ANNALES
POLONICI MATHEMATICI
XLIII (1983)

On the means of an entire Dirichlet series

by SATENDRA KUMAR VAISH (Roorkee, India)

Abstract. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ $(s = \sigma + it,\ \lambda_{n+1} > \lambda_n,\ \lambda_1 \geqslant 0,\ \lim_{n\to\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0)$ define an entire function. The Ritt-order ϱ and lower order for $f(s)$ are defined as:

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log M(\sigma)}{\sigma} = \frac{\varrho}{\lambda}, \quad 0 \leqslant \lambda \leqslant \varrho \leqslant \infty,$$

where $M(\sigma) = \sup\{|f(\sigma+it)|: -\infty < t < \infty\}$.

We have defined the following means of $f(s)$:

$$I_\delta(\sigma) = I_\delta(\sigma, f) = \lim_{T\to\infty} \left\{\frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^\delta dt\right\} \tag{1}$$

and

$$m_{\delta,k}(\sigma) = m_{\delta,k}(\sigma, f) = e^{-k\sigma}\int_0^\sigma I_\delta(x, f) e^{kx} dx, \tag{2}$$

where $0 < \delta < \infty$ and $0 < k < \infty$.

In this paper we have obtained a few properties of the product of any finite number of mean values, defined by (1) and (2). The main result of the paper is:

If $f_\alpha(s)$ $(\alpha = 1, 2, \ldots, n)$ are n entire functions of regular growth, of orders $\varrho_1, \varrho_2, \ldots, \varrho_n$, respectively, then so is $f(s)$, of order ϱ and

$$\varrho = \varrho_1 + \varrho_2 + \ldots + \varrho_n.$$

1. For an entire Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \tag{1.1}$$

$$(s = \sigma + it,\ \lambda_1 \geqslant 0,\ \lambda_{n+1} > \lambda_n,\ \lambda_n \to \infty \text{ with } n)$$

we define the maximum modulus over a vertical line $\mathrm{Re}(s) = \sigma$ as:

$$M(\sigma) = \sup\{|f(\sigma+it)|: -\infty < t < \infty\},$$

the maximum term as:

$$\mu(\sigma) = \max_{n \geq 1}\{|a_n| \exp(\sigma\lambda_n)\},$$

and the rank of the maximum term as:

$$\lambda_{\nu(\sigma)} = \max\{\lambda_n: \mu(\sigma) = |a_n| \exp(\sigma\lambda_n)\}.$$

Finally, the Ritt-order ϱ and lower order λ (in the sence of Ritt) are defined as [4], p. 78:

$$\lim_{\sigma \to \infty} \frac{\sup}{\inf} \frac{\log\log M(\sigma)}{\sigma} = \frac{\varrho}{\lambda}, \quad 0 \leq \lambda \leq \varrho \leq \infty.$$

Let us introduce the following mean values of $f(s)$:

$$I_\delta(\sigma) = I_\delta(\sigma, f) = \lim_{T \to \infty}\left\{\frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^\delta \, dt\right\} \tag{1.2}$$

and

$$m_{\delta,k}(\sigma) = m_{\delta,k}(\sigma, f) = e^{-k\sigma}\int_0^\sigma I_\delta(x, f)\, e^{kx}\, dx, \tag{1.3}$$

where $0 < \delta < \infty$ and $0 < k < \infty$.

It is known that [3]:

$$\lim_{\sigma \to \infty} \frac{\sup}{\inf} \frac{\log\log I_\delta(\sigma, f)}{\sigma} = \frac{\varrho}{\lambda} \tag{1.4}$$

and

$$\lim_{\sigma \to \infty} \frac{\sup}{\inf} \frac{\log\log m_{\delta,k}(\sigma, f)}{\sigma} = \frac{\varrho}{\lambda}. \tag{1.5}$$

In this paper we have obtained a few properties of the product of any finite number of mean values, defined by (1.2) and (1.3). Throughout this paper we shall assume that the function $f(s)$ is of finite non-zero order.

2. THEOREM 1. *Let $f(s)$ be an entire function of order ϱ and lower order λ. Then*

$$\lim_{\sigma \to \infty} \frac{\sup}{\inf} \frac{\log\{m'_{\delta,k}(\sigma, f)/m_{\delta,k}(\sigma, f)\}}{\sigma} = \frac{\varrho}{\lambda}, \tag{2.1}$$

where $m'_{\delta,k}(\sigma, f)$ is the derivative of $m_{\delta,k}(\sigma, f)$.

Proof. Since $\log m_{\delta,k}(\sigma,f)$ is an increasing convex function of σ ([6], Lemma 1), therefore $\log m_{\delta,k}(\sigma,f)$ is differentiable almost everywhere with an increasing derivative. This enables us to write $\log m_{\delta,k}(\sigma,f)$ in the following form:

$$\log m_{\delta,k}(\sigma,f) = \log m_{\delta,k}(\sigma_0,f)+\int_{\sigma_0}^{\sigma}\frac{m'_{\delta,k}(x,f)}{m_{\delta,k}(x,f)}dx, \qquad \sigma>\sigma_0. \tag{2.2}$$

Thus

$$\log m_{\delta,k}(\sigma,f) < \log m_{\delta,k}(\sigma_0,f)+\frac{m'_{\delta,k}(\sigma,f)}{m_{\delta,k}(\sigma,f)}(\sigma-\sigma_0).$$

Proceeding to limits, we get

$$\lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log\log m_{\delta,k}(\sigma,f)}{\sigma} \leqslant \lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log\{m'_{\delta,k}(\sigma,f)/m_{\delta,k}(\sigma,f)\}}{\sigma}. \tag{2.3}$$

Again, for an arbitrary $\eta>0$, we have

$$\begin{aligned}\log m_{\delta,k}(\sigma+\eta,f) &= \log m_{\delta,k}(\sigma,f)+\int_{\sigma}^{\sigma+\eta}\frac{m'_{\delta,k}(x,f)}{m_{\delta,k}(x,f)}dx\\ &> \frac{m'_{\delta,k}(\sigma,f)}{m_{\delta,k}(\sigma,f)}\eta.\end{aligned}$$

This gives

$$\lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log\log m_{\delta,k}(\sigma,f)}{\sigma} \geqslant \lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log\{m'_{\delta,k}(\sigma,f)/m_{\delta,k}(\sigma,f)\}}{\sigma} \tag{2.4}$$

From (2.3) and (2.4), we find

$$\lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log\log m_{\delta,k}(\sigma,f)}{\sigma} = \lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{\log\{m'_{\delta,k}(\sigma,f)/m_{\delta,k}(\sigma,f)\}}{\sigma}. \tag{2.5}$$

On using (1.5) in (2.5), we get (2.1).

COROLLARY 1. *For almost all values of* $\sigma>\sigma_0$,

$$m_{\delta,k}(\sigma,f)\,e^{(\lambda-\varepsilon)\sigma} < m'_{\delta,k}(\sigma,f) < m_{\delta,k}(\sigma,f)\,e^{(\varrho+\varepsilon)\sigma},$$

where ε *is an arbitrary small positive number.*

THEOREM 2. *Let* $f_1(s)$ *and* $f_2(s)$ *be two entire functions of orders* ϱ_1, ϱ_2 *and lower orders* λ_1, λ_2, *respectively. Then, if*

$$\log\log m_{\delta,k}(\sigma,f) \approx \log[\{\log m_{\delta,k}(\sigma,f_1)\}\{\log m_{\delta,k}(\sigma,f_2)\}], \tag{2.6}$$

the order ϱ *and lower order* λ *of the entire function* $f(s)$ *are such that*

$$\lambda_1+\lambda_2 \leqslant \lambda \leqslant \varrho \leqslant \varrho_1+\varrho_2, \tag{2.7}$$

and if

$$\text{(2.8)}\qquad \log\log m_{\delta,k}(\sigma,f) \approx [\{\log\log m_{\delta,k}(\sigma,f_1)\}\{\log\log m_{\delta,k}(\sigma,f_2)\}]^{1/2},$$

then

$$\text{(2.9)}\qquad (\lambda_1\lambda_2)^{1/2} \leqslant \lambda \leqslant \varrho \leqslant (\varrho_1\varrho_2)^{1/2},$$

where $m_{\delta,k}(\sigma,f)$, $m_{\delta,k}(\sigma,f_1)$ *and* $m_{\delta,k}(\sigma,f_2)$ *are the mean values of* $f(s)$, $f_1(s)$ *and* $f_2(s)$, *respectively.*

Proof. Since the entire functions $f_1(s)$ and $f_2(s)$ are of orders ϱ_1 and ϱ_2, therefore from (1.5), we have

$$\limsup_{\sigma\to\infty}\frac{\log\log m_{\delta,k}(\sigma,f_1)}{\sigma} = \varrho_1$$

and

$$\limsup_{\sigma\to\infty}\frac{\log\log m_{\delta,k}(\sigma,f_2)}{\sigma} = \varrho_2.$$

Hence, for any $\varepsilon > 0$ and $\sigma > \sigma_0$, we get

$$\text{(2.10)}\qquad \frac{\log\log m_{\delta,k}(\sigma,f_1)}{\sigma} < \varrho_1 + \tfrac{1}{2}\varepsilon$$

and

$$\text{(2.11)}\qquad \frac{\log\log m_{\delta,k}(\sigma,f_2)}{\sigma} < \varrho_2 + \tfrac{1}{2}\varepsilon.$$

Adding (2.10) and (2.11), we get

$$\text{(2.12)}\qquad \frac{\log[\{\log m_{\delta,k}(\sigma,f_1)\}\{\log m_{\delta,k}(\sigma,f_2)\}]}{\sigma} < \varrho_1+\varrho_2+\varepsilon.$$

Proceeding as above for the limit inferior, we find

$$\text{(2.13)}\qquad \frac{\log[\{\log m_{\delta,k}(\sigma,f_1)\}\{\log m_{\delta,k}(\sigma,f_2)\}]}{\sigma} > \lambda_1+\lambda_2-\varepsilon.$$

Now, if (2.6) holds, then from (2.12) and (2.13), for any $\varepsilon > 0$ and sufficiently large σ, we have

$$\lambda_1+\lambda_2-\varepsilon < \frac{\log\log m_{\delta,k}(\sigma,f)}{\sigma} < \varrho_1+\varrho_2+\varepsilon.$$

Taking limits and using (1.5), it leads to (2.7).

Again, multiplying (2.10) and (2.11), we get

$$\text{(2.14)}\qquad \frac{\{\log\log m_{\delta,k}(\sigma,f_1)\}\{\log\log m_{\delta,k}(\sigma,f_2)\}}{\sigma^2} < (\varrho_1+\tfrac{1}{2}\varepsilon)(\varrho_2+\tfrac{1}{2}\varepsilon)$$

for any $\varepsilon > 0$ and sufficiently large σ.

Similarly, we have

$$(2.15)\qquad \frac{\{\log\log m_{\delta,k}(\sigma, f_1)\}\{\log\log m_{\delta,k}(\sigma, f_2)\}}{\sigma^2} > (\lambda_1 - \tfrac{1}{2}\varepsilon)(\lambda_2 - \tfrac{1}{2}\varepsilon)$$

for any $\varepsilon > 0$ and sufficiently large σ.

Further, if (2.8) holds, then from (2.14) and (2.15), on taking limits and using (1.5), (2.9) follows.

COROLLARY 2. *If $f_\alpha(s)$ $(\alpha = 1, 2, \ldots, n)$ are n entire functions of orders $\varrho_1, \varrho_2, \ldots, \varrho_n$ and lower orders $\lambda_1, \lambda_2, \ldots, \lambda_n$ and having the mean values $m_{\delta,k}(\sigma, f_1)$, $m_{\delta,k}(\sigma, f_2), \ldots, m_{\delta,k}(\sigma, f_n)$, respectively. Then, if*

$$\log\log m_{\delta,k}(\sigma, f) \approx \log[\{\log m_{\delta,k}(\sigma, f_1)\}\{\log m_{\delta,k}(\sigma, f_2)\}\ldots\{\log m_{\delta,k}(\sigma, f_n)\}],$$

the order ϱ and lower order λ of the entire function $f(s)$ having the mean value $m_{\delta,k}(\sigma, f)$ are such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n \leqslant \lambda \leqslant \varrho \leqslant \varrho_1 + \varrho_2 + \ldots + \varrho_n,$$

and if

$$\log\log m_{\delta,k}(\sigma, f) \approx [\{\log\log m_{\delta,k}(\sigma, f_1)\}\{\log\log m_{\delta,k}(\sigma, f_2)\}\ldots\{\log\log m_{\delta,k}(\sigma, f_n)\}]^{1/n},$$

then

$$(\lambda_1\lambda_2\ldots\lambda_n)^{1/n} \leqslant \lambda \leqslant \varrho \leqslant (\varrho_1\varrho_2\ldots\varrho_2)^{1/n}.$$

COROLLARY 3. *If $f_\alpha(s)$ $(\alpha = 1, 2, \ldots, n)$ are n entire functions of regular growth, of orders $\varrho_1, \varrho_2, \ldots, \varrho_n$, respectively, then so is $f(s)$, of order ϱ and*

$$\varrho = \varrho_1 + \varrho_2 + \ldots + \varrho_n.$$

Remark. We know that $\log I_\delta(\sigma)$ is an increasing convex function of σ [3]. Hence, if we replace $m_{\delta,k}(\sigma, f)$, $m_{\delta,k}(\sigma, f_1), \ldots, m_{\delta,k}(\sigma, f_n)$ by $I_\delta(\sigma, f)$, $I_\delta(\sigma, f_1), \ldots, I_\delta(\sigma, f_n)$, respectively, in theorems first and second, the results remain same in view of (1.4).The details are omitted.

3. THEOREM 3. *Let $f(s)$ be an entire function of order ϱ and lower order λ. Then*

$$(3.1)\qquad \limsup_{\sigma\to\infty} \frac{\log m_{\delta,k}(\sigma, f)}{\sigma\lambda_{\nu(\sigma,f)}} \leqslant \delta(1 - \lambda/\varrho)$$

and

$$(3.2)\qquad \limsup_{\sigma\to\infty} \frac{\log m_{\delta,k}(\sigma, f)}{\lambda_{\nu(\sigma,f)}\log\lambda_{\nu(\sigma,f)}} \leqslant \delta(1/\lambda - 1/\varrho).$$

Proof. From (1.3), we have

$$m_{\delta,k}(\sigma, f) = e^{-k\sigma}\int_0^\sigma I_\delta(x, f)e^{kx}dx \leqslant \frac{1}{k} I_\delta(\sigma, f),$$

where $I_\delta(\sigma, f)$ is a positive increasing function of σ. Hence

$$\limsup_{\sigma\to\infty} \frac{\log m_{\delta,k}(\sigma, f)}{\sigma\lambda_{\nu(\sigma,f)}} \leqslant \limsup_{\sigma\to\infty} \frac{\log I_\delta(\sigma, f)}{\sigma\lambda_{\nu(\sigma,f)}}. \tag{3.3}$$

We know ([1], p. 15; [2], p. 45; [5], p. 84) that for $0 < \varrho < \infty$,

$$\limsup_{\sigma\to\infty} \frac{\log I_\delta(\sigma, f)}{\sigma\lambda_{\nu(\sigma,f)}} \leqslant \delta(1-\lambda/\varrho) \tag{3.4}$$

and

$$\limsup_{\sigma\to\infty} \frac{\log I_\delta(\sigma, f)}{\lambda_{\nu(\sigma,f)}\log\lambda_{\nu(\sigma,f)}} \leqslant \delta(1/\lambda - 1/\varrho). \tag{3.5}$$

Thus (3.1) follows from (3.3) and (3.4).

Similarly, we can easily derive (3.2), if we use (3.5) instead of (3.4).

THEOREM 4. *Let $f(s)$ be an entire function of order ϱ and lower order λ. Then*

$$\liminf_{\sigma\to\infty} \frac{m_{\delta,k}(\sigma, f)\log m_{\delta,k}(\sigma, f)}{m'_{\delta,k}(\sigma, f)} \leqslant \frac{1}{\varrho} \leqslant \frac{1}{\lambda} \leqslant \limsup_{\sigma\to\infty} \frac{m_{\delta,k}(\sigma, f)\log m_{\delta,k}(\sigma, f)}{m'_{\delta,k}(\sigma, f)}. \tag{3.6}$$

In order to prove this theorem, we need the following lemmas:

LEMMA 1. *If*

$$\limsup_{\sigma\to\infty} \frac{\log\varphi(\sigma)}{\sigma} = A, \tag{3.7}$$

where $\varphi(\sigma) > 0$ and continuous almost everywhere for $\sigma_0 < \sigma < \infty$, then

$$\liminf_{\sigma\to\infty} \frac{1}{\varphi(\sigma)}\int_{\sigma_0}^{\sigma} \varphi(x)\,dx \leqslant \frac{1}{A}. \tag{3.8}$$

LEMMA 2. *If $\varphi(\sigma)$ satisfies the conditions of Lemma 1 and*

$$\liminf_{\sigma\to\infty} \frac{\log\varphi(\sigma)}{\sigma} = B, \tag{3.9}$$

then

$$\limsup_{\sigma\to\infty} \frac{1}{\varphi(\sigma)}\int_{\sigma_0}^{\sigma} \varphi(x)\,dx \geqslant \frac{1}{B}. \tag{3.10}$$

The proofs of the above two lemmas are given by Srivastav [5], p. 88.

Proof of Theorem 4. Since $\log m_{\delta,k}(\sigma, f)$ is an increasing convex function of σ [6], the function $\{m'_{\delta,k}(\sigma, f)/m_{\delta,k}(\sigma, f)\}$ obviously satisfies the conditions of Lemma 1, i.e., $\{m'_{\delta,k}(\sigma, f)/m_{\delta,k}(\sigma, f)\}$ is a positive real function, continuous almost everywhere for $\sigma_0 < \sigma < \infty$. Hence, we can take

$$\varphi(\sigma) = \frac{m'_{\delta,k}(\sigma, f)}{m_{\sigma,k}(\sigma, f)}$$

in Lemmas 1 and 2. Then in accordance with Theorem 1, inequality (3.6) follows.

References

[1] S. K. Bose and S. N. Srivastava, *On the mean values of an integral function represented by Dirichlet series*, Ganita 25 (1974), p. 13–22.

[2] P. K. Kamthan, *A theorem on step function*, J. Gakugei, Tokushima University 13 (1962), p. 43–47.

[3] -- and P. K. Jain, *On the mean values of an entire function represented by Dirichlet series*, Notices Amer. Math. Soc. 15 (1968), p. 485.

[4] J. F. Ritt, *On certain points in the theory of Dirichlet series*, Amer. J. Math. 50 (1928), p. 73–86.

[5] R. P. Srivastav, *On the entire functions and their derivatives represented by Dirichlet series*, Ganita 9 (1958), p. 83–93.

[6] S. K. Vaish, *On the mean values of an entire Dirichlet series of order zero*, Ann. Polon. Math. 37 (1980), p. 149–155.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROORKEE
ROORKEE – 247667 (U.P.)
INDIA

Reçu par la Rédaction le 18.4.1979