and $T_1 \in PM(E')$. Since E' is of synthesis, we have $\langle T_1, f\rangle = 0$ for every $f \in A_0$ and T is a linear functional on $A(\bar{E})$ whose restriction to A_0 equals $\mu \in C_0^*$. Since every linear functional on $A(\bar{E})$ is an element of $PM(\bar{E})$, it follows that $A_0^* = C_0^*$ and, finally, $A_0 = C_0$. However, by (i), $C(E') \neq A(E')$ and, *a fortiori*, $C(\bar{E}) \neq A(\bar{E})$.

We are not able to construct an example proving Theorem 3 for a set E with uncountable closure (**P 942**) (2). The converse implication $A(\bar{E}) = C(\bar{E}) \Rightarrow A_0 = C_0$ holds trivially.

If E is a convergent sequence $t_n \to t_0$, then the three conditions, i.e. $E \in \mathrm{Sid}(G_d)$, $A_0 = C_0$ and $A(\bar{E}) = C(\bar{E})$, are equivalent. To see it we first state a well-known result (see, e.g., [4]) that for a countable compact set K to be Sidon in G_d is the same as to be Helson in G. In fact, every μ on K is atomic, and so

$$\hat{\mu}(x) = \sum_{t_n \in K} \mu(\{t_n\}) \cdot (t_n, x) \qquad (x \in \Gamma).$$

We are allowed to let x run over $\tilde{\Gamma}$. Hence the equivalence of the norms $\|\mu\|_{PM}$ and $\|\mu\|$, which is characteristic for Helson sets, is at the same time the very definition of the class $\mathrm{Sid}(G_d)$. Now, if $E \in \mathrm{Sid}(G_d)$, then

$$\bar{E} = E \cup \{t_0\} \in \mathrm{Sid}(G_d),$$

and so $C(\bar{E}) = A(\bar{E})$. In view of Theorem 1, nothing else is to prove.

THEOREM 4. *If K is a Helson set in G, E is a countable Sidon set in G_d and $E' \subset K$, then $E \cup K$ is Helson in G.*

Proof. It is enough to prove that $\bar{E}$ is Helson, for then Varopoulos' theorem ([7], p. 152) gives the assertion. So we must show the equivalence of norms $\|\mu\|$ and $\|\mu\|_{PM}$ for measures in $\bar{E}$. Since E is countable, the continuous part μ_c of μ is supported by E', and so $\|\mu_c\| \simeq \|\mu_c\|_{PM}$ because E' is Helson. From [2], Corollary 2, we have $\|\mu\|_{PM} \simeq \|\mu_c\|_{PM} + \|\mu_d\|_{PM}$. Since E' is Helson in G, it is Sidon in G_d. Consequently, $\bar{E} = E \cup E' \in \mathrm{Sid}(G_d)$ by Drury's theorem. Hence $\|\mu_d\|_{PM} \simeq \|\mu_d\|$. Obviously, $\|\mu\| = \|\mu_d\| + \|\mu_c\|$. So we infer that $\|\mu\|_{PM} \simeq \|\mu\|$ and the proof is complete.

We shall deduce some corollaries from Theorem 4. First we take for E a weakly isolated Sidon set Λ in a discrete group H and we consider

(2) Added in proof. Recently, Y. Meyer gave such an example.

this Λ as a subset of $\tilde{H}$, as we already did after proving Theorem 1. Functions on Λ which are extendable to elements of $AP(H)$ (almost periodic functions on H) are precisely those which are continuously extendable over the closure $\tilde{\Lambda}$ of Λ. The difference between two functions on Λ having the same extension on Λ' belongs to $c_0(\Lambda)$. If Λ' is a Helson set, then every continuous function on it can be extended to an element of $A(\tilde{H})$ and so, for every $f \in AP(H)$, the restriction $f|\Lambda$ differs for a $\varphi \in c_0(\Lambda)$ from a function which is extendable to an element of $A(\tilde{H})$ or, in other words, to an element of $|AP|(H)$ (almost periodic functions with absolutely convergent Fourier series, or else, Fourier transforms of atomic measures on $\hat{H}$). Thus in this case Theorem 4 yields

COROLLARY 1. *If Λ is a weakly isolated Sidon set in a discrete group H and if, for the restriction $f|\Lambda$ of any $f \in AP(H)$, there is an element $f_1 \in |AP|(H)$ such that $f|\Lambda - f_1|\Lambda \in c_0(\Lambda)$, then any $f|\Lambda$ $(f \in AP(H))$ is extendable to an element of $|AP|(H)$.*

In view of the fact that every function from $c_0(\Lambda)$ can be extended to the Fourier transform of a function belonging to $L_1(\hat{H})$ we can reformulate Corollary 1 in the following way:

COROLLARY 1′. *If Λ is a weakly isolated Sidon set in a discrete group H and if the restriction $f|\Lambda$ of any $f \in AP(H)$ can be extended to the Fourier transform of a measure without continuous singular part, then every function on Λ which is extendable to an almost periodic function on H can also be extended to the Fourier transform of an atomic measure.*

We do not know whether a compact set without synthesis can ever become a set of synthesis by adjoining an isolated Sidon set to it (**P 943**). In view of Theorem 2 the answer is "no" if we assume that Sidonicity of $E \setminus E'$ implies $A_0 = C_0$. Without assuming any conjecture we infer from Theorem 4 that a Helson-Körner set K (i.e., a Helson set without synthesis, see [5]) cannot become a set of synthesis by adjoining a Sidon set to it. In fact, the enlarged set would be again a Helson set and would carry a "true" pseudo-measure since K carries one. This remark gives raise to the following corollary which is related in some way to arithmetic properties of Helson-Körner sets:

COROLLARY 2. *For a compact $K \subset T$, let H_n be the set of numbers $2\pi j/n$ $(0 \leqslant j < n)$ which are outside K but in a distance less than $2\pi/n$ from K. Following Hewitt and Ross [3] we call a set $E \subset G$ dissociate if, for every finite subset $\{x_k\}_{k=1}^{N}$ of E, the equality*

$$\sum_{1}^{N} a_k x_k = 0 \quad \text{with } -2 \leqslant a_k \leqslant 2$$

does not hold unless $a_k = 0$ $(1 \leqslant k \leqslant N)$. Then, if K is a Helson-Körner *set, there is no infinite sequence $n_k \in Z^+$ for which the set $H = \bigcup_{k=1}^{\infty} H_{n_k}$ would be either void or dissociate.*

Proof. In virtue of the known theorem of Herz ([4], p. 58), if such a sequence existed, then, by adjoining H to K, we would get a set of synthesis. This, however, is impossible in view of the argument which precedes Corollary 2 and of the fact that a dissociate set is Sidon ([3], p. 427).

COROLLARY 2′. *If $K \subset \boldsymbol{T}$ is a Helson-Körner set, then there is no infinite sequence of pairwise relatively prime numbers n_k such that H_{n_k} are either void or dissociate.*

In fact, otherwise $\bigcup_k H_{n_k}$ would be void or dissociate.

The property of K, claimed in the assertion of Corollary 2, seems remarkable if K is independent, and so a Helson-Körner set can actually be [5].

REFERENCES

[1] P. C. Curtis and A. Figà-Talamanca, *Factorization theorems for Banach algebras. Functions algebras*, p. 169-185, Atlanta (1966).

[2] S. Hartman, *A remark on Fourier-Stieltjes transform*, this fascicle, p. 113-115.

[3] E. Hewitt and K. A. Ross, *Abstract harmonic analysis II*, Berlin 1970.

[4] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Berlin 1970.

[5] Th. Körner, *A pseudofunction on a Helson set, I*, Astérisque 5 (1973), p. 3-208.

[6] W. Rudin, *Fourier analysis on groups*, New York - London 1962.

[7] N. Th. Varopoulos, *Groups of continuous functions in harmonic analysis*, Acta Mathematica 125 (1970), p. 109-154.

[8] A. Zygmund, *Trigonometric series I*, Cambridge 1959.

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