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ON MULTIPLICATIVE LINEAR FUNCTIONALS

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In paper [2] and independently in paper [3] it was shown that if A is a commutative complex Banach algebra and if f is a linear functional defined on A (we do not assume f to be continuous, its continuity will follow from the property (1)) then f is a multiplicative and linear functional if (and only if) it satisfies the relation

$$f(x) \in \sigma(x) \tag{1}$$

for each $x \in A$, where $\sigma(x)$ denotes the spectrum of x in A. In paper [6] this result was extended onto non-commutative Banach algebras. In paper [3] it was shown also that a similar theorem is true for commutative complex multiplicatively-convex locally convex algebras (shortly: m-convex algebras; for the definitions see further down), provided the functional in question is continuous. Let us remark that this result is true also for non-commutative m-convex algebras since the proof given in [6] works in this case as well. It was also shown in [3] that such a theorem fails for general locally convex algebras, even in the case of a completely metrizable commutative algebra. It was not clear, however, whether the assumption of continuity of the functional f is necessary in the case of an m-convex algebra. The aim of this note [1] is to fulfillt his gap by showing that the assumption of continuity of the functional f is essential. To this end we construct in the algebra $\mathcal{E}$ of all entire functions of one complex variable a discontinuous linear functional which fulfills condition (1) but is non-multiplicative.

We recall now some necessary definitions and results. A locally convex topological algebra A is said to be *multiplicatively convex* (shortly *m-convex*) if its topology is given by means of a family of seminorms $\|x\|_a$, $a \in \mathfrak{A}$, satisfying the submultiplicativity condition

$$\|xy\|_a \leqslant \|x\|_a \|y\|_a \tag{2}$$

[1] Written during the author's stay at the University of Newcastle-upon-Tyne.

for all $x, y \in A$ and all $\alpha \in \mathfrak{A}$. A completely metrizable locally convex algebra is called a *B_0-algebra*. The algebra $\mathscr{E}$ of all entire functions of one complex variable with pointwise algebra operations is an m-convex B_0-algebra with seminorms

$$(3) \qquad \|\psi\|_n = \max_{|z|\leqslant n} |\psi(z)|, \qquad n = 1, 2, \ldots$$

For more detailed informations on m-convex algebras see [4] or [5].

If A is an m-convex algebra and $x \in A$, then the *spectrum* $\sigma(x)$ is defined as

$$(4) \qquad \sigma(x) = \{f(x): f \in \mathfrak{M}(A)\},$$

where $\mathfrak{M}(A)$ is the set of all non-zero continuous multiplicative linear functionals defined in A. It can be shown (cf., e.g., [4]) that for any $\psi \in \mathscr{E}$

$$(5) \qquad \sigma(\psi) = \{\psi(z): z \in C\} = \psi(C).$$

Our construction is based upon the following lemma due to Borel [1]:

LEMMA 1. *The set*

$$(6) \qquad S = \{\psi \in \mathscr{E}: \psi(z) = e^{\varphi(z)}, \varphi(0) = 0, \varphi \in \mathscr{E}\}$$

consists of linearly independent elements.

By this lemma the set S can be extended to a Hamel basis for the space $\mathscr{E}$, denoted by S_1. We define now a functional f by setting

$$(7) \qquad f(\psi) = \begin{cases} \psi(0) & \text{for } \psi \in S_1, \psi(z) \neq e^z, \\ \psi(1) = e & \text{for } \psi(z) = e^z, \end{cases}$$

and extending it by linearity onto the whole of $\mathscr{E}$. We show now that the functional f satisfies relation (1) for each $\psi \in \mathscr{E}$, where the spectrum $\sigma(\psi)$ is given by formula (5). Let $\psi \in \mathscr{E}$; if $\psi(C) = C$, then relation (1) is automatically satisfied. If $\psi(C) \neq C$, then ψ must be of the form $\psi(z) = a + be^{\varphi(z)}$, where $a, b \in C$ and $\varphi \in \mathscr{E}$ with $\varphi(0) = 0$. By the lemma of Borel this representation is unique. By formula (7) there is $f(\psi) = a + be^{\varphi(0)} = \psi(0)$ in the case where $\varphi(z) \neq z$, and $f(\psi) = a + be = \psi(1)$ in the case where $\varphi(z) = z$. In both cases there is $f(\psi) \in \psi(C) = \sigma(\psi)$, and so relation (1) holds for all ψ in $\mathscr{E}$.

On the other hand, the functional f is non-multiplicative since by relation (7) there is $(f(e^z))^2 = e^2 \neq 1 = f(e^{2z})$.

Thus we have obtained our result:

PROPOSITION. *There exists a commutative complex m-convex B_0-algebra A, namely the algebra $\mathscr{E}$ of all entire functions, and a linear (discontinuous) functional f defined on A such that $f(x) \in \sigma(x)$ for all $x \in A$, but f is not a multiplicative functional.*

Remark. Another lemma leading to the construction of a functional f satisfying this proposition is

LEMMA 2. *The space of all finite linear combinations of functions in S, where S is given by formula* (6), *is different from the whole space $\mathscr{E}$.*

The proof of this lemma, as well as of lemma 1, follows immediately from the following more general version of lemma 1 (cf. [1], p. 387):

LEMMA 3. *If $\varphi_1, \ldots, \varphi_n$ are different entire functions such that $\varphi_i(0) = 0$, and if $p_1, \ldots, p_n$ are polynomials, then the relation*

$$\sum_{i=1}^{n} p_i(z) e^{\varphi_i(z)} = 0$$

implies $p_1(z) = \ldots = p_n(z) = 0$.

Thus, e.g., $\psi(z) = z$ cannot be expressed in the form

$$z = \sum_{i=1}^{n} c_i e^{\varphi_i(z)}$$

where $c_i \in C$ and φ_i are entire functions satisfying $\varphi_i(0) = 0$.

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