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FOREWORD

The present paper is based on notes of the lectures of S. Mazur "Computable Analysis" in the academic year 1949-1950 at the Institute of Mathematics of the Polish Academy of Sciences in Warsaw. These lectures present a systematic exposition of the results obtained by S. Banach and S. Mazur in 1936-1939 and contain a detailed examination of recursive real numbers, recursive sequences, recursive functionals and recursive real functions. The authors never published their results, except for an abstract [2], because the Second World War made this impossible. The results concerning general recursive mathematical objects are obtained by Mazur himself after the war. In the meantime some of the results here presented were obtained independently and published by other authors. However, the clearness and simplicity of the lectures has induced us to elaborate the notes for publishing in the present form.

Helena Rasiowa and *Andrzej Grzegorczyk*

I. RECURSIVE AND COMPUTABLE NUMBERS

There exists a well known correspondence between the set of functions defined over natural numbers and assuming natural values, and the set of real numbers. A positive real number a can be represented by a function f such that

$$(1) \qquad \left| a - \frac{f(n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for any} \quad n = 0, 1, \ldots$$

Other representations are expressed by the formulae

$$(2) \qquad a = \sum \frac{f(n)}{10^n},$$

$$(3) \qquad a < \frac{p}{q+1} \equiv f(p, q) = 0.$$

It is understandable to call a number *primitive recursive* (or *general recursive*) if a function f satisfying (1), (2), or (3) is primitive recursive (or general recursive [1]). We start with a study of recursive numbers. The considerations of this section show that in the case of recursive numbers formulae (1), (2) and (3) lead to three different notions of recursive real numbers (Theorem 1.12, 1.13). In the case of computable numbers the three notions are identical (Theorem 1.26). In the sequel we shall be concerned with the recursive numbers defined by means of condition (1). All algebraic numbers are recursive (1.16). The set of recursive numbers, and the set of computable numbers constitute fields of numbers (1.3, 1.19). These two fields are algebraically closed in the field of real numbers. But these results require the notion of recursive (computable) sequence and are proved in the next section (2.56).

We shall use in the sequel the following operation-symbols and function-symbols:

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[1] To be short we shall say *recursive* instead of *primitive recursive*, and *computable* instead of *general recursive*.

$$(\mu x)(R\ldots) \overset{\mathrm{df}}{=} \begin{cases} 0 \text{ if there is no natural number } x \text{ such that } R \text{ holds,} \\ \text{the least natural number } x \text{ for which } R \text{ holds, if such} \\ \text{a number exists,} \end{cases}$$

$[a]$ = the integral part of the real number a,
$[a] = (\mu n)(n \in \mathcal{N} \text{ and } n+1 > a)$,

$$x \dot{-} y = \begin{cases} x-y & \text{if} \quad x > y, \\ 0 & \text{if} \quad y \geqslant x, \end{cases}$$

$x|y$ means: y is a divisor of x,
$\exp(m, k) = (\mu x)\big(\sim(m|kx^{+1})\big)$.

J, K, L and P, Q, R are two triples of pairing functions:

$$J(x, y) = (x+y)^2+x, \quad K(z) = z \dot{-} [\sqrt{z}]^2, \quad L(z) = [\sqrt{z}] \dot{-} K(z),$$

$$P(x, y) = 2^x(2y+1) \dot{-} 1, \quad Q(z) = \exp(z+1, 2), \quad R(z) = \frac{\dfrac{z+1}{2^{Q(z)}} - 1}{2}.$$

These functions satisfy the following equations:

$$J\big(K(z), L(z)\big) = z, \quad K\big(J(x, y)\big) = x, \quad L\big(J(x, y)\big) = y,$$

$$P\big(Q(z), R(z)\big) = z, \quad Q\big(P(x, y)\big) = x, \quad R\big(P(x, y)\big) = y,$$

$$\operatorname{sgn} a = \begin{cases} -1 & \text{if} \quad a < 0, \\ 0 & \text{if} \quad a = 0, \\ +1 & \text{if} \quad a > 0. \end{cases}$$

The letters x, y, z, k, l, m, n will stand for natural numbers. The letters a, b, c, d for real numbers. Some exceptions to this rule cause no misunderstanding.

The logical signs, quantifiers: $\prod_x$ (for all x), $\sum_x$ (for some x), and connectives: $\equiv, \vee, \wedge, \rightarrow$ (equivalence, alternative, conjunction and implication) will often be used.

1. Recursive numbers. Let $\mathcal{N}$ and $\mathbf{R}$ denote always the class of all non-negative integers and the class of all recursive functions, respectively.

1.1. *A real number a is called recursive if there exist two functions $f', f'' \in \mathbf{R}$ such that*

$$\left| a - \frac{f'(n) - f''(n)}{n+1} \right| < \frac{1}{n+1} \quad \textit{for every} \quad n \in \mathcal{N}.$$

The class of all real recursive numbers will be denoted by $\mathcal{R}$.

1.2. *For every real number a the following conditions are equivalent:*

(i) $a \in \mathscr{R}$;

(ii) *there exists a recursive function* $f \in \mathbf{R}$ *such that* $\left|\frac{f(n)}{n+1} - |a|\right| < \frac{1}{n+1}$ *for each* $n \in \mathscr{N}$;

(iii) *there exist two functions* $f, g \in \mathbf{R}$ *such that* $\left|\frac{f(n)}{g(n)+1} - |a|\right| < \frac{1}{n+1}$ *for each* $n \in \mathscr{N}$;

(iv) *there exist three functions* $f, g, h \in \mathbf{R}$ *such that* $\left|\frac{f(n)}{g(n)+1} - |a|\right| < \frac{1}{n+1}$ *for all non-negative integers* n, m *such that* $n \geqslant h(m)$.

(v) *there exist three functions* $f, g, h \in \mathbf{R}$ *and* $n_0 \in \mathscr{N}$ *such that* $\left|\frac{f(n)}{g(n)+1} - |a|\right| < \frac{n_0}{m+1}$ *for all integers* n, m *such that* $n \geqslant h(m)$.

To prove the implication (i) $\to$ (ii) it is sufficient to put $f(n) = (f'(n) \dot{-} f''(n)) + (f''(n) \dot{-} f'(n))$.

The implication (ii) $\to$ (i) is obvious.

To prove the implication (ii) $\to$ (iv) it is sufficient to remark that the recursive functions $f(n)$ (of (ii)), $g(n) = n$ and $h(m) = m$ satisfy (iv). In fact, we have

$$\left|\frac{f(n)}{g(n)+1} - |a|\right| < \frac{1}{n+1} \leqslant \frac{1}{m+1} \quad \text{for} \quad n \geqslant m.$$

In order to prove the implication (iii) $\to$ (ii) suppose that a real number a satisfies condition (iii). Then we have

$$\frac{f(2n+1)}{g(2n+1)+1} - \frac{1}{2(n+1)} < |a| < \frac{f(2n+1)}{g(2n+1)+1} + \frac{1}{2(n+1)}. \tag{1}$$

We shall prove that the function

$$f'(n) = \left[(n+1) \cdot \frac{f(2n+1)}{g(2n+1)+1} + \frac{1}{2}\right]$$

fulfils condition (ii). In fact, the function f' is recursive since $f, g \in \mathbf{R}$ and the function $[n/m]$ is a recursive one. Furthermore, from (1) (by adding $1/(n+1)$) we obtain

$$\frac{f(2n+1)}{g(2n+1)+1} + \frac{1}{2(n+1)} < |a| + \frac{1}{n+1}, \tag{2}$$

whence

$$(3)\quad \left[(n+1)\cdot\frac{f(2n+1)}{g(2n+1)+1}+\frac{1}{2}\right]$$
$$\leqslant (n+1)\cdot\frac{f(2n+1)}{g(2n+1)+1}+\frac{1}{2}<|a|(n+1)+1.$$

On the other hand, from (1) we obtain

$$(4)\qquad |a|-\frac{1}{n+1}<\frac{f(2n+1)}{g(2n+1)+1}-\frac{1}{2(n+1)}.$$

Hence

$$(5)\quad |a|(n+1)-1<\frac{f(2n+1)(n+1)}{g(2n+1)+1}-\frac{1}{2}\leqslant\left[\frac{f(2n+1)(n+1)}{g(2n+1)+1}+\frac{1}{2}\right].$$

Inequalities (3) and (5) give

$$(6)\qquad |a|(n+1)-1<f'(n)<|a|(n+1)+1,$$

whence

$$\left|\frac{f'(n)}{n+1}-|a|\right|<\frac{1}{n+1},$$

which completes the proof.

The implication (iv) $\to$ (v) is evident. To prove (v) $\to$ (iii) let us set $f'(m)=f\big(h\big(n_0(m+1)-1\big)\big)$, and $g'(m)=g\big(h\big(n_0(m+1)-1\big)\big)$. The functions f', g' satisfy (iii).

1.3. *The set $\mathscr{R}$ is a field of numbers.*

I. *The set $\mathscr{R}$ is closed under the operation of addition.*

Suppose that $a, b\in\mathscr{R}$. Then, by definition, there exist functions f', f'', g', $g''\in\boldsymbol{R}$ such that

$$(1)\quad \left|a-\frac{f''(n)-f'(n)}{n+1}\right|<\frac{1}{n+1}\quad\text{and}\quad\left|b-\frac{g''(n)-g'(n)}{n+1}\right|<\frac{1}{n+1}$$

for all $n\in\mathscr{N}$. Hence

$$(2)\quad \left|a+b-\frac{f''(n)-f'(n)+g''(n)-g'(n)}{n+1}\right|<\frac{2}{n+1}\quad\text{for all } n\in\mathscr{N}.$$

Thus, on account of 1.2 (v) $a+b\in\mathscr{R}$.

II. *The set $\mathscr{R}$ is closed under the operation of multiplication.*

Suppose that $a, b\in\mathscr{R}$. It follows from 1.2 (ii) that there exist two functions $f, g\in\boldsymbol{R}$ such that

$$(3)\quad \left||a|-\frac{f(n)}{n+1}\right|<\frac{1}{n+1}\quad\text{and}\quad\left||b|-\frac{g(n)}{n+1}\right|<\frac{1}{n+1}\quad\text{for all } n\in\mathscr{N}.$$

Consequently

$$(4)\qquad |a|-\frac{1}{n+1}<\frac{f(n)}{n+1}<|a|+\frac{1}{n+1},$$

$$(5)\qquad |b|-\frac{1}{n+1}<\frac{g(n)}{n+1}<|b|+\frac{1}{n+1}\quad \text{for all } n\in\mathcal{N}.$$

From these two inequalities we obtain

$$(6)\qquad |a\cdot b|-\frac{|a|+|b|}{n+1}-\frac{1}{(n+1)^2}<\frac{f(n)g(n)}{(n+1)^2}<|a\cdot b|+\frac{|a|+|b|}{n+1}+\frac{1}{(n+1)^2}.$$

Hence

$$(7)\qquad \left||a\cdot b|-\frac{f(n)\cdot g(n)}{(n+1)^2}\right|<\frac{|a|+|b|}{n+1}+\frac{1}{(n+1)^2}\leqslant\frac{[|a|+|b|]+2}{n+1}.$$

The number $[|a|+|b|]$ is constant. Hence, according to 1.2 (v), $a\cdot b\in\mathcal{R}$.

III. *The set $\mathcal{R}$ is closed under the operation of subtraction.*

This follows immediately from I, II and the remark that every rational number, in particular -1, is recursive since it fulfils 1.2 (iii). In fact, if $|a|=p/(q+1)$, where $p,q\in\mathcal{N}$, it is sufficient to put $f(n)=p$ and $g(n)=q$.

IV. *The set $\mathcal{R}$ is closed under the operation of division.*

Let the number $a\neq 0$ fulfil condition (iii). Then there is a number $n_0\in\mathcal{N}$ such that

$$(8)\qquad \frac{1}{n_0+1}<\frac{|a|}{2}.$$

Consequently

$$(9)\qquad |a|-\frac{f(n+n_0)}{n+n_0+1}<\frac{1}{n+n_0+1}<\frac{1}{n_0+1}<\frac{|a|}{2}\quad \text{for all } n\in\mathcal{N}.$$

Putting $g(n)=f(n+n_0)-1$ and $h(n)=n+n_0+1$ we infer from (9) that for all $n\in\mathcal{N}$

$$(10)\qquad \frac{g(n)+1}{h(n)}>\frac{|a|}{2}.$$

Hence, by (3),

$$(11)\qquad \left||a|-\frac{g(n)+1}{h(n)}\right|<\frac{1}{n+1}.$$

(10) and (11) imply

$$(12)\qquad \left|\frac{1}{|a|}-\frac{1}{\frac{g(n)+1}{h(n)}}\right|=\frac{\left|\frac{g(n)+1}{h(n)}-|a|\right|}{|a|\cdot\frac{g(n)+1}{h(n)}}<\frac{\frac{1}{n+1}}{|a|\cdot\frac{g(n)+1}{h(n)}}<\frac{\frac{1}{n+1}}{|a|\cdot\frac{|a|}{2}}.$$

Hence, on account of (8),

$$\left|\frac{1}{|a|}-\frac{h(n)}{g(n)+1}\right|<\frac{2}{(n+1)|a|^2}<\frac{2}{n+1}\,\frac{(n_0+1)^2}{4}<\frac{(n_0+1)^2}{n+1}.$$

Since $f, g \in \mathbf{R}$ the number $1/a$ fulfils condition 1.2 (v), which completes the proof.

We shall prove in the sequel (2.56) that the field of recursive numbers is algebraically closed in the field of real numbers.

The class $\mathscr{R}$ is identical with the class $\mathscr{R}_1$ of recursive numbers defined by Specker [49]. We assume besides the definition of the class $\mathscr{R}$ the following two definitions, analogous to the definitions of Specker:

1.4. *By $\mathscr{R}_2^k$ we shall mean the class of all real numbers a which fulfil the following condition: there exists a function $f \in \mathbf{R}$ such that*

$$|a| = \sum_{n=0}^{\infty} \frac{f(n)}{k^n} \quad \text{and} \quad 0 \leqslant f(n) < k \text{ for } n > 0,\ n \in \mathscr{N}.$$

1.5. *By $\mathscr{R}_3$ we shall mean the class of all real numbers a for which the following condition is satisfied: there exists a function $f \in \mathbf{R}$ such that*

$$|a| < \frac{p}{q+1} \quad \text{if and only if } f(p, q) = 0 \text{ for every } p, q \in \mathscr{N}.$$

The class $\mathscr{R}_2^{10}$ is identical with the class $\mathscr{R}_2$ of Specker (the class of numbers which have recursive decimal developments). The class $\mathscr{R}_3$ is identical with the class $\mathscr{R}_3$ of Specker (the class of numbers which determine the recursive section in the set of rationals).

In the sequel we shall consider for each real number $a \geqslant 0$ the functions $h_a(n) = [n \cdot a]$, $h'_a(n) = [k_0^n \cdot |a|]$. If these functions are recursive, a is a recursive number. Indeed, we have the following theorems:

1.6. *For every real number a and any $k \in \mathscr{N}$, $k > 1$, the following conditions are equivalent:*

(i) $a \in \mathscr{R}_2^k$,

(ii) $[k^n \cdot |a|] \in \mathbf{R}$.

I. If a is a rational number, then $|a| = \dfrac{p}{q+1}$ and $\left[k^n \cdot \dfrac{p}{q+1}\right]$ is obviously a recursive function. On the other hand, $|a| = \sum_{n=0}^{\infty} f(n)/k^n$, where f can be recursively defined as follows:

$$f(0) = \left[\frac{p}{q+1}\right], \quad g(0) = 0,$$

$$f(n+1) = \left[\left(\frac{p}{q+1} - \left[\frac{p}{q+1}\right]\right) \cdot k^{n+1} - g(n) \cdot k\right],$$

$$g(n+1) = g(n) \cdot k + f(n+1).$$

II. If a is not a rational number and

$$(1) \qquad |a| = \sum_{n=0}^{\infty} \frac{f(n)}{k^n}, \quad \text{where} \quad 0 \leqslant f(n) < k,\ f(n) \in \mathbf{R},$$

then for each $n \in \mathcal{N}$

$$(2) \qquad \left[\sum_{m=1}^{\infty} \frac{f(n+m)}{k^m}\right] = 0$$

because the function f represents the proper development of the number $|a|$ with the basis k.

From (1) we obtain

$$(3) \qquad |a| \cdot k^n = f(0) \cdot k^n + f(1) \cdot k^{n-1} + \ldots + f(n) + \sum_{m=1}^{\infty} \frac{f(n+m)}{k^m}.$$

From (2) and (3) it follows that for each n

$$(4) \qquad [|a| \cdot k^n] = f(0) \cdot k^n + f(1) \cdot k^{n-1} + \ldots + f(n),$$

whence

$$(5) \qquad [|a| \cdot k^{n+1}] = f(0) \cdot k^{n+1} f(1) \cdot k^n + \ldots + f(n+1).$$

In consequence, on account of (4),

$$(6) \qquad [|a| \cdot k^{n+1}] = [|a| \cdot k^n] \cdot k + f(n+1).$$

Now suppose that $[|a| \cdot k^n] \in \boldsymbol{R}$; thus also $f \in \mathbf{R}$ because according to (6) we can put

$$f(0) = a, \quad f(n+1) = [|a| \cdot k^{n+1}] \dot{-} [|a| \cdot k^n] \cdot k.$$

Conversely, if $f \in \mathbf{R}$, then on account of (4) the function $[|a| \cdot k^n]$ is recursive, for

$$[|a| \cdot k^n] = \sum_{i \leqslant n} f(i) \cdot k^{n \dot{-} i}$$

and the class $\mathbf{R}$ is closed under the operation of limited summation.

1.7. *For every real number a, the conditions*

(i) $a \in \mathcal{R}_3$,

(ii) $[|a| \cdot n] \in \mathbf{R}$

are equivalent.

(ii) $\rightarrow$ (i). The condition $|a| < p/(q+1)$, where $p, q \in \mathcal{N}$, is equivalent to $|a| \cdot (q+1) < p$. This holds if and only if $[|a| \cdot (q+1)] < p$. The last inequality is equivalent to the condition $\big([|a| \cdot (q+1)] + 1\big) \dot{-} p = 0$. Hence, if

$$[|a| \cdot n] \in \mathbf{R}, \quad \text{then} \quad a \in \mathcal{R}_3.$$

(i) $\to$ (ii). Suppose that there is an $f \in \mathbf{R}$ such that

(1) $$|a| < \frac{p}{q+1} \quad \text{if and only if} \quad f(p, q) = 0.$$

Since for all real non-negative numbers a and b there is

$$[a]+[b] \leqslant [a+b] \leqslant [a]+[b]+1,$$

we have

(2) $$[|a|\cdot n]+[|a|] \leqslant [|a|\cdot(n+1)] \leqslant [|a|\cdot n]+[|a|]+1.$$

Now, we distinguish two cases:

I. If $|a| \geqslant \dfrac{[n\cdot|a|]+[|a|]+1}{n+1}$, then $|a|\cdot(n+1) \geqslant [|a|\cdot n]+[|a|]+1$,

and

(3) $$[|a|\cdot(n+1)] \geqslant [|a|\cdot n]+[|a|]+1.$$

By (2) and (3) we infer that

(4) $$[|a|\cdot(n+1)] = [|a|\cdot n]+[|a|]+1.$$

II. If $|a| < \dfrac{[|a|\cdot n]+[|a|]+1}{n+1}$, then

(5) $$[|a|\cdot(n+1)] \leqslant |a|\cdot(n+1) < [|a|\cdot n]+[|a|]+1.$$

Hence

(6) $$[|a|\cdot(n+1)] \leqslant [|a|\cdot n]+[|a|].$$

From (2) and (6) it follows that

(7) $$[|a|\cdot(n+1)] = [|a|\cdot n]+[|a|].$$

From (4) and (7) we obtain

$$[|a|\cdot(n+1)] = [|a|\cdot n]+[|a|]+\begin{cases} 0 & \text{when } |a| < \dfrac{[|a|\cdot n]+[|a|]+1}{n+1}, \\ 1 & \text{in the other case.} \end{cases}$$

Hence

$$[|a|\cdot(n+1)] = [|a|\cdot n]+[|a|]+\operatorname{sgn}\big(f([|a|\cdot n]+[|a|]+1, n)\big).$$

Putting

$F(k, l) = k+[|a|]+\operatorname{sgn}\big(f(k+[|a|]+1, l)\big)$ we can define recursively the function $[|a|\cdot n]$:

$$[|a|\cdot 0] = 0, \quad [|a|\cdot(n+1)] = F([|a|\cdot n], n)$$

also $[|a|\cdot n] \in \mathbf{R}$ provided that $f \in \mathbf{R}$.

1.8. *For every real number a the following two conditions are equivalent:*

(i) $[|a|\cdot n]\in\mathbf{R}$,

(ii) $[|a|\cdot n!]\in\mathbf{R}$.

For any real number b we have $b=[b]+r$, where $0\leqslant r<1$. Hence, for $m>0$, $[m\cdot b]=m\cdot[b]+[m\cdot r]$, where $0\leqslant[m\cdot r]<n$. Thus $\left[\frac{mb}{m}\right]=[b]+\left[\frac{mr}{m}\right]$, where $\frac{[m\cdot r]}{m}<1$. Hence

$$[b]=\left[\frac{[b\cdot m]}{m}\right]\quad\text{for}\quad n>0. \tag{1}$$

From (1), putting $b=|a|\cdot n$, $m=(n\dot{-}1)!$, we obtain

$$[|a|\cdot n]=\left[\frac{[|a|\cdot n!]}{(n\dot{-}1)}\right]. \tag{2}$$

Hence the condition $[|a|\cdot n!]\in\mathbf{R}$ implies $[|a|\cdot n]\in\mathbf{R}$.

The converse implication is evident.

1.9. *By $\mathscr{R}_3^*$ we shall mean the class of all real numbers a which fulfil the following condition: there exists a function $f\in\mathbf{R}$ such that $|a|=\sum_{n=0}^{\infty}f(n)/n!$ where $0\leqslant f(n)<n$ for $n>0$.*

Clearly, the class $\mathscr{R}_3^*$ is the class of real numbers which have the Cantorian recursive developments.

1.10. *For every real number a, the conditions*

(i) $a\in\mathscr{R}_3^*$,

(ii) $[|a|\cdot n!]\in\mathbf{R}$

are equivalent.

We shall distinguish two cases.

I. If a is a rational number, then $|a|=\frac{p}{q+1}$, where $p,q\in\mathscr{N}$, $\left[\frac{p}{q+1}\cdot n!\right]\in\mathbf{R}$ and $|a|=\sum_{n=0}^{\infty}\frac{f(n)}{n!}$ where f can be defined recursively as follows:

$$f(0)=\left[\frac{p}{q+1}\right],\qquad g(0)=0,$$

$$f(n+1)=\left|\left(\frac{p}{q+1}-\left[\frac{p}{q+1}\right]\right)\cdot(n+1)!-g(n)\cdot(n+1)\right|,$$

$$g(n+1)=g(n)(n+1)+f(n+1).$$

II. If a is an irrational number and

$$(1) \qquad |a| = \sum_{n=0}^{\infty} \frac{f(n)}{n!} \quad \text{where} \quad 0 \leqslant f(n) < n \text{ for } n > 0,\ n \in \mathcal{N},$$

then for each n

$$(2) \qquad \left[\sum_{m=1}^{\infty} \frac{f(n+m)\,n!}{(n+m)!}\right] = 0.$$

It follows from (1) that

$$(3) \quad |a|\cdot n! = f(0)n! + f(1)\cdot\frac{n!}{1!} + f(2)\frac{n!}{2!} + \ldots + f(n) + \sum_{m=1}^{\infty} \frac{f(n+m)\cdot n!}{(n+m)!}.$$

Consequently, by (2),

$$(4) \qquad [|a|\cdot n!] = f(0)\cdot n! + f(1)\frac{n!}{1!} + \ldots + f(n) = \sum_{i\leqslant n} f(i)\frac{n!}{i!}.$$

Hence

$$(5) \qquad [|a|\cdot(n+1)!] = (n+1)\cdot[|a|\cdot n!] + f(n+1).$$

Thus, if $f \in \mathbf{R}$, then by (4) the function $[|a|\cdot n!]$ may be defined by means of the operation of limited summation which does not lead out of the class $\mathbf{R}$.

Conversely, if $[|a|\cdot n!] \in \mathbf{R}$, then on account of (5) the function $f(n)$ may be defined recursively as follows:

$$f(0) = [|a|],$$

$$f(n+1) = [|a|\cdot(n+1)!] - (n+1)\cdot[|a|\cdot n!].$$

Hence $f \in \mathbf{R}$.

1.11. $\mathcal{R}_3 = \mathcal{R}_3^*$.

This theorem is an immediate consequence of theorems 1.7, 1.8, 1.9, 1.10.

1.12. *$\mathcal{R}_3 \subset \mathcal{R}_2^k \subset \mathcal{R}$ for every $k \in \mathcal{N}$ and $k > 1$.*

Indeed, the condition $[|a|\cdot n] \in \mathbf{R}$ implies that $[|a|\cdot k^n] \in \mathbf{R}$. Hence, by 1.6 and 1.7 we have

$$\mathcal{R}_3 \subset \mathcal{R}_2^k.$$

Obviously for each $a \geqslant 0$

$$[a] - 1 < a < [a] + 1.$$

Hence

$$\frac{[|a|\cdot k^n]}{k^n} - \frac{1}{k^n} < |a| < \frac{[|a|\cdot k^n]}{k^n} + \frac{1}{k^n}.$$

Consequently

$$\left| |a| - \frac{[|a|\cdot k^n]}{k^n} \right| < \frac{1}{k^n} \leqslant \frac{1}{n+1}.$$

Since the condition $[|a|\cdot k^n]\in\mathbf{R}$ implies that the functions

$$f(n) = [|a|\cdot k^n] \quad \text{and} \quad g(n) = k^n \dot{-} 1$$

satisfy the condition of 1.2 (iii), we infer that $\mathscr{R}_2^k \subset \mathscr{R}$.

Specker has proved [33] that,

$$\mathscr{R}_3 \neq \mathscr{R}_2^k \neq \mathscr{R}.$$

It follows from the example of Specker [49] and from theorem 1.6 that

1.13. *There exists a positive real number $a\in\mathscr{R}$ such that*

$$[10^n\cdot a]\notin\mathbf{R}.$$

On the other hand, we have the theorem:

1.14. *Let $a\in\mathscr{R}$ and $k\in\mathscr{N}$, $k>1$. Then there exists a weakly increasing function $f\in\mathbf{R}$ such that $f(n)>0$, $f(n)\to\infty$ and $[k^{f(n)}\cdot|a|]\in\mathbf{R}$.*

For the proof of this theorem see Mostowski [37].

1.15. *The conditions*

(i) $a\in\mathscr{R}$, $k\in\mathscr{N}$ *and* $k>1$,

(ii) $|a| = \sum_{n=0}^{\infty} \frac{f(n)}{k^n}$,

(iii) $g\in\mathbf{R}$,

(iv) $g(n)\geqslant n+1$,

(v) $f(g(n)) < k-1$

imply that $f\in\mathbf{R}$ and $a\in\mathscr{R}_2^k$.

From 1.2 (ii) it follows that there exists a function $h\in\mathbf{R}$ such that

$$\left| \frac{h(n)}{n+1} - |a| \right| < \frac{1}{n+1} \quad \text{for all } n\in\mathscr{N}. \tag{1}$$

Hence

$$\left| \frac{h(k^n-1)}{k^n} - |a| \right| < \frac{1}{k^n}. \tag{2}$$

Thus

$$h(k^n-1)-1 < k^n\cdot|a| < h(k^n-1)+1, \tag{3}$$

$$[k^n\cdot|a|] \leqslant h(k^n-1) \leqslant [k^n\cdot|a|]+1. \tag{4}$$

Thus by substitution we obtain

$$\left[\frac{[k^{g(n)}\cdot|a|]}{k^{g(n)-n}}\right] \leqslant \left[\frac{h(k^{g(n)}-1)}{k^{g(n)-n}}\right] \leqslant \left[\frac{[k^{g(n)}\cdot|a|]+1}{k^{g(n)-n}}\right]. \tag{5}$$

Now it is possible to show that if $f(g(n)) < k-1$, then

$$\left[\frac{[k^{g(n)}\cdot|a|]+1}{k^{g(n)-n}}\right] \leqslant [k^n\cdot|a|]. \tag{6}$$

Indeed, from our assumption it follows that

$$k^n\cdot|a| = f(0)k^n+f(1)k^{n-1}+\ldots+f(n)+\sum_{m=1}^{\infty}\frac{f(m+n)}{k^m} \tag{7}$$

for every $n\in\mathcal{N}$.

Since $g(n)\geqslant n+1$ and $f(g(n)) < k-1$, there exists such an m_0 that $g(n) = n+m_0$, $m_0\geqslant 1$ and $f(m_0+n) < k-1$. Thus

$$\sum_{m=1}^{\infty}\frac{f(m+n)}{k^m} < 1. \tag{8}$$

By (7) and (8)

$$[k^n\cdot|a|] = f(0)k^n+f(1)k^{n-1}+\ldots+f(n) \quad \text{for every } n\in\mathcal{N}. \tag{9}$$

Hence, on account of $g(n)\geqslant n+1$ we infer that

$$\begin{aligned}[k^{g(n)}\cdot|a|] = {} & k^{g(n)-n}\big(f(0)k^n+f(1)k^{n-1}+\ldots+f(n)\big)+\\ & +k^{g(n)}\left(\frac{f(n+1)}{k^{n+1}}+\ldots+\frac{f(g(n))}{k^{g(n)}}\right).\end{aligned} \tag{10}$$

By (9) and (10)

$$[k^{g(n)}\cdot|a|] = k^{g(n)-n}\cdot[k^n\cdot|a|]+k^{g(n)}\left(\frac{f(n+1)}{k^{n+1}}+\ldots+\frac{f(g(n))}{k^{g(n)}}\right). \tag{11}$$

Thus

$$\begin{aligned}\frac{[k^{g(n)}\cdot|a|]+1}{k^{g(n)-n}} &= [k^n\cdot|a|]+k^n\left(\frac{f(n+1)}{k^{n+1}}+\ldots+\frac{f(g(n))}{k^{g(n)}}\right)+\frac{1}{k^{g(n)-n}} \\ &= [k^n\cdot|a|]+k^n\left(\frac{f(n+1)}{k^{n+1}}+\ldots+\frac{f(g(n))+1}{k^{g(n)}}\right) \\ &= [k^n\cdot|a|]+\frac{f(n+1)}{k}+\frac{f(n+2)}{k^2}+\ldots+\frac{f(g(n))+1}{k^{g(n)-n}} < [k^n\cdot|a|]+1.\end{aligned} \tag{12}$$

Formula (12) involves (6).

On the other hand, it follows from formula (1) of 1.8 that

$$[k^n \cdot |a|] = \left[\frac{[k^n \cdot |a| \cdot m]}{m}\right]. \tag{13}$$

Hence

$$[k^n \cdot |a|] = \left[\frac{[k^{g(n)} \cdot |a|]}{k^{g(n)-n}}\right]. \tag{14}$$

From (5), (6) and (14) it follows that

$$[k^n \cdot |a|] = \left[\frac{h(k^{g(n)}-1)}{k^{g(n)-n}}\right].$$

Consequently, on account of $g, h \in \mathbf{R}$, we infer that

$$[k^n \cdot |a|] \in \mathbf{R}.$$

This implies that $a \in \mathscr{R}_2^k$ and by 1.6 that $f \in \mathbf{R}$.

1.16. *Every algebraic number is recursive.*

Consider a polynomial

$$w(x) = A_0 x^k + A_1 x^{k-1} + \ldots + A_k.$$

Suppose that A_i are integers, $w(a) = 0$, and k is the degree of polynomial $w(x)$. This implies that the derivative $w'(a) \neq 0$. Suppose that $w'(a) > 0$ and $a \geqslant 0$. We shall prove that $a \in \mathscr{R}$.

The polynomial $w(x)$ will be represented as a difference of two polynomials $P(x)$, $Q(x)$ with non-negative coefficients:

$$w(x) = P(x) \dot{-} Q(x),$$

$$P(x) = P_0 x^k + P_1 x^{k-1} + \ldots + P_k, \qquad Q(x) = Q_0 x^k + Q_1 x^{k-1} + \ldots + Q_k,$$

where P_i, Q_i are non-negative integers.

Let $F(m, n)$ be a function defined as follows:

$$F(m, n) = \left(P_0 m^k + P_1 m^{k-1}(n+1) + \ldots + P_k (n+1)^k\right) \dot{-} \\ \dot{-} \left(Q_0 m^k + Q_1 m^{k-1}(n+1) + \ldots + Q_k (n+1)^k\right).$$

It is evident that for every rational number $m/(n+1) > 0$ we have

(i) $w\left(\dfrac{m}{n+1}\right) > 0$ if and only if $F(m, n) > 0$.

This follows immediately from the equation

$$F(m, n) = w\left(\frac{m}{n+1}\right)(n+1)^k.$$

If $w'(a) > 0$, there exists an open interval $\Delta = \left(\frac{s}{t+1}, \frac{p}{q+1}\right)$, where $p, q, s, t \in \mathcal{N}$, such that $w'(x) > 0$ for $\frac{s}{t+1} < x < \frac{p}{q+1}$ and $\frac{s}{t+1} < a < \frac{p}{q+1}$.

The function $w(x)$ increases in the interval Δ. Hence

(ii) $a < \frac{m}{n+1}$ if and only if $w\left(\frac{m}{n+1}\right) > 0$ for $\frac{s}{t+1} < \frac{m}{n+1} < \frac{p}{q+1}$.

From (i) and (ii) we infer that $a = |a| < \frac{m}{n+1}$ if and only if $F(m, n) > 0$ and $\frac{s}{t+1} < \frac{m}{n+1} < \frac{p}{q+1}$ or $\frac{m}{n+1} \geqslant \frac{p}{q+1}$.

Hence the relation $R \overset{\mathrm{Df}}{=} (\hat{m}, \hat{n})\left(|a| < \frac{m}{n+1}\right)$ is recursive, and $a \in \mathcal{R}_3$.

In the case of $w'(a) < 0$ the proof is similar.

2. Computable numbers. Let $\mathbf{C}$ always denote the class of all computable functions.

1.17. *A real number a is called computable if there exists two functions $f', f'' \in \mathbf{C}$ such that*

$$\left| a - \frac{f'(n) - f''(n)}{n+1} \right| < \frac{1}{n+1} \quad \textit{for every } n \in \mathcal{N}.$$

The class of all computable numbers will be denoted by $\mathscr{C}$.

By a similar method to that used for recursive numbers we can point out that

1.18. *For every real number a the following conditions are equivalent:*

(i) $a \in \mathscr{C}$;

(ii) *there exists a function $f \in \mathbf{C}$ such that*

$$\left| \frac{f(n)}{n+1} - |a| \right| < \frac{1}{n+1} \quad \textit{for every } n \in \mathcal{N};$$

(iii) *there exist two functions $f, g \in \mathbf{C}$ such that*

$$\left| \frac{f(n)}{g(n)+1} - |a| \right| < \frac{1}{n+1} \quad \textit{for every } n \in \mathcal{N};$$

(iv) *there exist three functions* $f, g, h \in \mathbf{C}$ *such that*

$$\left|\frac{f(n)}{g(n)+1} - |a|\right| < \frac{1}{m+1} \quad \textit{for all non-negative integers } n \geqslant h(m);$$

(v) *there exist functions* $f, g, h \in \mathbf{C}$ *and* $n_0 \in \mathcal{N}$ *such that*

$$\left|\frac{f(n)}{g(n)+1} - |a|\right| < \frac{n_0}{m+1} \quad \textit{for all integers } n, m \textit{ such that } n \geqslant h(m).$$

1.19. *The set* $\mathscr{C}$ *is a field of numbers.*

The proof is analogous to that of 1.3.

Now we shall introduce the notions of the classes $\mathscr{C}_2^k$ and $\mathscr{C}_3$ in a way similar to that of the classes $\mathscr{R}_2^k$ and $\mathscr{R}_3$.

1.20. *By* $\mathscr{C}_2^k$ *we shall mean the class of all real numbers* a *which fulfil the following condition: there exists a function* $f \in \mathbf{C}$ *such that* $|a| = \sum_{n=0}^{\infty} f(n)/k^n$ *and* $0 \leqslant f(n) < k$ *for* $n > 0$.

1.21. *By* $\mathscr{C}_3$ *we shall mean the class of all real numbers* a *for which the following condition is satisfied: there exists a function* $f \in \mathbf{C}$ *such that*

$$|a| < \frac{p}{q+1} \quad \textit{if and only if} \quad f(p, q) = 0 \quad \textit{for every } p, q \in \mathcal{N}.$$

1.22. *For every real number* a *and any* $k \in \mathcal{N}$ *and* $k > 1$ *the following conditions are equivalent:*

(i) $a \in \mathscr{C}_2^k$,

(ii) $[k^n \cdot |a|] \in \mathbf{C}$.

The proof is as in the case of 1.6.

1.23. *For every real number* a, *the conditions*

(i) $a \in \mathscr{C}_3$,

(ii) $[|a| \cdot n] \in \mathbf{C}$

are equivalent.

The proof is analogous to that of 1.7.

1.24. *Let* a *be a real number; the conditions*

(i) $a \in \mathscr{C}_3$,

(ii) $[|a| \cdot n!] \in \mathbf{C}$

are equivalent.

The proof is as in the case of 1.8.

1.25. $\mathscr{C}_3 \subset \mathscr{C}_2^k \subset \mathscr{C}$.

The proof is analogous to that of 1.12.

1.26. $\mathscr{C} = \mathscr{C}_2^k = \mathscr{C}_3$.

In order to prove this theorem it is sufficient to show that $\mathscr{C} \subset \mathscr{C}_3$.

Every rational number a belongs to $\mathscr{C}_3$. This follows from the fact that the relation $R \overset{\mathrm{Df}}{=} (\hat{p}, \hat{q})\left(\frac{m}{n+1} < \frac{p}{q+1}\right)$ is computable.

Let a be an arbitrary non-rational number. Then for every rational number $p/(q+1)$ (where $p, q \in \mathscr{N}$) there exists such an $n \in \mathscr{N}$ that

$$(1) \qquad \frac{p}{q+1} < |a| - \frac{2}{n+1} \quad \text{or} \quad |a| + \frac{2}{n+1} < \frac{p}{q+1}.$$

Suppose that $a \in \mathscr{C}$. Then there exists such an $f \in \mathbf{C}$ that for all $n \in \mathscr{N}$

$$(2) \qquad |a| - \frac{1}{n+1} < \frac{f(n)}{n+1} < |a| + \frac{1}{n+1}.$$

Hence

$$(3) \qquad |a| - \frac{2}{n+1} < \frac{f(n)-1}{n+1} \quad \text{and} \quad \frac{f(n)+1}{n+1} < |a| + \frac{2}{n+1}.$$

It follows from (1) and (2) that for all $p, q \in \mathscr{N}$ there exists such an $n \in \mathscr{N}$ that

$$(4) \qquad \frac{p}{q+1} < \frac{f(n)-1}{n+1} \quad \text{or} \quad \frac{f(n)+1}{n+1} < \frac{p}{q+1}.$$

Thus the following function g is computable:

$$(5) \qquad g(p, q) = (\mu n)\left(\frac{p}{q+1} < \frac{f(n)-1}{n+1} \text{ or } \frac{f(n)+1}{n+1} < \frac{p}{q+1}\right).$$

Indeed, by (4) the operation of minimum is effective.

By (5)

$$(6) \qquad \frac{p}{q+1} < \frac{f(g(p, q))-1}{g(p, q)+1} \quad \text{or} \quad \frac{f(g(p, q))+1}{g(p, q)+1} < \frac{p}{q+1}.$$

Now by (2)

$$(7) \qquad \frac{f(g(p, q))-1}{g(p, q)+1} < |a| < \frac{f(g(p, q))+1}{g(p, q)+1}.$$

Formulae (6) and (7) yield

$$|a| < \frac{p}{q+1} \quad \text{if and only if} \quad \frac{f(g(p, q))+1}{g(p, q)+1} < \frac{p}{q+1}.$$

Hence the relation

$$R \overset{\mathrm{Df}}{=} (\hat{p}, \hat{q})\left(|a| < \frac{p}{q+1}\right)$$

is computable and $a \in \mathscr{C}_3$. This completes the proof.

It is evident that $\mathscr{R} \subset \mathscr{C}$. We shall prove that

1.27. $\mathscr{R} \neq \mathscr{C}$.

Let $h(n)$ be a computable function for which the following conditions are satisfied:

(i) $h \notin \mathbf{R}$,

(ii) h assumes only two values: 0 and 1.

We then put

$$j(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ h\left(\frac{n}{2}\right) & \text{if } n \text{ is even,} \end{cases}$$

and

$$a = \sum_{n=0}^{\infty} \frac{f(n)}{2^n}.$$

The number a is computable but not recursive. To prove that $a \notin \mathscr{R}$, suppose that $a \in \mathscr{R}$. Then from 1.15, putting $g(n) = 2n+1$, we infer that $j \in \mathbf{R}$. Indeed, $j(g(n)) = 0 < 2-1$ and $g(n) \geqslant n+1$.

On the other hand, since $h(n) = j(2n)$, h is recursive. Hence $a \notin \mathscr{R}$.

1.28. *There exist two functions $f, f' \in \mathbf{R}$, for which the following conditions are satisfied:*

(i) *putting* $a_n = \dfrac{f(n)}{f'(n)}$ *we have* $a_0 < a_1 < a_2 < \ldots$,

(ii) $\lim\limits_{n=\infty} a_n = a$,

(iii) $a \notin \mathscr{C}$.

There exists a recursive function $h(m, n)$ such that the function

$$j(m) = (\mu x)\big(h(m, x) = 0\big) \tag{1}$$

is not computable. Now we put:

$$g(m, n) = 1 \dot{-} \prod_{i \leqslant n} h(m, i), \tag{2}$$

$$\sigma(m) = \lim_{n \to \infty} g(m, n); \tag{3}$$

the functions g, j, σ have the following properties:

$$\text{if} \quad g(m, n) = 1 \quad \text{then} \quad g(m, n+1) = 1, \tag{4}$$

$$\sigma(m) = \begin{cases} 0 & \text{if } g(m, n) = 0 \ \big(h(m, n) \neq 0\big) \text{ for all } n \in \mathscr{N}, \\ 1 & \text{if } g(m, n) = 1 \ \big(h(m, n) = 0\big) \text{ for some } n \in \mathscr{N}. \end{cases} \tag{5}$$

From (1) and (5) it follows that

(6) $$j(m) = (\mu x)\big(g(m, x) = 1\big) = (\mu x)\big(g(m, x) = \sigma(m)\big),$$

(7) for all $m \in \mathcal{N}$ there exists such an x that $g(m, x) = \sigma(m)$.

Consequently, j may be defined by means of the operation of effective minimum, by the use of g and σ. Hence j is computable provided that σ is computable. Thus σ is not a computable function. We put $\bar{f}(n) = \sum_{k=0}^{n} \big(g(k, n)\cdot 2^{n-k}\big)$. Obviously $\bar{f}(n) \in \mathbf{R}$. Let

$$a = \sum_{m=0}^{\infty} \frac{\sigma(m)}{2^n} \quad \text{and} \quad a_n = \sum_{k=0}^{n} \frac{g(k, n)}{2^k} = \frac{1}{2^n}\bar{f}(n).$$

On account of 1.22 and 1.20 we infer that $a \in \mathcal{C}$. Now we shall demonstrate that $a = \lim_{n=\infty} a_n$. Indeed, it follows from (3) and (4) that $g(m, x) = \sigma(m)$ implies $g(m, x+1) = \sigma(m)$. Hence, by (6),

$$g(m, n) = \sigma(m) \quad \text{for} \quad n \geqslant j(m).$$

Put $x(m)$ = the maximal value of the function $j(k)$ for $k \leqslant m$. Thus $g(k, n) = \sigma(k)$ for each $m \in \mathcal{N}$, $k \in \mathcal{N}$ and $k \leqslant m$, and $n \in \mathcal{N}$ and $n > x(m)$. Hence $(a - a_n) \leqslant 1/2^m$ and $a = \lim_{n\to\infty} a_n$.

(3) and (4) imply that $g(k, n)$ is a non-decreasing function of n and $\sigma(k)$ = the maximal value of the function $g(k, n)$ for all $n \in \mathcal{N}$. Hence

$$a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant \ldots \leqslant a.$$

Let $f^*(n) = \sum_{k=0}^{n} \frac{\bar{f}(k)\cdot 2^{n-k}}{n+1}$ for $n \in \mathcal{N}$, and $a_n^* = \frac{f^*(n)}{2^n}$. It is easy to see that $a_n^* = \frac{a_0 + a_1 + \ldots + a_n}{n+1}$ and

(i) $\lim_{n\to\infty} a_n^* = a$,

(ii) there exists an $n_0 \in \mathcal{N}$ such that $a_i^* < a_{i+1}^*$ for every $i \in \mathcal{N}$ and $i \geqslant n_0$.

Put $f(n) = f^*(n + n_0)$, $f'(n) = 2^{n_0+n}$; clearly, $f, f' \in \mathbf{R}$.

The functions $f(n), f'(n)$ fulfil the conditions of the theorem. This completes the proof (1).

(1) It remained some problems not solved in Mazur lectures concerning recursive numbers and sequences. This provocated two papers of A. Mostowski [37], [39] and indirectly of R. S. Lehman [33] and A. H. Lachlan [30].

II. RECURSIVE AND COMPUTABLE SEQUENCES

Analogously to the three definitions of recursive (computable) numbers we introduce in this section three notions of *recursive* (*computable*) *sequences* (Definitions 2.1, 2.19, 2.20). These notions are non-identical even in the case of computability. In the sequel we shall be concerned with recursive functions defined by means of a condition similar to condition 1.1 (p. 5) for recursive numbers. The set of computable sequences is closed under the four arithmetical operations: $+$, $-$, $\cdot$, $:$ (2.7, 2.10, 2.11, 2.18). However, the set of recursive sequences is not closed under the operation of division (2.13).

In this section the notion of *recursive* (*computable*) *convergence* is also considered (2.31). The four arithmetical operations performed on the sequences recursively (computably) convergent give the sequences recursively (computably) convergent to the limits obtained by the four arithmetical operations (2.35). The limit of a sequence recursively (computably) convergent is recursive (computable) provided that the sequence is recursive (computable) (2.38). There exists a recursive sequence which converges to 0 in a non-computable manner (2.36). But for each computable sequence convergent to a computable number there exists a computable subsequence computably convergent (2.43). There exists also a recursive monotonic convergent sequence which does not converge to a computable number (1.28). Hence the theorem of Bolzano-Weierstrass is not satisfied in the domain of computable sequences and numbers. There are also many other anomalies in the set of recursive (computable) sequences. For example the functions "signum" and the "integral part" lead out of the class of computable sequences (2.28, 2.30). These anomalies are studied in this section. Some of them are open problems.

2.1. *A sequence* $\{a_k\}$ *of real numbers is called recursive* (*computable*) *if there exist two functions* $f', f'' \in \mathbf{R}$ $(\mathbf{C})$ *such that*

$$\left| a_k - \frac{f'(k,n) - f(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \textit{for every } k, n \in \mathcal{N}.$$

The class of all recursive (computable) sequences will be denoted by $\boldsymbol{R}$ $(\boldsymbol{C})$.

2.2. *For every sequence* $\{a_k\}$ *of non-negative real numbers the following conditions are equivalent:*

(i) $\{a_k\} \in \mathbf{R}\ (C)$,

(ii) *there exists a function* $f \in \mathbf{R}\ (\mathbf{C})$ *such that* $\left|\frac{f(k,n)}{n+1} - a_k\right| < \frac{1}{n+1}$ *for every* $k, n \in \mathcal{N}$,

(iii) *there exist two functions* $f, g \in \mathbf{R}\ (\mathbf{C})$ *such that* $\left|\frac{f(k,n)}{g(k,n)+1} - a_k\right| < \frac{1}{n+1}$ *for every* $k, n \in \mathcal{N}$,

(iv) *there exist three functions* $f(k,n)$, $g(k,n)$, $h(m) \in \mathbf{R}\ (\mathbf{C})$ *such that* $\left|\frac{f(k,n)}{g(k,m)+1} - a_k\right| < \frac{1}{m+1}$ *for every* $k \in \mathcal{N}$ *and* $n > h(m)$,

(v) *there exist three functions* $f(k,n)$, $g(k,n)$, $h(m) \in \mathbf{R}\ (\mathbf{C})$ *and* $n_0 \in \mathcal{N}$ *such that* $\left|\frac{f(k,n)}{g(k,n)+1} - a_k\right| < \frac{n_0}{n+1}$ *for every* $k \in \mathcal{N}$ *and* $n > h(m)$.

The proof of this theorem is analogous to that of 1.2.

2.3. *For every sequence* $\{a_k\}$ *of real numbers the following conditions are equivalent:*

(i) $\{a_k\} \in \mathbf{R}\ (C)$,

(ii) *there exist functions* f', f'', g, $j \in \mathbf{R}\ (\mathbf{C})$ *such that* $\left|\frac{f'(k,n)-f(k,n)}{g(k,n)+1} - a_k\right| < \frac{j(k)}{n+1}$ *for every* $k, n \in \mathcal{N}$,

(iii) *there exist functions* f', f'', g, h, $j \in \mathbf{R}\ (\mathbf{C})$ *such that* $\left|\frac{f'(k,n)-f(k,n)}{g(k,n)+1} - a_k\right| < \frac{j(k)}{m+1}$ *for every* $k \in \mathcal{N}$ *and* $n \geqslant h(m)$.

The proofs of the implications (i) $\to$ (iii) and (iii) $\to$ (ii) in the case of $j(k) = 1$ are analogous to those of (ii) $\to$ (iv) and (iv) $\to$ (iii) in 1.2.

We prove the implication (ii) $\to$ (i) first for $j(k) = 1$ for any $k \in \mathcal{N}$. Suppose that $\{a_k\}$ fulfils condition (ii). Then we have

$$(1)\qquad \frac{f'(k,2n+1)-f(k,2n+1)}{g(2n+1)+1} - \frac{1}{2(n+1)} < a_k < \frac{f'(k,2n+1)-f(k,2n+1)}{g(2n+1)+1} + \frac{1}{2(n+1)}.$$

We shall prove that

$$\left[(n+1)\cdot\frac{f'(k,2n+1)-f''(k,2n+1)}{g(2n+1)+1} + \frac{1}{2}\right] = f_1(k,n) - f_2(k,n).$$

where $f_1, f_2 \in \mathbf{R}$ (C). Obviously

$$\left[\frac{(n+1)f'(k, 2n+1)-f(k, 2n+1)}{g(2n+1)+1}+\frac{1}{2}\right]=\left[\frac{f_1'(k, n)-f_2'(k, n)}{g'(k, n)}\right],$$

where $f_1', f_2', g' \in \mathbf{R}$ (C) and $g'(k, n) \neq 0$. It is easy to see that

$$\left[\frac{f_1'(k, n)-f_2'(k, n)}{g'(k, n)}\right]=\left[\frac{f_1'(k, n)\dot{-}f_2'(k, n)}{g'(k, n)}\right]-\left(\left[\frac{f_2'(k, n)-f_1'(k, n)}{g'(k, n)}\right]+\delta(k, n)\right)$$

where

$$\delta(k, n)=\begin{cases} 0 & \text{if } \dfrac{f_2'(k, n)\dot{-}f_1'(k, n)}{g'(k, n)}=\left[\dfrac{f_2'(k, n)-f_1'(k, n)}{g'(k, n)}\right], \\ 1 & \text{in the opposite case.}\end{cases}$$

It remains to show that $\delta(k, n) \in \mathbf{R}$ (C). Indeed, let

$$h(k, n)=\left\{\left(f_2'(k, n)\dot{-}f_1'(k, n)\right)\dot{-}g'(k, n)\cdot\left[\frac{f_2'(k, n)\dot{-}f_1'(k, n)}{g'(k, n)}\right]\right\}+$$
$$+\left\{g'(k, n)\cdot\left[\frac{f_2'(k, n)\dot{-}f_1'(k, n)}{g'(k, n)}\right]\dot{-}\left(f_2'(k, n)\dot{-}f_1'(k, n)\right)\right\}.$$

Hence $\delta(k, n) \in \mathbf{R}$ (C). That is

$$f_1(k, n)=\left[\frac{f_1'(k, n)\dot{-}f_2'(k, n)}{g'(k, n)}\right]$$

and

$$f_2(k, n)=\left[\frac{f_2'(k, n)\dot{-}f_1'(k, n)}{g'(k, n)}\right]+\delta(k, n).$$

From (1) by adding $1/(n+1)$ we obtain

$$\frac{f'(k, 2n+1)-f''(k, 2n+1)}{g(2n+1)+1}+\frac{1}{2(n+1)}<a_k+\frac{1}{n+1}. \tag{2}$$

Hence

$$\varphi'(k, n)-\varphi''(k, n)=\left[(n+1)\frac{\left(f'(k, 2n+1)-f''(k, 2n+1)\right)}{g(2n+1)+1}+\frac{1}{2}\right] \tag{3}$$
$$\leqslant (n+1)\cdot\frac{\left(f'(k, 2n+1)-f''(k, 2n+1)\right)}{g(2n+1)+1}+\frac{1}{2}<a_k(n+1)+1.$$

On the other hand, from (1) we obtain

$$a_k-\frac{1}{n+1}<\frac{\left(f'(k, 2n+1)-f''(k, 2n+1)\right)}{g(2n+1)+1}+\frac{1}{2(n+1)}. \tag{4}$$

Hence

$$(5)\quad a_k(n+1)-1 < \frac{(f'(k,2n+1)-f''(k,2n+1))(n+1)}{g(2n+1)+1}-\frac{1}{2}$$

$$\leqslant\left[\frac{(n+1)f'(k,2n+1)-f''(k,2n+1)}{g(2n+1)+1}+\frac{1}{2}\right]=\varphi'(k,n)-\varphi''(k,n).$$

Inequalities (3) and (5) give

$$(6)\qquad a_k(n+1)-1<\varphi'(k,n)-\varphi''(k,n)<a_k(n+1)+1.$$

Hence

$$\left|\frac{\varphi'(k,n)-\varphi''(k,n)}{n+1}-a_k\right|<\frac{1}{n+1}\quad\text{for every } k,n\in\mathcal{N},$$

which completes the proof. For $j(k)$ non-constant we prove conditions (ii) and (iii) in the same way as formula (v) from (iii) in 1.2.

2.4. *If a sequence* $\{a_k\}\in\boldsymbol{R}$ (C), *then* $a_k\in\mathscr{R}$ $(\mathscr{C})$ *for every* $k\in\mathcal{N}$.

This follows immediately from 1.1. and 2.1.

2.5. *A function* $f(k)$ *where* k *runs over* $\mathcal{N}$ *is recursive* (*computable*) *if and only if the sequence* $\{a_k\}=\{f(k)\}$ *is recursive* (*computable*).

Necessity. Suppose that $f(k)\in\mathbf{R}$ (C). Let $\{a_k\}=\{f(k)\}$. Then the function $g(k,n)=f(k)\cdot(n+1)\in\mathbf{R}$ (C) and we have

$$\left|\frac{g(k,n)}{n+1}-|a_k|\right|<\frac{1}{n+1}\quad\text{for every } k,n\in\mathcal{N}.$$

Hence the sequence $\{a_k\}=\{f(k)\}\in\boldsymbol{R}$ (C).

Sufficiency. Suppose that $\{a_k\}=\{f(k)\}\in\boldsymbol{R}$ (C); then there exists a function $h(k,n)\in\mathbf{R}$ (C) such that

$$\left|f(k)-\frac{h(k,n)}{n+1}\right|<\frac{1}{n+1}\quad\text{for every } k,n\in\mathcal{N}.$$

Hence

$$h(k,n)-1<f(k)(n+1)<h(k,n)+1.$$

Consequently,

$$f(k)\cdot(n+1)=h(k,n).$$

This implies that $f(k)\in\mathbf{R}$ (C).

2.6. *Let* $a_k=c$ *for every* $k\in\mathcal{N}$, *where* $c\in\mathscr{R}$ $(\mathscr{C})$. *Then the sequence* $\{a_k\}\in\boldsymbol{R}$ (C).

This follows immediately from 2.1.

2.7. *If sequences* $\{a_k\}$ *and* $\{b_k\}$ *are recursive* (*computable*), *then the sequence* $\{a_k+b_k\}$ *is recursive* (*computable*).

By 2.1 and the hypothesis there exist four functions f', f'', g', $g'' \in \mathbf{R}$ (C) such that

$$(1) \qquad \left| a_k - \frac{f'(k,n) - f''(k,n)}{n+1} \right| < \frac{1}{n+1}$$

and

$$(2) \qquad \left| b_k - \frac{g'(k,n) - g''(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for every } k, n \in \mathcal{N}.$$

Hence

$$(3) \qquad \left| a_k + b_k - \frac{f'(k,n) - f''(k,n) + g'(k,n) - g''(k,n)}{n+1} \right| < \frac{2}{n+1}.$$

Hence, by 2.3 (ii), the sequence $\{a_k + b_k\} \in \boldsymbol{R}$ (C).

2.8. *If a sequence* $\{a_k\} \in \boldsymbol{R}$ (C), *then the sequence* $\{|a_k|\} \in \boldsymbol{R}$ (C).

Suppose that $\{a_k\} \in \boldsymbol{R}$ (C). Then there exist two functions $f', f'' \in \mathbf{R}$ (C) such that

$$(1) \qquad \left| a_k - \frac{f'(k,n) - f(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for every } k, n \in \mathcal{N}.$$

Let $g(k,n) = \big(f'(k,n) \dot{-} f''(k,n)\big) + \big(f''(k,n) \dot{-} f'(k,n)\big)$. By (1)

$$\left| |a_k| - \frac{g'(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for every } k, n \in \mathcal{N}.$$

Then, by 2.2 (ii), the sequence $\{|a_k|\} \in \boldsymbol{R}$ (C).

The inverse of this theorem is not true.

2.9. *There exists a sequence* $\{a_k\}$ *such that* $\{a_k\} \notin \boldsymbol{R}$ (C) *but* $\{|a_k|\} \in \boldsymbol{R}$ (C).

Let $f(k)$ be a function where k runs over $\mathcal{N}$ and the values are 0 and 2. Let $f \notin \mathbf{R}$ (C). Let $a_k = f(k) - 1$ for every $k \in \mathcal{N}$. Then a_k assumes only two values: -1 and $+1$. Thus $|a_k| = 1$ for every $k \in \mathcal{N}$ and, by 2.6, $\{|a_k|\} \in \boldsymbol{R}$ (C). The sequence $\{a_k\} \notin \boldsymbol{R}$ (C), according to 2.5.

2.10 *If sequences* $\{a_k\}$ *and* $\{b_k\}$ *are recursive* (*computable*), *then the sequence* $\{a_k \cdot b_k\}$ *is recursive* (*computable*).

By 2.1 and the hypothesis there exist four functions $f', f'', g', g'' \in \mathbf{R}$ (C) such that

$$(1) \qquad \left| a_k - \frac{f'(k,n) - f''(k,n)}{n+1} \right| < \frac{1}{n+1},$$

$$(2) \qquad \left| b_k - \frac{g'(k,n) - g''(k,n)}{n+1} \right| < \frac{1}{n+1}.$$

It follows from (1) and (2) that

$$|a_k| < f'(k, 0)+f''(k, 0)+2, \tag{3}$$

$$|b_k| < g'(k, 0)+g''(k, 0)+2. \tag{4}$$

Putting $h(k) = f'(k, 0)+f''(k, 0)+2$, $j(k) = g'(k, 0)+g''(k, 0)+2$ we have

$$|a_k| < h(k), \tag{3}$$

$$|b_k| < j(k). \tag{4}$$

Using (1) and (2), we obtain

$$\left|\frac{f'(k, n)-f''(k, n)}{n+1}\right| < h(k), \tag{5}$$

$$\left|\frac{g'(k, n)-g''(k, n)}{n+1}\right| < j(k). \tag{6}$$

Let

$$W(k, n) = \frac{f'(k, n)-f''(k, n)}{n+1}, \qquad W'(k, n) = \frac{g'(k, n)-g''(k, n)}{n+1}.$$

Now we shall estimate the product $a_k \cdot b_k$. We have

$$\begin{aligned} (7)\quad & a_k \cdot b_k - W(k, n)\cdot W'(k, n) \\ &= a_k \cdot b_k - a_k W'(k, n) + a_k \cdot W'(k, n) - W(k, n)\cdot W'(k, n) \\ &\qquad = a_k\big(b_k - W'(k, n)\big) + W'(k, n)\big(a_k - W(k, n)\big). \end{aligned}$$

With the aid of (3) and (6) we obtain

$$\begin{aligned} (8)\quad & |a_k \cdot b_k - W(k, n)\cdot W'(k, n)| \\ &\leqslant |a_k \cdot \big(b_k - W'(k, n)\big)| + |W'(k, n)\big(a_k - W(k, n)\big)| \\ &\leqslant h(k)|b_k - W'(k, n)| + j(k)|a_k - W(k, n)|. \end{aligned}$$

Consequently, by (1) and (2),

$$|a_k \cdot b_k - W(k, n)\cdot W'(k, n)| < \big(h(k)+j(k)\big)\cdot \frac{1}{n+1}. \tag{9}$$

Now, let

$$r'(k, n) = f'(k, n)\cdot g'(k, n) + f''(k, n)\cdot g''(k, n),$$

$$r''(k, n) = f'(k, n)\cdot g''(k, n) + f''(k, n)\cdot g'(k, n).$$

Hence formula (9) gives

$$\left| a_k \cdot b_k - \frac{r'(k,n) - r''(k,n)}{(n+1)^2} \right| < \frac{h(k)+j(k)}{n+1} \quad \text{for every } k, n \in \mathcal{N},$$

which completes the proof according to 2.3 (ii).

2.11. *If sequences $\{a_k\}$ and $\{b_k\}$ are recursive (computable), then the sequence $\{a_k - b_k\}$ is recursive (computable).*

This follows immediately from 2.7 and 2.10.

2.12. *There exists a function f with arguments running over $\mathcal{N}$ and the values belonging to $\mathcal{N}$ such that $f \notin \mathbf{R}$ but the sequence $\{a_k\} = \{1/f(k)\} \in \mathbf{R}$.*

It follows from the general theories of recursive and computable functions that there exists a recursive function $g(m, n)$ for which the following conditions are satisfied:

(i) the values of $g(m, n)$ are 0 and 1,

(ii) $g(m, 0) = 1$,

(iii) if $g(m, x) = 0$, then $g(m, x+1) = 0$,

(iv) for every $m \in \mathcal{N}$ there exists an $x \in \mathcal{N}$ such that $g(m, x) = 0$,

(v) the function $f(m) = (\mu x)\big(g(m, x) = 0\big)$ is not recursive.

Indeed, it is known that there exists a recursive function $h(m, n)$ for which the following conditions are satisfied: for every m there exists an x such that $h(m, x) = 0$, and $j(m) = (\mu x)\big(h(m, x) = 0\big) \notin \mathbf{R}$.

Let $h^*(m, n)$ be the function defined as follows:

$$h^*(m, 0) = 1, \qquad h^*(m, x+1) = h(m, x).$$

Clearly, $h^*(m, n) \in \mathbf{R}$. Moreover, for every m there exists an x such that $h(m, x) = 0$. The function $j^*(m) = (\mu x) h^*(m, x) = 0$ is not recursive. This follows from the fact that $j(m) = j^*(m) - 1$ and $j(n) \notin \mathbf{R}$.

Let

$$g^*(m, n) = 1 \dot{-} \prod_{i \leqslant n} h^*(m, i).$$

Obviously, $g^*(m, n)$ assumes only two values, 0 and 1. $g^*(m, 0) = 0$. If $g^*(m, x) = 1$, then $g^*(m, x+1) = 1$.

Now, let

$$g(m, n) = \begin{cases} 0 & \text{if} \quad g^*(m, n) = 1, \\ 1 & \text{if} \quad g^*(m, n) = 0. \end{cases}$$

It is clear that $g(m, n)$ fulfils conditions (i)-(iv). To show (v) it is sufficient to remark that

$$\begin{aligned} f(m) &= (\mu x)\big(g(m, x) = 0\big) = (\mu x)\big(g^*(m, x) = 1\big) \\ &= (\mu x)\big(h^*(m, x) = 0\big) = j^*(m). \end{aligned}$$

Clearly,

$$g(m, x) = \begin{cases} 1 & \text{for} \quad x < f(m), \\ 0 & \text{for} \quad x \geqslant f(m). \end{cases}$$

Moreover, $f(m) > 0$ and $f(m) \in \mathbf{C}$.

We shall demonstrate that the sequence $\{1/f(k)\}$ is recursive. Since the function $\left[\frac{n+1}{g(k,0)+\ldots+g(k,n)}\right]$ is recursive, it is sufficient to show on account of 2.2 (ii)) that

$$(1) \qquad \left|\frac{1}{f(k)} - \frac{n+1}{g(k,0)+\ldots+g(k,n)}\right| < \frac{1}{n+1} \text{ for every } k, n \in \mathcal{N}.$$

Consider two cases:

I. $n < f(k)$. Then $g(k, 0) = g(k, 1) = \ldots = g(k, n) = 1$ and $\frac{1}{n+1} \geqslant \frac{1}{f(k)}$. Hence

$$\left|\frac{1}{f(k)} - \frac{\left[\frac{n+1}{g(k,0)+\ldots+g(k,n)}\right]}{n+1}\right| = \left|\frac{1}{f(k)} - \frac{1}{n+1}\right|$$

$$= \frac{1}{n+1} - \frac{1}{f(k)} < \frac{1}{n+1}.$$

II. $n \geqslant f(k)$. Then $g(k, 0) = \ldots = g(k, f(k)) = 1$ and $g(k, f(k)+1) = \ldots = g(k, n) = 0$. Hence

$$\left|\frac{1}{f(k)} - \frac{\left[\frac{n+1}{g(k,0)+\ldots+g(k,n)}\right]}{n+1}\right| = \left|\frac{1}{f(k)} - \frac{1}{n+1}\left[\frac{n+1}{f(k)}\right]\right|$$

$$= \left|\frac{1}{n+1}\left(\frac{n+1}{f(k)} - \left[\frac{n+1}{f(k)}\right]\right)\right| < \frac{1}{n+1}.$$

In this way we have proved that (1) holds for every $k, n \in \mathcal{N}$. This completes the proof.

2.13. *There exists a sequence $\{a_k\} \in \mathbf{R}$ such that the sequence $\{1/a_k\} \notin \mathbf{R}$.*

Let $f(k)$ be the function described in 2.12. We have established in 2.12 that the sequence $\{a_k\} = \{1/f(k)\}$ is recursive. Since the function $f(k) \notin \mathbf{R}$, it follows from 2.5 that the sequence $\{1/a_k\} = \{f(k)\} \notin \mathbf{R}$.

2.14. *If a sequence $\{a_k\} \in \mathbf{R}$ ($\mathbf{C}$), then there exist three functions f', f'', $g \in \mathbf{R}$ ($\mathbf{C}$) such that*

(i) $g(k, n) \neq 0$,

(ii) $\left| a_k - \dfrac{f'(k, n) - f''(k, n)}{g(k, n)} \right| < \dfrac{1}{n+1}$ *for every* $k, n \in \mathcal{N}$,

(iii) $f'(k, n) - f''(k, n) \neq 0$.

Indeed, since $\{a_k\} \in \boldsymbol{R}$ (C), there exist two functions $g'(k, n)$, $g''(k, n) \in \mathbf{R}$ (C) such that

$$(1) \qquad \left| a_k - \frac{g'(k, n) - g''(k, n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for every } k, n \in \mathcal{N}.$$

Let $\varepsilon(k, n)$ be the function defined as follows:

$$\varepsilon(k, n) = \begin{cases} 0 & \text{in the case of } g'(k, n) - g''(k, n) \neq 0, \\ 1 & \text{in the case of } g'(k, n) - g''(k, n) = 0. \end{cases}$$

Obviously $\varepsilon(k, n) \in \mathbf{R}$ (C). By (1)

$$(2) \qquad \left| a_k - \frac{g'(k, 2n+1) - g''(k, 2n+1)}{2(n+1)} \right| < \frac{1}{2(n+1)}.$$

Hence

$$(3) \qquad \frac{g'(k, 2n+1) - g''(k, 2n+1) + 1}{2(n+1)} - \frac{1}{n+1} < a_k$$
$$< \frac{g'(k, 2n+1) - g''(k, 2n+1) + 1}{2(n+1)} + \frac{1}{n+1}.$$

From (2) and (3) it follows that

$$(4) \qquad \left| a_k - \frac{g'(k, 2n+1) - g''(k, 2n+1) + \varepsilon(k, 2n+1)}{2(n+1)} \right| < \frac{1}{n+1} \text{ for } j, n \in \mathcal{N}.$$

Let $f'(k, n)$, $f''(k, n)$ and $g(k, n)$ be the functions defined by the following formulae:

$$f'(k, n) = g'(k, 2n+1) + \varepsilon(k, 2n+1),$$

$$f''(k, n) = g''(k, 2n+1),$$

$$g(k, n) = 2(n+1).$$

Obviously conditions (i) and (ii) are satisfied.

2.14 is a lemma for the following theorem:

2.15. *Let* $\{a_k\} \in \boldsymbol{R}$ (C) *and* $a_k \neq 0$ *for every* $k \in \mathcal{N}$. *Then* $\{1/a_k\} \in \boldsymbol{R}$ (C) *if and only if there exists a function* $h(k) \in \mathbf{R}$ (C) *such that* $1/|a_k| \leqslant h(k)$ *for every* $k \in \mathcal{N}$.

Necessity. Suppose $\{1/a_k\} \in \boldsymbol{R}\ (\boldsymbol{C})$. Then there exist three functions $f'(k, n), f''(k, n),\ g(k, n) \in \mathbf{R}\ (\mathbf{C})$ such that

$$\left|\frac{1}{a_k} - \frac{f'(k,n)-f''(k,n)}{g(k,n)+1}\right| < \frac{1}{n+1} \quad \text{for every } k, n \in \mathcal{N}.$$

Hence

$$\left|\frac{1}{a_k}\right| - \left|\frac{f(k,n)-f''(k,n)}{g(k,n)+1}\right| < \frac{1}{n+1}.$$

Consequently,

$$\frac{1}{|a_k|} < 1 + |f'(k, n) - f''(k, n)|$$

for every $k, n \in \mathcal{N}$. Putting $n = 0$ we have $1/|a_k| < 1 + |f'(k, 0) - f''(k, 0)|$. Thus the function $h(k)$, defined by the formula

$$h(k) = 1 + |f'(k, 0) - f''(k, 0)|,$$

fulfils the condition of theorem 2.15.

Sufficiency. Suppose that there exists a function $h(k) \in \mathbf{R}\ (\mathbf{C})$ such that $1/|a_k| \leqslant h(k)$ for every $k \in \mathcal{N}$. Since the sequence $\{a_k\} \in \boldsymbol{R}\ (\boldsymbol{C})$, we see by 2.14 that there exist three functions $f'(k, n),\ f''(k, n),\ g(k, n) \in \mathbf{R}\ (\mathbf{C})$ such that $g(k, n) \neq 0$ and

$$\left|a_k - \frac{f'(k,n)-f''(k,n)}{g(k,n)}\right| < \frac{1}{n+1} \quad \text{for every } k, n \in \mathcal{N}. \tag{1}$$

Putting

$$A(k, n) = \frac{f'(k,n)-f''(k,n)}{g(k,n)} \quad \text{for every } k, n \in \mathcal{N} \tag{2}$$

we have by (1)

$$|a_k| - |A(k, n)| < \frac{1}{n+1}. \tag{3}$$

Thus

$$|A(k, n)| > |a_k| - \frac{1}{n+1}. \tag{4}$$

Since $1/|a_k| \leqslant h(k)$ for every $k \in \mathcal{N}$, (4) gives

$$|A(k, n)| > \frac{1}{h(k)} - \frac{1}{n+1}. \tag{5}$$

Hence

$$\begin{aligned} |A(k, n+h(k))| &> \frac{1}{h(k)} - \frac{1}{n+h(k)+1} > \frac{1}{h(k)} - \frac{1}{h(k)+1} \\ &= \frac{1}{h(k)\cdot(h(k)+1)}. \end{aligned} \tag{6}$$

Now we shall estimate $|1/a_k - 1/A(k,n)|$. Since

$$\left|\frac{1}{a_k} - \frac{1}{A(k,n)}\right| = \frac{|a_k - A(k,n)|}{|a_k|\cdot|A(k,n)|},$$

we have, by (1) and (2),

$$\left|\frac{1}{a_k} - \frac{1}{A(k,n)}\right| < \frac{1}{n+1}\cdot\frac{1}{|a_k|\cdot|A(k,n)|}, \tag{7}$$

From (6) and (7) it follows that

$$\left|\frac{1}{a_k} - \frac{1}{A\big(k, n+h(k)\big)}\right| < \frac{1}{n+1}\cdot\frac{1}{|a_k|}\, h(k)\cdot\big(h(k)+1\big). \tag{8}$$

Consequently, on account of $1/|a_k| \leqslant h(k)$ for every $k \in \mathcal{N}$,

$$\left|\frac{1}{a_k} - \frac{1}{A\big(k, n+h(k)\big)}\right| < \frac{1}{(n+1)}\cdot h^2(k)\big(h(k)+1\big) \quad \text{for every } k, n \in \mathcal{N}. \tag{9}$$

Hence, according to 2.3, $\{1/a_k\} \in \boldsymbol{R}\ (C)$.

2.16. *For every sequence $\{a_k\}$ of real numbers, if $\{a_k\} \in \boldsymbol{R}$ and $|a_k| \geqslant \geqslant \varepsilon > 0$, then $\{1/a_k\} \in \boldsymbol{R}$.*

This follows immediately from 2.15.

2.17. *If $\{a_k\} \in C$ and $a_k \neq 0$, then there exists a function $h(k) \in \mathbf{C}$ such that $1/|a_k| \leqslant h(k)$ for every $k \in \mathcal{N}$.*

Since $\{a_k\} \in C$, there exist by 2.14 three functions $f', f'', g \in \mathbf{C}$ such that $g(k,n) \neq 0$,

$$f'(k,n) - f''(k,n) \neq 0, \qquad \left|a_k - \frac{f'(k,n) - f''(k,n)}{g(k,n)}\right| < \frac{1}{n+1}$$

for every $k, n \in \mathcal{N}$. Hence

$$|a_k| > \frac{|f'(k,n) - f''(k,n)|}{g(k,n)} - \frac{1}{n+1} \tag{1}$$

$$= \frac{(n+1)\cdot|f'(k,n) - f''(k,n)| - g(k,n)}{(n+1)\cdot g(k,n)}.$$

For every $k \in \mathcal{N}$ there exists such a positive n_0 that

$$\frac{|f'(k,n) - f''(k,n)|}{g(k,n)} - \frac{1}{n+1} > 0 \quad \text{for every} \quad n > n_0.$$

Consequently

$$(n+1)\cdot|f'(k,n) - f''(k,n)| - g(k,n) > 0 \quad \text{for every} \quad n > n_0. \tag{2}$$

The relation

$$S(k, n) = (n+1)|f'(k, n)-f''(k, n)|-g(k, n) > 0$$

is computable.

Let $h^*(k)$ be a function defined by the formula

$$h^*(k) = (\mu n) S(k, n).$$

Obviously, $h^*(k) \in \mathbf{C}$. It is easy to see that putting

$$h(k) = \frac{\big(h^*(k)+1\big)\cdot|f'\big(k, h^*(k)\big)-f''\big(k, h^*(k)\big)|-g\big(k, h(k)\big)}{\big(h^*(k)+1\big)\cdot g\big(k, h^*(k)\big)}$$

we obtain, by (1), $|a_k| > h(k)$ for every $k \in \mathcal{N}$, which completes the proof, since $h(k) \in \mathbf{C}$.

2.18. *If a sequence $\{a_k\}$ of real numbers is computable and $a_k \neq 0$ for $k \in \mathcal{N}$, then the sequence $\{1/a_k\}$ is also computable.*

This follows immediately from 2.15 and 2.17.

Now we shall introduce the notions of the classes $\boldsymbol{R}_2^q$ ($\boldsymbol{C}_2^q$), $\boldsymbol{R}_3$ ($\boldsymbol{C}_3$) of recursive (computable) sequences by a method similar to that used for the classes $\mathscr{R}_2^k$ ($\mathscr{C}_2^k$) and $\mathscr{R}_3$ ($\mathscr{C}_3$) of recursive (computable) numbers.

2.19. *By $\boldsymbol{R}_2^q$ ($\boldsymbol{C}_2^q$) we shall mean the class of all sequences $\{a_k\}$ of real numbers which fulfil the following condition: there exist two functions $f, g \in \mathbf{R}$ ($\mathbf{C}$) such that*

$$a_k = \sum_{i=0}^{\infty}\left(\frac{f(k, i)}{q^i}\right)-g(k).$$

2.20. *By $\boldsymbol{R}_3$ ($\boldsymbol{C}_3$) we shall mean the class of all sequences $\{a_k\}$ of real numbers which satisfy the following condition: there exists a function $f \in \mathbf{R}$ ($\mathbf{C}$) such that $a_k < \dfrac{p-q}{r+1}$ if and only if $f(k, p, q, r) = 0$ for every $k, p, q, r \in \mathcal{N}$.*

2.21. *If $[a_k \cdot n]$ is a recursive (computable) function, then $\{a_k\} \in \boldsymbol{R}_3$ ($\boldsymbol{C}_3$).*

It is evident that

$$(1) \qquad \left(a_k < \frac{p-q}{r+1}\right) \equiv \big((r+1)\cdot a_k < p-q\big) \equiv \big([a_k\cdot(r+1)] < p-q\big).$$

Let $[a_k \cdot n]$ be recursive (computable) and let us put

$$(2) \qquad g(k, r) = [a_k \cdot r], \qquad g \in \mathbf{R}\ (\mathbf{C}).$$

Hence, from (1) and (2)

$$\left(a_k < \frac{p-q}{r+1}\right) \equiv \big(g(k, r+1) \dot{-} p+q+1 = 0\big),$$

Putting $h(k, p, q, r) = g(k, r+1) \dot{-} p + q + 1$, we obtain $\left(a_k < \frac{p-q}{r+1}\right) \equiv$ $\equiv \big(h(k, p, q, r) = 0\big)$ where $h \in \mathbf{R}$ $(\mathbf{C})$. Thus $\{a_k\} \in \boldsymbol{R}$ $(\boldsymbol{C})$.

2.22. *If* $\{a_k\} \in \boldsymbol{C}_3$ *and* $a_k \geqslant 0$ *for* $k \in \mathcal{N}$, *then* $[a_k \cdot n] \in \mathbf{C}$.

Since $\{a_k\} \in \boldsymbol{C}_3$, there exists a function $f \in \mathbf{C}$ such that

$$\left(a_k < \frac{p-q}{r+1}\right) \equiv \big(f(k, p, q, r) = 0\big). \tag{1}$$

By a method similar to that used for theorem 1.7 we obtain

$$[a_k \cdot (n+1)] = [a_k \cdot n] + [a_k] + \begin{cases} 0 & \text{if} \quad a_k < \dfrac{[a_k \cdot n] + [a_k] + 1}{n+1}, \\ 1 & \text{in other case.} \end{cases} \tag{2}$$

The function $[a_k]$ is computable. Indeed, we can put $[a_k] = (\mu p) f(k, p, 0, 0) = 0$. The function $[a_k \cdot n]$ is also computable, since it can be defined recursively by means of the formulae $[a_k \cdot 0] = 0$ and (2), and the condition $a_k < \frac{[a_k \cdot n] + [a_k] + 1}{n+1}$ can be expressed by means of the function $f \in \mathbf{C}$.

2.23. $\boldsymbol{C}_3 \subset \boldsymbol{C}_2^q$ *for* $q = 2, 3, \ldots$

Suppose that $P \in \mathbf{C}$ and

$$a_k < \frac{p-m}{r+1} \quad \text{if and only if} \quad F(k, p, m, r) = 0. \tag{1}$$

Let $p(k)$ and $m(k)$ be two functions defined as follows:

$$p(k) = (\mu p)\big(F(k, p, 0, 0) = 0\big), \tag{2}$$

$$m(k) = (\mu m)\big(F\big(k, p(k), m, 0\big) = 0\big). \tag{3}$$

Now we put

$$f(k, 0) = p(k), \tag{4}$$

$$f(k, l+1) = (\max n < q)\left(-a_k < \sum_{i=0}^{l} \frac{f(k, i)}{q^i} + \frac{n}{q^{l+1}} - m(k)\right).$$

The relation which appears under the sign of the operation of limited maximum can easily be expressed by means of the function F. Hence, on account of $m(k) \in \mathbf{C}$, it is evident that $f(k, l) \in \mathbf{C}$. Clearly we have $a_k = \sum_{l=0}^{\infty} f(k, l)/q^l - m(k)$, which completes the proof.

2.24. $\boldsymbol{R}_2^q$ $(\boldsymbol{C}_2^q) \subset \boldsymbol{R}$ $(\boldsymbol{C})$ *for* $k = 2, 3, \ldots$

Since $\{a_k\} \in \boldsymbol{R}_2^q$ $(\boldsymbol{C}_2^q)$, there exist two functions f, $g \in \mathbf{R}$ (C) such that

$$a_k = \sum_{i=0}^{\infty} \frac{f(k, i)}{q^i} - g(k).$$

Then we put

$$g'(k, l) = f(k, 0) \cdot q^l + f(k, 1) q^{l-1} + \ldots + f(q, l),$$

$$g''(k, l) = g(k) \cdot q^l.$$

Hence

$$\left| a_k - \frac{g'(k, l) - g''(k, l)}{q^l} \right| < \frac{1}{q^l} \leqslant \frac{1}{l+1}.$$

It follows immediately from 2.23 and 2.24 that

2.25. $\boldsymbol{C}_3 \subset \boldsymbol{C}$.

2.26. *The conditions*

(i) $\{a_k\} \in \boldsymbol{C}$,

(ii) a_k *are non-rational numbers for* $k \in \mathcal{N}$

imply $\{a_k\} \in \boldsymbol{C}_3$.

Since a_k are non-rational numbers, then for each rational number $\frac{p-q}{r+1}$ there exists such an $m \in \mathcal{N}$ that

$$\frac{p-q}{r+1} < a_k - \frac{2}{m+1} \quad \text{or} \quad a_k + \frac{2}{m+1} < \frac{p-q}{r+1}. \tag{1}$$

Since $\{a_k\} \in \boldsymbol{C}$, there exist two functions $f', f'' \in \mathbf{C}$ such that

$$a_k - \frac{1}{m+1} < \frac{f'(k, m) - f''(k, m)}{m+1} < a_k + \frac{1}{m+1}. \tag{2}$$

Hence, for all $k, p, q, r \in \mathcal{N}$ there exists such an $m \in \mathcal{N}$ that

$$\frac{p-q}{r+1} < \frac{\big(f'(k, m) - f''(k, m)\big) - 1}{m+1} \quad \text{or} \quad \frac{\big(f'(k, m) - f''(k, m)\big) + 1}{m+1} < \frac{p-q}{r+1}. \tag{3}$$

Thus the function g defined by the formula

$$g(k, p, q, r) = (\mu m) \quad \big(m \text{ satisfies condition (3)}\big) \tag{4}$$

is computable. By (1), (2), (3), (4) we obtain

$$\left(a_k < \frac{p-q}{r+1}\right) \equiv \left(\frac{f'\big(k, g(k, p, q, r)\big) - f''\big(k, g(k, p, q, r)\big)}{g(k, p, q, r) + 1} < \frac{p-q}{r+1}\right).$$

Hence the relation

$$S = (\hat{k}, \hat{p}, \hat{q}, \hat{r})\left(a_k < \frac{p-q}{r+1}\right)$$

is computable and $\{a_k\} \in \boldsymbol{C}_3$.

2.27. *Let $\{a_k\}$ be a sequence satisfying*

(i) $a_k \geqslant 0$ *for each* $k \in \mathcal{N}$,

(ii) a_k *are non-rational numbers.*

Then $\{a_k\} \in \boldsymbol{C}$ *if and only if* $[a_k \cdot n] \in \mathbf{C}$.

The necessity follows from 2.26 and 2.22, the sufficiency — by 2.21 and 2.25.

2.28. *There exists a sequence $\{a_k\}$ of real numbers for which the following conditions are satisfied:*

(i) $\{a_k\} \in \boldsymbol{R}$,

(ii) $0 \leqslant a_k \leqslant 1$ *for every* $k \in \mathcal{N}$,

(iii) $\{\operatorname{sgn} a_k\} \notin \boldsymbol{C}$.

Let $g(k, n)$ be the function defined in 1.28, by formula (2). This function has the following properties: (1) $g(k, n) \in \mathbf{R}$, (2) $g(k, n)$ assumes only two values, 0 and 1, (3) if $g(k, n) = 1$ then $g(k, n+1) = 1$, (4) the function $\sigma(k) = \lim\limits_{n\to\infty} g(k, n) \notin \mathbf{C}$.

Let $\{e_k\}$ be the sequence defined as follows:

$$a_k = \sum_{n=0}^{\infty} \frac{g(k, n)}{2^{n+1}} = \frac{g(k, 0)}{2} + \frac{g(k, 1)}{2^2} + \ldots$$

It is evident that $0 \leqslant a_k \leqslant 1$ for every $k \in \mathcal{N}$, and, on account of 2.19 and 2.24, $\{a_k\} \in \boldsymbol{R}$.

It is easy to show that

$$\sigma(k) = \operatorname{sgn} a_k.$$

Indeed, if $a_k = 0$, then $\sum\limits_{n=0}^{\infty} g(k, n)/2^{n+1} = 0$. Hence $g(k, n) = 0$ for all $n \in \mathcal{N}$ and in consequence $\sigma(k) = \lim\limits_{n\to\infty} g(k, n) = 0$. If $a_k \neq 0$, then $\sum\limits_{n=0}^{\infty} g(k, n)/2^{n+1} \neq 0$. Hence $g(k, n) = 1$ for $n \geqslant m$, where m is some positive integer. This implies that

$$\sigma(k) = \lim_{n\to\infty} g(k, n) = 1.$$

Since $\sigma(k) \notin \mathbf{C}$, by 2.5 the sequence $\{\sigma(k)\} = \{\operatorname{sgn} a_k\} \notin \mathbf{C}$.

It is easy to see that the sequence $\{a_k\}$ cannot be expressed in the form $\{a_k\} = \left\{\frac{f_1(k)}{f_2(k)}\right\}$ where $f_1, f_2 \in \mathbf{C}$. In fact, suppose that $\{a_k\} = \left\{\frac{f_1(k)}{f_2(k)}\right\}$ where $f_1, f_2 \in \mathbf{C}$. Then $\operatorname{sgn} f_1(k) \in \mathbf{C}$. Hence $\sigma(k) = \operatorname{sgn} a_k = \operatorname{sgn} f_1(k) \in \mathbf{C}$, contrary to the supposition.

2.29. *Is $\{a_k\} \in \boldsymbol{R}$ $(\boldsymbol{C})$ and if there exists a real number $\varepsilon > 0$ such that $|a_k| \geqslant \varepsilon$ for every $k \in \mathcal{N}$, then $\{\operatorname{sgn} a_k\} \in \boldsymbol{R}$ $(\boldsymbol{C})$.*

Since $\{a_k\} \in \boldsymbol{R}$ $(\boldsymbol{C})$, there exist two functions $f', f'' \in \mathbf{R}$ $(\mathbf{C})$ such that

$$(1) \qquad \left| a_k - \frac{f'(k,n) - f''(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for all} \quad k, n \in \mathcal{N}.$$

Let m be a positive integer such that $\frac{1}{m+1} < \varepsilon$. Then either $a_k \geqslant \varepsilon > \frac{1}{m+1}$ or $a_k \leqslant -\varepsilon < -\frac{1}{m+1}$. In the first case we have by (1)

$$(2) \qquad \frac{f'(k,m) - f''(k,m)}{m+1} + \frac{1}{m+1} > a_k > \frac{1}{m+1}.$$

In the other case

$$(3) \qquad \frac{f'(k,m) - f''(k,m)}{m+1} - \frac{1}{m+1} < a_k < -\frac{1}{m+1}.$$

Hence we have

$$(4) \qquad \frac{f'(k,m) - f''(k,m)}{m+1} > 0 \quad \text{in the first case,}$$

$$(5) \qquad \frac{f'(k,m) - f''(k,m)}{m+1} < 0 \quad \text{in the other case.}$$

In consequence

$$(6) \qquad \operatorname{sgn} a_k = \operatorname{sgn}\big(f'(k,m) - f''(k,m)\big)$$

and

$$(7) \qquad \operatorname{sgn} a_k = \operatorname{sgn}\big(f'(k,n) - f''(k,n)\big) \quad \text{for every } n \geqslant m.$$

Let g', g'' be the functions defined by the formulae

$$g'(k,n) = \begin{cases} 1+n & \text{if} \quad f'(k,n) \dot{-} f''(k,n) > 0, \\ 0 & \text{if} \quad f'(k,n) \dot{-} f''(k,n) = 0, \end{cases}$$

$$g''(k,n) = \begin{cases} 0 & \text{if} \quad f'(k,n) \dot{-} f''(k,n) > 0, \\ 1+m & \text{if} \quad f'(k,n) \dot{-} f''(k,n) = 0. \end{cases}$$

Evidently, $g'(k, n)-g''(k, n) = \big(\operatorname{sgn}\big(f'(k, n)-f''(k, n)\big)\big)(n+1)$ for $n \geqslant m$, since $f'(k, n)-f''(k, n) \neq 0$ for $n \geqslant m$. Hence, by (7),

$$(8) \qquad \left|\operatorname{sgn} a_k - \frac{g'(k, n)-g''(k, n)}{n+1}\right| < \frac{1}{n+1} \quad \text{for} \quad k \in \mathcal{N} \text{ and } n \geqslant m.$$

Since $g', g'' \in \mathbf{R}$ (**C**) this completes the proof.

2.30. *There exists a sequence $\{b_k\}$ for which the following conditions are satisfied:*

(i) $0 \leqslant b_k \leqslant 1$,

(ii) $b_k \in \mathcal{R}$,

(iii) $\{[b_k]\} \notin C$.

There exists by 2.28 a sequence $\{a_k\}$ such that $\{a_k\} \in \boldsymbol{R}$, $0 \leqslant a_k \leqslant 1$ for all $k \in \mathcal{N}$ and $\{\operatorname{sgn} a_k\} \notin C$. Consequently

$$(1) \qquad \{1-\operatorname{sgn} a_k\} \notin C.$$

Consider the sequence $\{b_k\} = \{1-\operatorname{sgn} a_k\}$. Obviously $\{b_k\}$ fulfils conditions (i) and (ii) of 2.30. Moreover, it is easy to see that

$$(2) \qquad [b_k] = 1-\operatorname{sgn} a_k.$$

(1) and (2) imply that $\{[b_k]\} \notin C$, which completes the proof.

Now, we shall be concerned with the notion of recursive (computable) convergence, which we are going to define:

2.31. *The sequence $\{a_k\}$ is said to be recursively (computably) convergent to a if there exists a function $h \in \mathbf{R}$ (**C**) such that*

$$|a_k - a| < \frac{1}{m+1} \quad \textit{for} \quad k > h(m).$$

Then we shall write $\{a_k\} \underset{R}{\to} a$ ($\{a_k\} \underset{C}{\to} a$).

2.32. *Given a sequence $\{a_k\}$ we have $\{a_k\} \underset{R}{\to} a$ ($\{a_k\} \underset{C}{\to} a$) if and only if there exist a positive integer n_0 and a function $h \in \mathbf{R}$ (**C**) such that*

$$|a_k - a| < \frac{n_0}{m+1} \quad \textit{for} \quad k > h(m).$$

The neccesity of this condition is obvious.

Sufficiency. Let $h^*(m) = h\big(n_0(m+1)-1\big)$. Clearly, $h^*(m) \in \mathbf{R}$ (**C**) and

$$|a_k - a| < \frac{n_0}{\big(n_0(m+1)-1\big)+1} = \frac{1}{m+1} \quad \text{for} \quad k > h^*(m).$$

Obviously we have

2.33. *The inequality* $k > h(m)$ *in* 2.31 *and in* 2.32 *can be replaced by* $k \geqslant h(m)$.

2.34. *The function* h *of* 2.31 *and of* 2.32 *can be regarded as a strictly increasing one.*

The necessity is obvious.

Sufficiency. Suppose that conditions of 2.32 are satisfied. Let h^* be the function defined as follows:

$$h^*(0) = h(0),$$

$$h^*(n+1) = \max\big(h(n+1),\ h^*(n)+1\big).$$

Since the function $\max(n, m)$ (the largest of the integers n, m) is recursive, the function $h^* \in \mathbf{R}\ (\mathbf{C})$. It is evident that

$$h^*(n+1) \geqslant h^*(n)+1 > h^*(n).$$

Hence, h^* is strictly increasing. Moreover, since $h^*(0) \geqslant h(0)$ and $h^*(n+1) \geqslant h(n+1)$, we have

$$h^*(n) \geqslant h(n) \quad \text{for every } n \in \mathcal{N}.$$

In consequence

$$|a_k - a| < \frac{1}{m+1} \quad \text{for} \quad k > h^*(m),$$

which completes the proof.

2.35. *If* $\{a_k\} \underset{\mathbf{R}(\mathbf{C})}{\longrightarrow} a$ *and* $\{b_k\} \underset{\mathbf{R}(\mathbf{C})}{\longrightarrow} b$, *then*

(i) $\{a_k + b_k\} \underset{\mathbf{R}(\mathbf{C})}{\longrightarrow} a+b$,

(ii) $\{a_k \cdot b_k\} \underset{\mathbf{R}(\mathbf{C})}{\longrightarrow} a \cdot b$,

(iii) $\{a_k - b_k\} \underset{\mathbf{R}(\mathbf{C})}{\longrightarrow} a-b$,

(iv) $\{a_k / b_{k+k_0}\} \underset{\mathbf{R}(\mathbf{C})}{\longrightarrow} a / b$ *for some* $k_0 \in \mathcal{N}$, *provided that* $b \neq 0$,

(v) $\{|a_k|\} \underset{\mathbf{R}(\mathbf{C})}{\longrightarrow} |a|$.

Suppose that $h_1, h_2 \in \mathbf{R}\ (\mathbf{C})$, h_1 is strictly increasing and

$$|a_k - a| < \frac{1}{m+1} \quad \text{for} \quad k \geqslant h_1(m), \tag{1}$$

$$|b_k - b| < \frac{1}{m+1} \quad \text{for} \quad k \geqslant h_2(m). \tag{2}$$

Let $h(m) = \max\big(h_1(m), h_2(m)\big)$. Obviously $h(m) \in \mathbf{R}\ (\mathbf{C})$. (1) and (2) imply that

$$|(a_k + b_k) - (a+b)| < \frac{2}{m+1} \quad \text{for} \quad k \geqslant h(m). \tag{3}$$

(i) follows from (3) and 2.32. By (1)

$$|a_k| < |a|+1 \quad \text{for} \quad k \geqslant h_1(0).$$

Let $M = \max[|a_0|, |a_1|, \ldots, |a_{h_1(0)}|, |a|+1]+1$. Obviously $|a_k| \leqslant M$ for every $k \in \mathcal{N}$. Now we prove in a familiar way:

$$|a_k \cdot b_k - a \cdot b| = |a_k \cdot (b_k - b) + b \cdot (a_k - a)| \leqslant |a_k| \cdot |b_k - b| + |b| \cdot |a_k - a|$$

$$\leqslant M \cdot |b_k - b| + ([|b|]+1) \cdot |a_k - a| < (M + [|b|]+1) \cdot \frac{1}{m+1} \quad \text{for} \quad k \geqslant h(m).$$

Hence, on account of 2.32, (ii) holds. (iii) follows from (i) and (ii).

To prove (iv) suppose that $b \neq 0$. Then for some integer $n_0 > 0$

$$\frac{1}{n_0} < |b|. \tag{4}$$

By (2), $|b_k - b| < \dfrac{1}{2n_0+1}$ for $k > h(2n_0)$. Hence $|b| - |b_k| \leqslant |b - b_k| < \dfrac{1}{2n_0}$. Thus, on account of (4),

$$|b_k| > \frac{1}{2n_0}. \tag{5}$$

(2), (4) and (5) involve that

$$\left|\frac{1}{b_k} - \frac{1}{b}\right| = \frac{|b - b_k|}{|b_k| \cdot |b|} < \frac{1}{m+1} \cdot \frac{1}{|b_k| \cdot |b|} < \frac{2n_0^2}{m+1}$$

for $k > \max\{h_2(2n_0), h_2(m)\}$. Putting $k_0 = h(2n_0)$ we have $b_{k+k_0} \neq 0$ and

$$\frac{1}{b_{k+k_0}} \underset{\mathbf{R}(C)}{\longrightarrow} \frac{1}{b}. \tag{6}$$

Formula (iv) follows from (ii) and (6). In order to prove (v) it is sufficient to remark that

$$\big||a_k| - |a|\big| \leqslant |a_k - a| < \frac{1}{m+1} \quad \text{for} \quad k > h_1(m).$$

2.36. *There exists a function $f \in \mathbf{R}$ for which the following conditions are satisfied:*

(i) $f(k) > 0$,

(ii) $\{1/f(k)\} \to 0$,

(iii) $\{1/f(k)\} \underset{C}{\not\to} 0$.

Let g be the function defined by formula (2) of the proof of theorem 1.28, and let g^* be the function defined as follows:

$$g^*(m, 0) = 0, \quad g^*(m, 1) = g(m, 0), \quad \text{and}$$

$$\text{for} \quad n \geqslant 1, \quad g^*(m, n+1) = \begin{cases} g(m, n) & \text{if} \quad g(m, n-1) = 0, \\ 0 & \text{if} \quad g(m, n-1) = 1. \end{cases}$$

Obviously

(1) $$g^* \in \mathbf{R}$$

and

(2) if m is an arbitrary but fixed non-negative integer, then either $g^*(m, n) = 0$ for all $n \in \mathcal{N}$ or there exists such an n_m that $g^*(m, n_m) = 1$ and $g^*(m, n) = 0$ for $n \neq n_m$.

The function

(3) $$h(m) = \begin{cases} 0 & \text{if} \quad g^*(m, n) = 0 \text{ for all } \quad n \in \mathcal{N}, \\ 1 & \text{if} \quad g^*(m, n) \neq 0 \text{ for some } n \in \mathcal{N} \end{cases}$$

is not computable. In fact,

$$h(m) = \sigma(m),$$

where $\sigma(m)$ is the function defined by formula (5) appearing in the proof of 1.28, and $\sigma(m)$ is not computable, which was proved in 1.28.

Now consider the function

$$f(k) = 1 - g^*(0, k) + \big(1 - g^*(0, k)\big)\big(1 - g^*(1, k)\big) + \ldots + \\ + \big(1 - g^*(0, k)\big)\big(1 - g^*(1, k)\big)\ldots\big(1 - g^*(k, n)\big).$$

Clearly, $f(k) \in \mathbf{R}$ and $f(k) > 0$. For any integer x of the closed interval $\langle 0, k\rangle$ we have

(4) $$f(k) \geqslant x+1 \quad \text{if and only if} \quad g^*(0, k) = \ldots = g^*(x, k) = 0.$$

Let x be an arbitrary integer ($x \geqslant 0$). Since for an established m the function $g^*(m, k)$ assumes the value 1 at most once, there exists such a $k \geqslant x$ that

$$g^*(0, k) = g^*(1, k) = \ldots = g^*(x, k) = 0.$$

Hence $f(k) \geqslant x+1$. This implies that $\{f(k)\} \to \infty$. Consequently $\{1/f(k)\} \to 0$.

To show (iii), suppose that there exists a function $h^*(m) \in \mathbf{C}$ such that

$$\frac{1}{f(k)} < \frac{1}{m+1} \quad \text{for} \quad k > h^*(m).$$

That is

(5) $$f(k) \geqslant m+1 \quad \text{for} \quad k > h^*(m)$$

and $h^*(m) \geqslant m-1$. Indeed, $f(k) \leqslant k+1$. Since $f(k) \geqslant m+1$ for $k > h^*(m)$ we have $k \geqslant m$ for $k > h^*(m)$. Consequently, $h^*(m) \geqslant m-1$.

By (4) and (5)

$$g^*(0, k) = g^*(1, k) = \ldots = g^*(m, k) = 0 \quad \text{for every } k > h^*(m).$$

This implies

(6) $$g^*(x, k) = 0 \quad \text{for} \quad k > h^*(x).$$

Consequently,

(7) $$h(x) = g^*(x, 0) + g^*(x, 1) + \ldots + g^*\big(x, h^*(x)\big).$$

On account of (1) and (7) we have $h(x) \in \mathbf{C}$, contrary to (3).

The next theorem shows that the notion of recursive convergence is narrower than that of computable convergence.

2.37. *There exists a function $f \in \mathbf{R}$ for which the following conditions are satisfied*:

(i) $f(k) > 0$,

(ii) $\left\{\dfrac{1}{f(k)}\right\} \underset{C}{\to} 0$,

(iii) non $\left\{\dfrac{1}{f(k)}\right\} \underset{R}{\to} 0$,

(iv) *if $h(k)$ is a strictly increasing function and $h \in \mathbf{R}$, then* non $\left\{\dfrac{1}{f(h(k))}\right\} \underset{R}{\to} 0$.

Let a be a number of the open interval $(0, 1)$ such that

(1) $a \in \mathscr{R}$ but the function $[10^n \cdot a] \notin \mathbf{R}$.

A number satisfying those conditions exists by 1.13. In consequence, the sequence $\{[10^n \cdot a]\} \notin \mathbf{R}$ (by 2.5). On the other hand, by 1.14 there exists a weakly increasing function $g \in \mathbf{R}$, such that $g(n) > 0$, $g(n) \to \infty$ and $[10^{g(n)} \cdot a] \in \mathbf{R}$.

We shall prove that the function $f(k) = [10^{g(k)} \cdot a]$ fulfils conditions (i)-(iv).

(i) is obvious.

(ii) Let $\bar{h}(n)$ be the function defined by the formula

(2) $$\bar{h}(n) = (\mu k)\left(\frac{1}{[10^{g(k)} \cdot a]} < \frac{1}{n+1}\right).$$

Since the relation $\frac{1}{[10^{g(k)}\cdot a]} < \frac{1}{n+1}$ is computable, $\bar{h}(n)\in\mathbf{C}$. Obviously,

$$\frac{1}{[10^{g(k)}\cdot a]} < \frac{1}{n+1} \quad \text{for} \quad k > \bar{h}(n), \text{ that is } \left\{\frac{1}{[10^{g(k)}\cdot a]}\right\} \xrightarrow[\sigma]{} 0.$$

(iii) follows immediately from (iv).

(iv) To prove (iv) suppose that

$$\left\{\frac{1}{[10^{g(h(k))}\cdot a]}\right\} \xrightarrow[\mathbf{R}]{} 0.$$

This implies that

$$\frac{1}{[10^{g(h(k))}\cdot a]} < \frac{1}{n+1} \quad \text{for} \quad k \geqslant \bar{\bar{h}}(n)\in\mathbf{R}.$$

Hence

$$\frac{1}{[10^{g(h(k))}\cdot a]} < \frac{1}{10^n} \quad \text{for} \quad k \geqslant \bar{\bar{h}}(10^n-1). \tag{3}$$

Let $O(n) = g\big(h\big(\bar{\bar{h}}(10^n-1)\big)\big)$. Since $g, h, \bar{\bar{h}}(10^n-1)$ are recursive functions, $O(n)$ is also one. By (3), $[10^{g(h(k))}\cdot a] > 10^n$ for $k \geqslant \bar{\bar{h}}(10^n-1)$, that is $[10^{O(n)}\cdot a] > 10^n$ for every $n\in\mathcal{N}$. Hence, $10^{O(n)}\cdot a > 10^n$. Since $0 < a < 1$, this implies that $O(n) > n$. In consequence, by 1.8 (1),

$$[10^n\cdot a] = \left[\frac{[10^{O(n)}\cdot a]}{10^{O(n)-n}}\right].$$

Hence the function $[10^n\cdot a]$ is recursive, contrary to (1).

2.38. *If* $\{a_k\}\in\boldsymbol{R}$ $(\boldsymbol{C})$ *and* $\{a_k\} \xrightarrow[\boldsymbol{R}(\boldsymbol{C})]{} a$, *then* $a\in\mathcal{R}$ $(\mathcal{C})$.

Since $\{a_k\}\in\boldsymbol{R}$ $(\boldsymbol{C})$, there exist two recursive (computable) functions $f'(k,n)$ and $f''(k,n)$ such that

$$\left|a_k - \frac{f'(k,n)-f''(k,n)}{n+1}\right| < \frac{1}{n+1} \quad \text{for} \quad k, n\in\mathcal{N}. \tag{1}$$

On the other hand, since $\{a_k\} \xrightarrow[\boldsymbol{R}(\boldsymbol{C})]{} a$, there exists a function $h(n)\in\mathbf{R}$ $(\mathbf{C})$ such that

$$|a_k - a| < \frac{1}{m+1} \quad \text{for} \quad k > h(m). \tag{2}$$

It follows from (1) and (2) that

$$\left|\frac{f'(k,n)-f''(k,n)}{n+1} - a\right| < \frac{1}{n+1} + \frac{1}{m+1} \tag{3}$$

for $k > h(m)$ and $n\in\mathcal{N}$.

Let $f_1(n) = f'(n, n)$ and $f_2(n) = f''(n, n)$. Then, by (3),

$$\left|\frac{f_1(n)-f_2(n)}{n+1} - a\right| < \frac{2}{m+1} \quad \text{for} \quad n > h(m)+m. \tag{4}$$

Hence $a \in \mathscr{R}(\mathscr{C})$.

The problem arises under what conditions a computable sequence which converges to a computable number is convergent in a computable way. An answer to this question will be given in theorem 2.40. We first establish the following lemma:

2.39. *Let $\{a_k\}$ be a sequence which fulfils the following conditions:*

(i) $\{a_k\} \in C$,

(ii) $\{a_k\}$ *is monotonic,*

(iii) $\{a_k\} \to 0$.

Then $\{a_k\} \underset{C}{\to} 0$.

Suppose that $a_k \geqslant a_{k+1}$ for $k \in \mathscr{N}$. Obviously $a_k \geqslant 0$. By (i) there exists a function $f \in \mathbf{C}$ such that

$$\left|a_k - \frac{f(k, n)}{n+1}\right| < \frac{1}{n+1} \quad \text{for every} \quad n \in \mathscr{N}. \tag{1}$$

Consequently

$$|(n+1)a_k - f(k, n)| < 1. \tag{2}$$

Hence $f(k, n) < 1 + (n+1)\cdot a_k$. If n is a fixed non-negative integer, then $\lim\limits_{k=\infty} (n+1)\cdot a_k = 0$.

This implies that there exists such a $k_n \in \mathscr{N}$ that

$$f(k, n) \leqslant 1 \quad \text{for} \quad k > k_n. \tag{3}$$

Let $h(n)$ be the function defined by means of the following formula:

$$h(n) = (\mu k)\big(f(k, n) \leqslant 1\big). \tag{4}$$

Since $f \in \mathbf{C}$ and by (3), the operation of minimum is effective, $h(n) \in \mathbf{C}$. Clearly we have

$$\frac{f(k, n)}{n+1} < \frac{1}{n+1} \quad \text{for} \quad k > h(n). \tag{5}$$

By (1) and (5)

$$a_k < \frac{2}{n+1} \quad \text{for} \quad k > h(n).$$

Hence $\{a_k\} \underset{C}{\to} 0$.

Now consider the case of $a_k \leqslant a_{k+1}$. Evidently $a_k \leqslant 0$. Obviously $\{-a_k\} \in C$, $-a_k \geqslant -a_{k+1}$, $-a_k \geqslant 0$. Thus $\{-a_k\} \underset{C}{\to} 0$. By 2.35 (ii) $\{a_k\} \underset{C}{\to} 0$.

2.40. *Let* $\{a_k\}$ *be a sequence satisfying the following conditions*:

(i) $\{a_k\} \in \boldsymbol{C}$,

(ii) $\{a_k\}$ *is monotonic*,

(iii) $\{a_k\} \to a$,

(iv) $a \in \mathscr{C}$.

Then $\{a_k\} \underset{\boldsymbol{C}}{\to} a$.

Since the sequence $\{a_k\}$ is monotonic, the sequence $\{a_k - a\}$ is also monotonic. By 2.6 and 2.11 the sequence $\{a_k - a\}$ is computable. Moreover, $\{a_k - a\} \to 0$. Hence, on account of 2.39, $\{a_k - a\} \underset{\boldsymbol{C}}{\to} 0$. Consequently, by 2.35 (i), $\{a_k\} \underset{\boldsymbol{C}}{\to} a$.

The following two lemmas will be used in the proof of theorem 2.43.

2.41. *For every sequence* $\{a_k\}$ *satisfying the conditions*

(i) $\{a_k\} \in \boldsymbol{C}$,

(ii) $\{a_k\} \to 0$,

(iii) $a_k \geqslant 0$,

there exists a strictly increasing function $f \in \mathbf{C}$ *such that* $\{a_{f(k)}\} \underset{\boldsymbol{C}}{\to} 0$.

It follows from (i) that there exists a function $g \in \mathbf{C}$ such that

$$\left| a_k - \frac{g(k, n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for} \quad k, n \in \mathscr{N}. \tag{1}$$

Hence $g(k, n) < 1 + a_k(n+1)$. Since $a_k \to 0$, it is easy to see that for an established n there exists a $k_n \in \mathscr{N}$ such that

$$g(k, n) \leqslant 1 \quad \text{for} \quad k > k_n.$$

Let f be the function defined as follows:

$$f(0) = 1,$$

$$f(n+1) = (\mu x)\big((g(x, n+1) \leqslant 1) \wedge (x > f(n))\big).$$

The relation $R(x, n) = \big(g(x, n+1) \leqslant 1\big) \wedge \big(x > f(n)\big)$ is computable. Thus $f \in \mathbf{C}$.

It follows from the definition of f that

$$g\big(f(n), n\big) \leqslant 1 \quad \text{for} \quad n > 0.$$

Hence, on account of (1),

$$a_{f(n)} \cdot (n+1) < 1 + g\big(f(n), n\big) \leqslant 2.$$

Consequently

$$a_{f(n)} < \frac{2}{n+1} \quad \text{for} \quad n > 0.$$

Consequently, by 2.32, $\{a_{f(n)}\} \underset{C}{\rightarrow} 0$.

2.42. *Condition* (iii) *of* 2.41 *can be omitted.*

Indeed, consider a sequence $\{b_k\} \in C$ which converges to 0. Hence, by 2.8, $\{|b_k|\} \in C$. Moreover, $|b_k| \geqslant 0$ and $\{|b_k|\} \rightarrow 0$. By 2.41 there exists an increasing function $f \in \mathbf{C}$ such that

$$\{|b_{f(k)}|\} \underset{C}{\rightarrow} 0,$$

or

$$|b_{f(k)}| < \frac{1}{m+1} \quad \text{for} \quad k > h(m) \in \mathbf{C}.$$

Hence, $\{b_{f(k)}\} \underset{C}{\rightarrow} 0$.

2.43. *For every sequence* $\{a_k\}$ *which fulfils the conditions*

(i) $\{a_k\} \in C$,

(ii) $\{a_k\} \rightarrow a$,

(iii) $a \in \mathscr{C}$,

there exists a strictly increasing function $f \in \mathbf{C}$ *such that* $\{a_{f(k)}\} \underset{C}{\rightarrow} a$.

Suppose that $\{a_k\}$ fulfils the conditions of this theorem. Let $b_k = a$ for $k \in \mathscr{N}$. The sequence $\{a_k - b_k\} \in \mathscr{C}$ by 2.6 and 2.11. Moreover, $\{a_k - b_k\} \rightarrow 0$. Then, by 2.42, there exists a function $f \in \mathbf{C}$ such that $\{a_{f(k)} - b_{f(k)}\} \underset{C}{\rightarrow} 0$. Evidently $\{b_{f(k)}\} \underset{C}{\rightarrow} a$. Hence, by 2.35 (i), $\{a_{f(k)}\} \underset{C}{\rightarrow} a$.

In view of further applications we shall need the following lemmas.

2.44. *If* $\{a_k\} \in \boldsymbol{R}\ (C)$ *and* $\{b_k\} \in \boldsymbol{R}\ (C)$, *where* $b_k \in \mathscr{N}$, *then* $\{c_k\} = \{a_{b_k}\} \in \boldsymbol{R}\ (C)$.

Indeed, since $\{a_k\} \in \boldsymbol{R}\ (C)$, there exist two functions $f', f'' \in \mathbf{R}\ (\mathbf{C})$ such that

$$(1) \qquad \left| a_k - \frac{f'(k, n) - f''(k, n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for} \quad k, n \in \mathscr{N}.$$

By 2.5 the function $h(k) = b_k$ is recursive (computable). Let $g'(k, n) = f'(h(k), n)$ and $g''(k, n) = f''(h(k), n)$. Obviously $g', g'' \in \mathbf{R}\ (\mathbf{C})$ and by (1)

$$\left| c_k - \frac{g'(k, n) - g''(k, n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for} \quad k, n \in \mathscr{N}.$$

2.45. *If* $\{a_k\} \in C$, $a \in \mathscr{C}$ *and* $\{a_k\} \not\rightarrow a$, *then there exist such an* $m_0 \in \mathscr{N}$ *and such an increasing computable function* $h(k)$ *that*

$$|a_{h(k)} - a| > \frac{1}{m_0+1} \quad \textit{for every} \quad k \in \mathscr{N}.$$

Consider the sequence $\{b_k\} = \{a_k - a\}$. Obviously $\{b_k\} \not\to 0$ and, by 2.6 and 2.11, $\{b_k\} \in C$. Hence, there exists such an $m_0 \in \mathcal{N}$ that for every $m \in \mathcal{N}$ there exists an $l \in \mathcal{N}$ and $l > m$ such that

$$(1) \qquad |b_l| > \frac{1}{m_0+1}.$$

On the other hand, there exists a function $f \in \mathbf{C}$ such that

$$(2) \qquad \left| |b_q| - \frac{f(q, n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for every } q, n \in \mathcal{N}.$$

Consequently,

$$(3) \qquad \frac{f(q, n) \dot{-} 1}{n+1} < |b_q| < \frac{f(q, n)+1}{n+1} \quad \text{for} \quad q, n \in \mathcal{N}.$$

Hence, for every $m \in \mathcal{N}$, there exist $k \in \mathcal{N}$, $l > m$ and an $n \in \mathcal{N}$ such that

$$(4) \qquad \frac{f(l, n) \dot{-} 1}{n+1} > \frac{1}{m_0+1}.$$

Evidently (4) implies (1).

Let $l = K(z)$, $n = L(z)$, $z = J(l, n)$ be the pairing functions of Robinson and let

$$(5) \qquad g(m) = K\left\{(\mu z)\left((K(z) > m) \cdot \wedge \left(\frac{f(K(z), L(z) \dot{-} 1}{L(z)+1} > \frac{1}{m_0+1}\right)\right)\right\}.$$

Obviously, $g \in \mathbf{C}$ and g assumes arbitrarily large values.

Let $h(k)$ be the function defined as follows:

$$h(0) = g(0),$$
$$h(k+1) = g(\mu m)\big(g(m) > h(k)\big).$$

Clearly, $h(k)$ is a computable increasing function, and

$$|b_{h(k)}| > \frac{1}{m_0+1} \quad \text{for every} \quad k \in \mathcal{N}.$$

Consequently $|a_{h(k)} - a| > \dfrac{1}{m_0+1}$ for every $k \in \mathcal{N}$.

2.46. *For $\{a_k\} \in C$ if the relation $R \overset{\mathrm{df}}{=} (\hat{k})(a_k = 0)$ is computable, then the relation $S \overset{\mathrm{df}}{=} (\hat{k})(a_k < 0)$ is also computable.*

From the definition 2.1 it follows that there exist two functions $f', f'' \in \mathbf{C}$ such that for all $n, k \in \mathcal{N}$

$$(1) \qquad \left| a_k - \frac{f'(k, n) - f''(k, n)}{n+1} \right| < \frac{1}{n+1}.$$

It is evident that

(2) if $a_k \neq 0$, then there exists such an n that either

$$a_k < -\frac{1}{n+1} \quad \text{or} \quad \frac{1}{n+1} < a_k.$$

From (1) and (2) it follows that the following condition is satisfied:

(3) for every $k \in \mathcal{N}$, there exists an $n \in \mathcal{N}$ such that the condition $a_k \neq 0$ implies $1 \neq |f'(k, n) - f''(k, n)| \neq 0$.

Then the function

(4) $$g(k) = (\mu n)\big(R(k) \vee 1 \neq |f'(k, n) - f''(k, n)| \neq 0\big)$$

is computable. Thus it is evident that

(5) $a_k < 0$ if and only if non $R(k)$ and $f'\big(k, g(k)\big) - f''\big(k, g(k)\big) < -1$.

Indeed, let us notice that according to (1)

(6) $$\left| a_k - \frac{f'\big(k, g(k)\big) - f''\big(k, g(k)\big)}{g(k)+1} \right| < \frac{1}{g(k)+1}.$$

If $a_k < 0$, then $a_k \neq 0$ and non $R(k)$, whence, according to (3), $1 \neq |f'\big(k, g(k)\big) - f''\big(k, g(k)\big)| \neq 0$; but if $f'\big(k, g(k)\big) - f''\big(k, g(k)\big) > 1$, then $a_k > \frac{1}{g(k)+1} > 0$ by (6). Hence it is possible only that $f'\big(k, g(k)\big) - f''\big(k, g(k)\big) < -1$. Consequently, if $f'\big(k, g(k)\big) - f''\big(k, g(k)\big) < -1$, then from (6) it follows that $a_k < -\frac{1}{g(k)+1} < 0$.

From (5) it follows that the relation S is computable. The implication inverse to 2.46 is not true.

2.47. *If $\{a_k\} \in C$ and $a_k \neq 0$ for any $k \in \mathcal{N}$, then the relation $S \overset{\mathrm{df}}{=} (\hat{k})(a_k < 0)$ is computable.*

Indeed, if $a_k \neq 0$ for any $k \in \mathcal{N}$, then the relation $R \overset{\mathrm{df}}{=} (\hat{k})(a_k = 0)$ is always false and hence computable. Thus 2.47 is an immediate consequence of 2.46. The condition $a_k \neq 0$ is necessary (see the example of 2.28). From 2.47 it follows for example that if $W(x)$ is a polynomial with irrational roots and with computable coefficient, then the relation

$$S \overset{\mathrm{df}}{=} (\hat{p}, \hat{q}) W\left(\frac{p}{q+1}\right) < 0$$

is computable.

2.48. *If $\{a_k\} \in C$ and $b < c$, then there exists a computable number d such that $b < d < c$ and $d \neq a_k$ for any $k \in \mathcal{N}$.*

Let m_0 be such a number that $c-b > \frac{12}{m_0+1}$. Suppose that $0 \leqslant b$. Let

$$b < \frac{p_0}{m_0+1} < \frac{p_0+10}{m_0+1} < c. \tag{1}$$

We set

$$\begin{aligned} j(0) &= J\big((\mu x)(p_0 < x < p_0+10 \wedge |f(0, m)-x| > 2), m_0\big), \\ j(n+1) &= (\mu x)\Big(L(x) > L(j(n)) \wedge |f(n+1, L(x))-K(x)| > 2 \wedge \\ &\qquad \frac{K(j(n))-1}{L(j(n))+1} < \frac{K(x)-1}{L(x)+1} < \frac{K(x)+1}{L(x)+1} < \frac{K(j(n))+1}{L(j(n))+1}\Big), \\ g(n) &= K(j(n)), \quad h(n) = L(j(n)), \end{aligned} \tag{2}$$

where J, K, L are the pairing functions and the function $f \in \mathbf{C}$ satisfies the condition

$$\left| |a_k| - \frac{f(k, n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for any } k, n \in \mathcal{N}. \tag{3}$$

Let us set

$$d = \lim_{n\to\infty} \frac{g(n)}{h(n)+1}. \tag{4}$$

From definition (2) it follows that $b < d < c$ and

$$\left| d - \frac{g(n)}{h(n)+1} \right| < \frac{1}{h(n)+1} \leqslant \frac{1}{n+1}, \tag{5}$$

$$\left| \frac{f(k, h(k))}{h(k)+1} - \frac{g(k)}{h(k)+1} \right| > \frac{1}{h(k)+1}. \tag{6}$$

(3), (5) and (6) imply that $a_k \neq d$ for any k. From (5) it follows that $d \in \mathcal{C}$.

2.49. *If* $\{a_k\} \in C\ (\boldsymbol{R})$ *and* $b_k = \sup(a_0, \ldots, a_k)$, *then* $\{b_k\} \in \boldsymbol{R}\ (C)$.

For any two sequences $\{a_k\}$ and $\{d_k\}$ it is true that

$$|\sup(c_0, \ldots, c_k) - \sup(d_0, \ldots, d_k)| \leqslant \sup(|c_0-d_0|, \ldots |c_k-d_k|),$$

whence if $\left| a_k - \frac{f'(n, k)-f(n, k)}{n+1} \right| < \frac{1}{n+1}$, then

$$\left| b_k - \sup\left(\frac{f'(n, 0)-f''(n, 0)}{n+1}, \ldots, \frac{f'(n, k)-f''(n, k)}{n+1}\right) \right| < \frac{1}{n+1}.$$

Thus

$$\left| b_k - \frac{\max\big(f'(n,0)-f''(n,0),\ldots,f'(n,k)-f''(n,k)\big)}{n+1} < \frac{1}{n+1},\right.$$

which completes the proof.

2.50. *If* $\{a_k\}\in C$, $a_k < a$ *for all* $k\in\mathcal{N}$ *and* $a = \lim\limits_{n\to\infty}\{a_k\}$, *then there exists a rational computable increasing sequence* $b_k = \dfrac{g(k)}{h(k)+1}$ *such that* $a = \lim\limits_{k\to\infty}\{b_k\}$.

We set $a'_k = \sup(a_0,\ldots,a_k)$. Hence $\{a'_k\}$ is non-decreasing. We also set $c_k = \dfrac{1}{k+1}(a'_0+\ldots+a'_k)$. The sequence $\{c_k\}$ is strictly increasing, $a = \lim\limits_{k\to\infty}\{c_k\}$ and $\{c_k\}\in C$. Suppose that

$$\left| c_k - \frac{f(k,n)}{n+1}\right| < \frac{1}{n+1}, \tag{1}$$

where $f\in \mathbf{C}$. We set

$$\begin{aligned} h(k) &= (\mu n)\big(f(k,n)+1 < f(k+1,n)-1\big),\\ g(k) &= f\big(k,h(k)\big)+1. \end{aligned} \tag{2}$$

From (1) and (2) it follows that $c_k < \dfrac{g(k)}{h(k)+1} < c_{k+1}$, which completes the proof.

2.51. *If* $\{a_k\}\in C$ *and* $\{b_k\}\in C$, $a = \lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} b_k$ *and* $a_k < a < b_k$ *for all* $k\in\mathcal{N}$, *then* $a\in\mathscr{C}$.

From 2.50 it follows that there exist four functions $h, h', g, g'\in \mathbf{C}$ such that

$$\frac{g(k)}{h(k)+1} < \frac{g(k+1)}{h(k+1)+1} < a < \frac{g'(k+1)}{h'(k+1)+1} < \frac{g'(k)}{h'(k)+1}, \tag{1}$$

$$a = \lim_{k\to\infty}\frac{g(k)}{h(k)+1} = \lim_{k\to\infty}\frac{g'(k)}{h'(k)+1}. \tag{2}$$

From (2) it follows that the following condition is satisfied: for every $n\in\mathcal{N}$ there exists a $k\in\mathcal{N}$ such that

$$\left|\frac{g(k)}{h(k)+1} - \frac{g'(k)}{h'(k)+1}\right| < \frac{1}{n+1}. \tag{3}$$

Hence the following function is computable:

$$(4)\qquad j(n) = (\mu k)\left(\left|\frac{g(k)}{h(k)+1} - \frac{g'(k)}{h'(k)+1}\right| < \frac{1}{n+1}\right).$$

From (1)-(4) it follows that

$$(5)\qquad \left|\frac{g(j(n))}{h(j(n))+1} - a\right| < \frac{1}{n+1}.$$

From (5) by 1.18 it follows that $a \in \mathscr{C}$.

2.52. *If $\{a_k\} \in C$ and there exist infinitely many different numbers of the sequence $\{a_k\}$, then there exists a sequence $\{b_k\} \in C$ such that $\{b_k\}$ enumerates the elements of the sequence $\{a_k\}$ without repetitions.*

Let us suppose that $a_k > 0$, whence there exists a function $f \in C$ such that for any $n, k \in \mathscr{N}$

$$(1)\qquad \left|a_k - \frac{f(n,k)}{n+1}\right| < \frac{1}{n+1}.$$

The following equivalence is true for every k:

(2) for every i such that $i < k$ we have $a_i \neq a_k$ if and only if there exists an n such that for every $i < k$ we have $|f(n,i) - f(n,k)| \geqslant 2$.

Indeed, if for some $i < k$, $a_i = a_k$ then, according to (1), $|f(n,i) - f(n,k)| < 2$ for any $n \in \mathscr{N}$. Conversely, if $a_i \neq a_k$ for $i < k$, then there exists such an $n \in \mathscr{N}$ that, for $i < k$, $|a_i - a_k| > \frac{10}{n+1}$, whence from (1) it follows that $|f(n,i) - f(n,k)| \geqslant 2$.

From (2) it follows that the set K of the numbers k such that $a_i \neq a_k$ for $i < k$ is recursively enumerable, and hence can be enumerated without repetitions by a computable function. Thus there exists a function $g \in C$ such that

(3) for every $i < k$ we have $a_i \neq a_k$ if and only if there exists an n such that $g(n) = K$.

Moreover, the function g enumerates K without repetitions. Now we set $b_k = a_{f(k)}$. From (3) it follows that $b_i \neq b_k$ for $i < k$.

2.53. *If $\{a_n\} \in \boldsymbol{R}$ (C) and $b_n = \max(0, a_n)$, then $\{b_n\} \in \boldsymbol{R}$ (C).*

Suppose that

$$(1)\qquad \left|a_n - \frac{f(n,k) - f'(n,k)}{k+1}\right| < \frac{1}{k+1},$$

where $f, f' \in \mathbf{R}$ (C). We put

(2) $$g(n, k) = f(n, k) \dot{-} f'(n, k).$$

From (1) and (2) it follows that

(3) $$\left| b_n - \frac{g(n, k)}{k+1} \right| < \frac{1}{n+1}.$$

Indeed, if $a_n < 0$, then $b_n = 0$. From (1) it follows for all $k \in \mathcal{N}$ that

(4) $$f(n, k) \leqslant f'(n, k)$$

and

(5) $$g(n, k) = 0 \quad \text{(from (2), (4))}.$$

Formula (3) is also fulfilled.

If $a_n \geqslant 0$, then $b_n = a_n$. From (1) it follows that for all $k \in \mathcal{N}$

(6) $$f(n, k) \geqslant f'(n, k).$$

Hence according to (2)

(7) $$g(n, k) = f(n, k) - f'(n, k).$$

Formula (3) is also fulfilled.

Now we shall prove that the field of recursive (computable) numbers is algebraically closed in the field of real numbers. We start with the following two lemmas:

2.54. *Let* $f(n, p_1, \ldots, p_m) \in \mathbf{R}$ (C) *and* $g(n) \in \mathbf{R}$ (C). *Let*

$$F_n(x_1, \ldots, x_n) = \sum_{p_1 + \ldots + p_m \leqslant g(n)} f(n, p_1, \ldots, p_m) \cdot x_1^{p_1} \ldots x_m^{p_m}$$

and let $a_1, \ldots, a_n \in \mathcal{R}$ (C).

Then $\{F_n(a_1, \ldots, a_n)\} \in \mathbf{R}$ (C).

It follows from the theorem on the mean value that

(1) $$F_n(x_1 \ldots x_m) - F_n(a_1, \ldots, a_m) = \sum_{i=1}^{m} (x_i - a_i) \left(\frac{\partial F_n}{\partial x_i} \right)_{(\bar{x}_1, \ldots, \bar{x}_m)}.$$

It is easy to see that

$$|x_i - a_i| \leqslant 1 \quad \text{implies} \quad \left| \left(\frac{\partial F_n}{\partial x_i} \right)_{(\bar{x}_1, \ldots, \bar{x}_m)} \right| \leqslant h(n),$$

where $h(n) \in \mathbf{R}$ (C). Indeed, suppose that

$$|x_i - a_i| \leqslant 1 \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Hence

$$|x_i| - |a_i| \leqslant 1 \quad \text{and} \quad |x_i| \leqslant |a_i| + 1 \leqslant n_0 \in \mathcal{N}.$$

Obviously, also $|\bar{x}_i| \leqslant n_0$

$$(2)\quad \left|\left(\frac{\partial F_n}{\partial x_i}\right)(\bar{x}_1, \ldots, \bar{x}_m)\right| = \left|\sum_{p_1+\ldots+p_m\leqslant g(n)} f(n, p_1, \ldots, p_m)\cdot p_i \bar{x}_1^{p_1}\ldots\bar{x}_{i-1}^{p_{i-1}}\cdot \bar{x}_i^{p_i-1}\cdot \bar{x}_{i+1}^{p_{i+1}}\ldots\bar{x}_n^{p_n}\right| \leqslant \sum_{p_1+\ldots+p_m\leqslant g(n)} f(n, p_1, \ldots, p_m)\cdot p_i\cdot n_0^*,$$

where n_0^* is a suitable positive integer. This integer can be chosen on account of $|\bar{x}_i| \leqslant n_0$.

Since $f(n, p_1, \ldots, p_m), g(n) \in \mathbf{R}$ (**C**),

$$(3)\qquad h(n) = \sum_{p_1+\ldots+p_m\leqslant g(n)} f(n, p_1, \ldots, p_m)\cdot p_i n_0^* \in \mathbf{R}\ (\mathbf{C}).$$

Consequently, by (1), (2), (3),

$$(4)\qquad |F_n(x_1, \ldots, x_m) - F_n(a_1, \ldots, a_m)| \leqslant h(n)\sum_{i=1}^{m} |x_i - a_i|.$$

Since $a_i \in \mathscr{R}$ ($\mathscr{C}$) $(i = 1, \ldots, m)$, there exist $f_i', f_i'' \in \mathbf{R}$ (**C**) such that

$$(5)\qquad \left|a_i - \frac{f_i'(k) - f_i''(k)}{k+1}\right| < \frac{1}{k+1} \quad \text{for} \quad k \in \mathscr{N}.$$

On account of (4) and (5) we have

$$(6)\qquad \left|F_n\left(\frac{f_1'(k)-f_1''(k)}{k+1}, \ldots, \frac{f_m'(k)-f_m''(k)}{k+1}\right) - F_n(a_1, \ldots, a_m)\right| \leqslant h(n)\frac{m}{k+1}.$$

Let $h^*(n) = m(n+1)\cdot h(n)$. Putting $h^*(n)$ for k in (6) we obtain

$$(7)\qquad \left|F_n\left(\frac{f_1'(h^*(n))-f_1''(h^*(n))}{h^*(n)+1}, \ldots, \frac{f_m'(h^*(n))-f_m''(h^*(n))}{h^*(n)+1}\right) - F_n(a_1, \ldots, a_m)\right| \leqslant h(n)\cdot\frac{m}{h^*(n)+1} < \frac{1}{n+1}.$$

On the other hand,

$$(8)\qquad F_n\left(\frac{f_1'(h^*(n))-f_1''(h^*(n))}{h^*(n)+1}, \ldots, \frac{f_m'(h^*(n))-f_m''(h^*(n))}{h^*(n)+1}\right) = \sum_{p_1+\ldots+p_n\leqslant g(n)} f(n, p_1, \ldots, p_m)\cdot\left(\frac{f_1'(h^*(n))-f_1''(h^*(n))}{h^*(n)+1}\right)^{p_1}\ldots \left(\frac{f_m'(h^*(n))-f_m''(h^*(n))}{h^*(n)+1}\right)^{p_m} = \frac{f_1^*(n)-f_2^*(n)}{g^*(n)}, \quad \text{where} \quad f_1^*, f_2^*, g^* \in \mathbf{R}\ (\mathbf{C}).$$

It follows from (7) and (8) that $\{F_n(a_1, \ldots, a_n)\} \in \boldsymbol{R}$ ($\boldsymbol{C}$).

2.55. *Let $W(x)$ be a function of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are two polynomials satisfying the following conditions:*

(i) *the coefficients of $P(x)$ and $Q(x)$ are recursive (computable) numbers,*

(ii) *the degree of $P(x)$ is not smaller than the degree of $Q(x)$,*

(iii) $|Q(x)| \geqslant \varepsilon$ *for* $x \in \Delta$, *where ε is a positive real number and Δ is an interval.*

Let $\{\xi_n\}$ be a sequence defined by the formulae

$$\xi_0 = \xi, \quad \xi_{n+1} = W(\xi_n)$$

such that $\xi_n \in \Delta$ for every $n \in \mathcal{N}$, and $\xi \in \mathcal{R}$ $(\mathcal{C})$.

Then $\{\xi_n\} \in \mathbf{R}$ (C).

Let

$$P(x) = a_0' x^p + a_1' x^{p-1} + \ldots + a_p'$$

and

$$Q(x) = a_0'' x^q + a_1'' x^{q-1} + \ldots + a_p'',$$

where $a_0', \ldots, a_p', a_0'', \ldots, a_q'' \in \mathcal{R}$ $(\mathcal{C})$, and $p \geqslant q$. Consider two sequences of polynomials defined as follows:

$$F_0'(x_0', \ldots, x_p', x_0'', \ldots, x_q'', x) = x_0' x^p + x_1' x^{p-1} + \ldots + x_p',$$

$$F_0''(x_0', \ldots, x_p', x_0'', \ldots, x_q'', x) = x_0'' x^q + x_1'' x^{q-1} + \ldots + x_q'',$$

$$F_{n+1}'(x_0', \ldots, x_p', x_0'', \ldots, x_q'' \cdot x) = F_n''^q (x_0' F_n'^p + x_1' F_n'^{p-1} F_n'' + \ldots + x_p' F_n''^p),$$

$$F_{n+1}''(x_0', \ldots, x_p', x_0'', \ldots, x_q'', x) = F_n''^p (x_0'' F_n'^q + x_1'' F_n'^{q-1} F_n'' + \ldots + x_q'' F_n''^q).$$

These sequences can be defined by means of some recursive functions:

$$f'(n, m_0', \ldots, m_p', m_0'', \ldots, m_q'', m), \quad f''(n, m_0', \ldots, m_p', m_0'', \ldots, m_q'', m),$$

as in the theorem 2.54.

Let

$$g'(0) = p+1, \quad g''(0) = q+1,$$

$$g'(n+1) = g''(n) \cdot q + g'(n) \cdot p + 1, \quad g''(n+1) = g''(n) \cdot p + g'(n) \cdot q + 1.$$

Obviously, $g'(n)$, $g''(n) \in \mathbf{R}$. These functions determine the degree of polynomials F_n' and F_n'', respectively. The functions f' and f'', the values of which determine the coefficients, can also be defined in a recursive way. We shall show by induction that

$$\frac{F_n'(a_0', \ldots, a_p', a_0'', \ldots, a_q'', \xi)}{F_n''(a_0', \ldots, a_p', a_0'', \ldots, a_q'', \xi)} = \xi n + 1.$$

Indeed,

$$\frac{F_0'(a_0', \ldots, a_p', a_0'', \ldots, a_q'', \xi)}{F_0''(a_0', \ldots, a_p', a_0'', \ldots, a_q', \xi)} = \frac{P(\xi)}{Q(\xi)} = W(\xi) = \xi_1.$$

Suppose that for some n

$$\frac{F'_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)}{F''_n(a'_0, \ldots, a'_p. a''_0, \ldots, a''_q, \xi)} = \xi_{n+1}.$$

Hence

$$\frac{F'_{n+1}(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)}{F''_{n+1}(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)} = \frac{\frac{1}{F''^p_n}(a'_0 F'^p_n + a'_1 F'^{p-1}_n F''_n + \ldots + a'_p F''^p_n)}{\frac{1}{F''^q_n}(a''_0 F'^q_n + a''_1 F'^{q-1}_n F''_n + \ldots + a''_q F''^q_n)}$$

$$= \frac{a'_0\left(\frac{F'_n}{F''_n}\right)^p + a'_1\left(\frac{F'_n}{F''_n}\right)^{p-1} + \ldots + a'_p}{a''_0\left(\frac{F'_n}{F''_n}\right)^q + a''_1\left(\frac{F'_n}{F''_n}\right)^{q-1} + \ldots + a''_q} = \frac{a'_0\xi^p_{n+1} + a'_1\xi^{p-1}_{n+1} + \ldots + a'_p}{a''_0\xi^q_{n+1} + a''_1\xi^{q-1}_{n+1} + \ldots + a''_q} =$$

$$= \frac{P(\xi_{n+1})}{Q(\xi_{n+1})} = W(\xi_{n+1}) = \xi_{n+2}.$$

On account of theorem 2.54 the sequences

$$\{F'_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)\} \quad \text{and} \quad \{F''_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)\}$$

are recursive (computable). We can assume that $|Q(x)| > \eta > 1$. In fact, $W(x) = \frac{P(x)}{Q(x)}$ where $|Q(x)| > \varepsilon$. If $\varepsilon \leqslant l$, then $W(x) = \frac{\lambda P(x)}{\lambda Q(x)}$, where $\lambda, \varepsilon > 1$, and $\lambda \in \mathscr{R}$. Then $|\lambda Q(x)| > 1$. We shall show by induction that

$$|F''_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)| > \eta.$$

In fact,

$$|Q(\xi)| = |a''_0\xi^q + \ldots + a''_q| > \eta > 1.$$

That is

$$|F''_0(a'_0, \ldots, a'_p, \ldots, a''_0, a''_q, \xi)| > \eta.$$

Since $\xi_n \in \Delta$, we have

$$|a''_0\xi^q_n + a''_1\xi^{q-1}_n + \ldots + q''| > \eta > 1,$$

consequently

$$\left|\frac{1}{F''^q_n}(a_0 F'^q_n + a''_1 F'^{q-1}_n F''_n + \ldots a''_q F''^q_n)\right| > \eta > 1.$$

On the other hand, by the induction assumption $|F''_n| > \eta > 1$. Hence $|F''^{p+q}_n| > \eta > 1$. That is

$$|F''^p_n(a''_0 F'^q_n + \ldots + a''_q F''^q_n)| > \eta.$$

Thus

$$|F''_{n+1}(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)| > \eta.$$

Since $\{F'_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)\}$ and $\{F''_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)\}$ are recursive (computable) sequences, and $|F''_{n+1}(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)| > \eta$, then by 2.15 and 2.18 the sequence

$$\frac{F'_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)}{F''_n(a'_0, \ldots, a'_p, a''_0, \ldots, a''_q, \xi)}$$

is a recursive (computable) one. Consequently, $\{\xi_n\} \in \boldsymbol{R}$ (C).

2.56. *Any root of a polynomial whose coefficients are recursive (computable) numbers is also a recursive (computable) one.*

Let us consider a polynomial with no multiple roots. Consider $w(x) = a_0x^k + a_1x^{k-1} + \ldots + a_k$, where $a_0 \neq 0$, $a_0, \ldots, a_k \in \mathscr{R}$ $(\mathscr{C})$.

Suppose that $k \geqslant 2$ is the lowest degree of this polynomial and $w(\xi) = 0$. It follows from the assumptions made that

(1) $$w'(\xi) \neq 0$$

and

(2) $$w''(\xi) \neq 0.$$

Suppose that

(3) $$w'(\xi) > 0.$$

In the case of $w'(\xi) < 0$ the proof of the theorem is analogical.

Consider an interval $\Delta = \langle a, b\rangle$ for which the following conditions are satisfied:

(i) a, b are rational numbers,

(ii) $a < \xi < b$,

(iii) $w'(x) > 0$ for $x \in \Delta$,

(iv) $w''(x) \neq 0$ for $x \in \Delta$,

(v) $\frac{M}{2m}(b-\xi) < 1$ and $\frac{M}{2m}(\xi - a) < 1$, where $M = \sup_{\Delta} |w''(x)|$ and $m = \inf_{\Delta} w'(x)$.

An interval $\langle a, b\rangle$ satisfying conditions (i)-(iv) always exists on account of $w'(\xi) > 0$, $w''(\xi) \neq 0$ and the continuity of $w'(x)$ and $w''(x)$. The fulfilment of condition (v) can always be obtained by a suitable diminuation of the interval assumed, satisfying conditions (i)-(iv).

Now we shall consider the case where

I. $w''(x) > 0$ for $x \in \Delta$.

In the case where

II. $w''(x) < 0$ for $x \in \Delta$

the proof is similar.

Let

$$\xi_0 = b, \quad \xi_{n+1} = \xi_n - \frac{w(\xi_n)}{w'(\xi_n)}. \tag{4}$$

Now we shall prove by induction that

$$\xi < \xi_n \leqslant b \quad \text{for} \quad n \in \mathcal{N}. \tag{5}$$

For $n = 0$, inequality (5) is obvious. Suppose that (5) holds for some n. If follows from (4) that

$$\xi - \xi_{n+1} = \xi - \xi_n + \frac{w(\xi_n)}{w'(\xi_n)}. \tag{6}$$

Applying the formula of Taylor to the function $w(x)$ we obtain

$$w(\xi) = w\big(\xi_n + (\xi - \xi_n)\big) = w(\xi_n) + (\xi - \xi_n)w'(\xi_n) + \frac{1}{2}(\xi - \xi_n)^2 \cdot w''(\bar{x}_n), \tag{7}$$

where $\xi < \bar{x}_n < \xi_n$. Since $w(\xi) = 0$, it follows from (7) that

$$\xi - \xi_n + \frac{w(\xi_n)}{w'(\xi_n)} = -\frac{1}{2}(\xi - \xi_n)^2 \cdot \frac{w''(\bar{x}_n)}{w'(\xi_n)}. \tag{8}$$

Making use of the inductive assumption, we infer that $\xi < \bar{x}_n < \xi_n \leqslant b$. Hence $w''(\bar{x}_n) > 0$. On the other hand, since $\xi < \xi_n \leqslant b$,

$$w'(\xi_n) > 0.$$

Hence

$$\xi - \xi_n + \frac{w(\xi_n)}{w'(\xi_n)} < 0. \tag{9}$$

By (6) and (9) we obtain

$$\xi - \xi_{n+1} < 0; \tag{10}$$

that is

$$\xi < \xi_{n+1}. \tag{11}$$

It is easy to see that

$$\xi_{n+1} < \xi_n. \tag{12}$$

Indeed,

$$\xi_{n+1} = \xi_n - \frac{w(\xi_n)}{w'(\xi_n)}, \quad \text{where} \quad \xi < \xi_n \leqslant b.$$

Hence $w'(\xi_n) > 0$ and $w(\xi_n) > 0$. That is $\xi_{n+1} < \xi_n$. By (11), (12) and the inductive assumption

$$\xi < \xi_{n+1} < \xi_n \leqslant b,$$

which completes the proof of $\xi < \xi_n \leqslant b$ for $n \in \mathcal{N}$.

Now we shall prove that

$$\{\xi_n\} \underset{\mathbf{R}(C)}{\longrightarrow} \xi. \tag{13}$$

It follows from (4), (8) and the definitions of m and M that

$$|\xi-\xi_{n+1}| = \frac{1}{2}(\xi-\xi_n)^2\cdot\frac{w''(\bar{x}_n)}{w'(\xi_n)} \leqslant \frac{1}{2}(\xi-\xi_n)^2\cdot\frac{M}{m}. \tag{14}$$

Let $\frac{1}{2}\cdot\frac{M}{m} = \omega$, that is

$$|\xi-\xi_{n+1}| \leqslant \omega(\xi-\xi_n)^2. \tag{15}$$

Hence, it is easy to see that

$$\begin{aligned}(16)\quad |\xi-\xi_{n+1}| &\leqslant \omega^{1+2+2^2+\ldots+2^n}(\xi-b)^{2^{n+1}}\\ &= \omega^{2^{n+1}-1}(\xi-b)^{2^{n+1}} = \frac{1}{\omega}\big(\omega(\xi-b)\big)^{2^{n+1}} = \frac{1}{\omega}\left(\frac{M}{2m}(\xi-b)\right)^{2^{n+1}}.\end{aligned}$$

By (v), $\frac{M}{2m}(b-\xi) < 1$. Hence

$$\lim_{n\to\infty}\frac{1}{\omega}\left(\frac{M}{2m}(\xi-b)\right)^{2^{n+1}} = 0. \tag{17}$$

Let

$$c_n = \frac{1}{\omega}\left(\frac{M}{2m}(\xi-b)\right)^{2^{n+1}}.$$

We shall show that $\{c_n\} \underset{\mathbf{R}(C)}{\longrightarrow} 0$. Let $\omega(b-\xi) = \varrho$. Consider two cases: (a) $\omega\log\frac{1}{\varrho} \geqslant 1$, (b) $0 < \omega\log\frac{1}{\varrho} < 1$.

In case (a) we put $h_1(k) = k+1$. In case (b) we put $h_2(k) = (k+1)\vartheta$, where

$$\vartheta = \left[\frac{1}{\omega\log\frac{1}{\varrho}}+1\right].$$

Obviously, $h_1, h_2 \in \mathbf{R}(\mathbf{C})$. Let $n > h_1(k)$ in case (a), and let $n > h_2(k)$ in case (b). Then in both cases

$$n > \frac{k+1}{\omega\log\frac{1}{\varrho}} = \frac{\frac{k+1}{\omega}}{\log\frac{1}{\varrho}} > \frac{\log\frac{k+1}{\omega}}{\log\frac{1}{\varrho}}. \tag{18}$$

Hence

$$n\log\frac{1}{\varrho} > \log\frac{k+1}{\omega},$$

that is

$$\log\left(\frac{1}{\varrho}\right)^n > \log\frac{k+1}{\omega}.$$

Consequently

$$\omega\cdot\left(\frac{1}{\varrho}\right)^n > k+1.$$

Hence, on account of $\varrho < 1$,

$$\frac{1}{k+1} > \frac{1}{\omega}\cdot\varrho^n > \frac{1}{\omega}\,\varrho^{2^{n+1}} = C_n.$$

Thus it is proved that

$$C_n < \frac{1}{k+1} \quad \text{for} \quad \begin{cases} n > h_1(k) \text{ in case (a).} \\ n > h_2(k) \text{ in case (b).} \end{cases}$$

Hence $\{C_n\} \underset{\mathbf{R}(\vec{C})}{\longrightarrow} 0$. It is easy to establish that $\{\xi_n\} \underset{\mathbf{R}(\vec{C})}{\longrightarrow} \xi$. Indeed, since $\{C_n\} \underset{\mathbf{R}(\vec{C})}{\longrightarrow} 0$, there exists an $h \in \mathbf{R}$ such that $C_n < \dfrac{1}{k+1}$ for $n > h(k)$. Hence, on account of $|\xi - \xi_{n+1}| \leqslant C_n$ we infer that

$$|\xi - \xi_{n+1}| < \frac{1}{k+1} \quad \text{for} \quad n > h(k).$$

In the last part of the proof of 2.56 we shall show that

$$\{\xi_n\} \in \boldsymbol{R}\ (\boldsymbol{C}). \tag{19}$$

The sequence $\{\xi_n\}$ fulfils the conditions of theorem 2.55. Indeed, let

$$W(x) = \frac{x\cdot w'(x) - w(x)}{w'(x)} = \frac{P(x)}{Q(x)}.$$

Then

$$\xi_0 = b, \quad \xi_{n+1} = W(\xi_n).$$

The coefficients of polynomials $P(x)$ and $Q(x)$ are recursive (computable):

$$|Q(x)| = |w'(x)| \geqslant \varepsilon = m.$$

On account of (5), $\xi_n \in \Delta$ for $n \in \mathcal{N}$. Hence, by 2.55, $\{\xi_n\} \in \boldsymbol{R}\ (\boldsymbol{C})$.

It follows from (13), (19) and 2.38 that $\xi \in \mathscr{R}\ (\mathscr{C})$. This completes the proof [1].

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[1] See footnote on p. 21.

III. RECURSIVE AND COMPUTABLE FUNCTIONALS

A functional Φ defined over natural computable functions and assuming real values is computable if for a computable sequence $\{f_n\}$ of functions the sequence $\{a_n\}$ of values $a_n = \Phi\langle f_n\rangle$ is computable. Starting from such a definition of computable functionals and from a similar definition of recursive functionals, we prove many properties of these notions. In the first place many auxiliary theorems are proved. Namely the theorems stating that the class of computable functionals is closed under the same operations which do not lead out of the class of computable functions on natural numbers (3.10-3.13). In the second place we prove that computable functionals are continuous in such a sense that if $\{f_n\} \to f$ then $\{\Phi\langle f_n\rangle\} \to \Phi\langle f\rangle$ (3.36). This theorem implies the theorem stating that a computable functional depends on a finite number of values of the function considered as an argument (3.43) and the theorem which presents a kind of a "general form" of computable functionals (3.43). For the recursive functionals it is possible only to prove the weaker theorems (3.33, 3.47, 3.40). For other notions of recursive functional see footnote on p. 79.

The notion of recursive (computable) functional is necessary for proving some theorems on the recursive (computable) real functions.

3.1. *A sequence $\{f_k\}$ of recursive (computable) functions of one natural argument is said to be recursive (computable) provided that there exists a function $F \in \mathbf{R}$ (C) of two arguments such that*

$$f_k(n) = F(k, n) \quad \textit{for all } n, k \in \mathcal{N}.$$

The class of recursive (computable) sequences of functions will be denoted by the same symbols as the class of recursive (computable) functions.

3.2. *A functional Φ defined over the set of recursive (computable) natural functions of one argument and assuming real numbers as values is said to be recursive (computable) if for each sequence $\{f_k\} \in \mathbf{R}$ (C) there exists such a sequence $\{a_k\} \in \mathbf{R}$ (C) that*

$$\Phi\langle f_k\rangle = a_k \quad \textit{for all } k \in \mathcal{N}.$$

The class of recursive (computable) functionals will be denoted by $\mathfrak{F}_C$ ($\mathfrak{F}_R$).

The class of such functionals $\Phi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$) that $\Phi\langle f\rangle \in \mathcal{N}$ for all $f \in \mathbf{R}$ (**C**) will be denoted by $\mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$).

3.3. *$\Phi \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) if and only if for each $\{f_k\} \in \mathbf{R}$ (**C**) there exists such a $g \in \mathbf{R}$ (**C**) that*

$$\Phi\langle f_k\rangle = g(k) \quad \textit{for all} \quad k \in \mathcal{N}.$$

It follows from 2.5 and 3.2 that if $a_k = g(k)$, then $\{a_k\} \in \boldsymbol{R}$ ($\boldsymbol{C}$) if and only if $g \in \mathbf{R}$ (**C**).

3.4. *If $n_0 \in \mathcal{N}$ and $\{a_k\} \in \boldsymbol{R}$ ($\boldsymbol{C}$), and $\Phi\langle f\rangle = af(n_0)$ for all $f \in \mathbf{R}$ (**C**), then $\Phi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$).*

If $\{f_k\} \in \mathbf{R}$ (**C**), then $\Phi\langle f_k\rangle = af_k(n_0)$. This means that for some F, $g \in \mathbf{C}$ (**R**), $g(k) = f_k(n_0) = F(k, n_0)$ and $\Phi\langle f_k\rangle = a_{g(k)}$. Hence $\{a_{g(k)}\} \in \boldsymbol{R}$ ($\boldsymbol{C}$), and $\Phi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$) by 3.2. For instance if $n_0 \in \mathcal{N}$ and $\Phi\langle f\rangle = f(n_0)$, then $\Phi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$).

3.5. *Another, more complicated, example of functionals $\mathfrak{F}_R$ ($\mathfrak{F}_C$) is the functional Φ_1 defined as follows:*

$$\Phi_1\langle f\rangle = \sum_{x=0}^{\infty} \frac{|f(x) \text{ red mod } 10|}{10^x}.$$

If $\{f_k\} \in \boldsymbol{R}$ ($\boldsymbol{C}$), then, by 2.19 and 2.24, $\Phi_1\langle f_k\rangle \in \mathbf{R}$ (**C**), whence, by 3.2, $\Phi_1 \in \mathfrak{F}_R$ ($\mathfrak{F}_C$).

3.6. *Let Φ, $\Psi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$); then*

$$\Phi\langle f\rangle + \Psi\langle f\rangle \in \mathfrak{F}_R \ (\mathfrak{F}_C), \qquad \Phi\langle f\rangle - \Psi\langle f\rangle \in \mathfrak{F}_R \ (\mathfrak{F}_C),$$

$$\Phi\langle f\rangle \cdot \Psi\langle f\rangle \in \mathfrak{F}_R \ (\mathfrak{F}_C), \qquad |\Phi\langle f\rangle| \in \mathfrak{F}_R \ (\mathfrak{F}_C).$$

From: 2.7, 2.8, 2.10, 2.11 and 3.2.

3.7. *A sequence $\{\Phi_n\}$ of functionals is said to be computable (recursive) ($\{\Phi_n\} \in \mathfrak{F}_R$ ($\mathfrak{F}_C$)) provided that for each sequence $\{f_k\} \in \mathbf{R}$ (**C**) there exists a sequence $\{a_k\} \in \boldsymbol{R}$ ($\boldsymbol{C}$) such that*

$$\Phi_n\langle f_k\rangle = a_{P(n,k)} \quad \textit{for each} \quad n, k \in \mathcal{N},$$

where $P(n, k) = 2^n(2k+1)-1$.

*If $\Phi_n\langle f\rangle \in \mathcal{N}$ for each $f \in \mathbf{R}$ (**C**), then we say that $\{\Phi_n\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$).*

3.8. *The double sequence $\{\Phi_{n,m}\}$ is said to be recursive (computable) provided that there exist a sequence $\{\Psi_n\} \in \mathfrak{F}_R$ ($\mathfrak{F}_C$) and a function $g \in \mathbf{R}$ (**C**) such that*

$$\Phi_{n,m}\langle f\rangle = \Psi_{g(n,m)}\langle f\rangle.$$

3.9. *Let* $\Phi_n\langle f\rangle \in \mathcal{N}$ *for all* $f \in \mathbf{R}$ (**C**) *and* $n \in \mathcal{N}$; *then* $\{\Phi_n\} \in \mathfrak{F}_R$ ($\mathfrak{F}_C$) *if and only if for each sequence* $\{f_k\} \in \mathbf{R}$ (**C**) *there exists a function* $g \in \mathbf{R}$ (**C**) *such that*

$$\Phi_n\langle f_k\rangle = g(n, k) \quad \textit{for each } n, k \in \mathcal{N}. \tag{1}$$

From 3.7 and 2.5.

3.10. *If* $\{\Psi_{n,m}\}, \Omega \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) *and*

$$\Phi_0\langle f\rangle = \Omega\langle f\rangle; \quad \Phi_{n+1}\langle f\rangle = \Psi_{n,\Phi_n\langle f\rangle}\langle f\rangle, \tag{1}$$

then $\{\Phi_n\} \in \mathfrak{F}_R$ ($\mathfrak{F}_C$).

If $\{f_k\} \in \mathbf{R}$ (**C**), then there exist functions $h, l \in \mathbf{R}$ (**C**) such that $\Psi_{n,m}\langle f_k\rangle = h(m, n, k)$ and $\Omega\langle f_k\rangle = l(k)$ according to 3.3. We shall prove that there exists a function $j \in \mathbf{R}$ (**C**) such that $\Phi_n\langle f_k\rangle = j(n, k)$. Indeed, such a function can be defined recursively:

$$j(0, k) = l(k), \quad j(n+1, k) = h\big(n, j(n, k), k\big). \tag{2}$$

From formulae (1) and (2) it follows that

$$\Phi_0\langle f_k\rangle = \Omega\langle f_k\rangle = l(k) = j(0, k). \tag{3}$$

Now suppose that $\Phi_n\langle f_k\rangle = j(n, k)$; then

$$\Phi_{n+1}\langle f_k\rangle = \Psi_{n,\Phi_n\langle f_k\rangle}\langle f_k\rangle = h\big(n, j(n, k), k\big) = j(n+1, k). \tag{4}$$

From (3) and (4) it follows by induction that for each $n \in \mathcal{N}$

$$\Phi_n\langle f_k\rangle = j(n, k).$$

Hence, according to 3.9, $\{\Phi_n\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$).

3.11. *If* $\{\Phi_n\} \in \mathfrak{F}_C^*$, *and for each* $f \in \mathbf{C}$ *there exists such an* $n \in \mathcal{N}$ *that* $\Phi_n\langle f\rangle = 0$, *then the functional*

$$\Psi\langle f\rangle = (\mu n)[\Phi_n\langle f\rangle = 0] \tag{1}$$

belongs to $\mathfrak{F}_C^*$.

From 3.9 it follows that for each $\{f_k\} \in \mathbf{C}$ there exists a function $g \in \mathbf{C}$ such that

$$\Phi_n\langle f_k\rangle = g(n, k). \tag{2}$$

Let us put

$$h(k) = (\mu n)[g(n, k) = 0]. \tag{3}$$

From (2) it follows that for each k there exists such an n that $g(n, k) = 0$. Hence $h \in \mathbf{C}$. On the other hand, from (1), (2) and (3) it follows that

$$\Psi\langle f_k\rangle = h(k).$$

Hence $\Psi \in \mathfrak{F}_C^*$ according to 3.3

3.12. *If* $f, g \in \mathbf{R}$ (**C**) *and* $\Phi_n\langle h\rangle = f\big(h(g(n))\big)$ *for all* $h \in \mathbf{R}$ (**C**), *then* $\{\Phi_n\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$).

Indeed, if $\{h_k\} \in \boldsymbol{R}$ (C), then there exists a function $j \in \mathbf{R}$ (**C**) such that $\Phi_n\langle h_k\rangle = j(n, k)$. Namely we can put $j(n, k) = f\big(h_k(g(n))\big) = f\big(h(k, g(n))\big)$ when $h(k, n) = h(n)$. Then $\{\Phi_n\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) by 3.9.

3.13. *If* $f \in \mathbf{R}$ (**C**) *and* $\{\Phi_n\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) *and*

$$\Psi_n\langle h\rangle = f(\Phi_n\langle h\rangle), \tag{1}$$

then

$$\{\Psi_n\} \in \mathfrak{F}_R \ (\mathfrak{F}_C).$$

Suppose that $\{h_k\} \in \mathbf{R}$ (**C**); then by 3.9 there exists a function $g \in \mathbf{R}$ (**C**) such that

$$\Phi_n\langle h_k\rangle = g(n, k). \tag{2}$$

Hence $\Psi_n\langle h_k\rangle = f\big(g(n, k)\big)$ by (1) and (2). Thus $\{\Psi_n\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) according to 3.9.

3.14. *If* $\{\Psi_{k,m}\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) *and* $\Phi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$), *setting for each* $f \in \mathbf{R}$ (**C**) $\Psi'_m\langle f\rangle(k) = \Psi_{k,m}\langle f\rangle$ *and* $\Omega_m\langle f\rangle = \Phi\big\langle\Psi'_m\langle f\rangle\big\rangle$, *we have* $\{\Omega_m\} \in \mathfrak{F}_R$ ($\mathfrak{F}_C$).

If $\{\Psi_{k,m}\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$), then by 3.8 there exist such $\{\Psi_n^*\} \in \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) and $j \in \mathbf{R}$ (**C**) that

$$\Psi_{k,m}\langle f\rangle = \Psi^*_{j(k,m)}\langle f\rangle. \tag{1}$$

If $\{f_m\} \in \mathbf{R}$ (**C**), then by 3.9 there exists a function $h \in \mathbf{R}$ (**C**) such that $\Psi_k^*\langle f_n\rangle = h(k, n)$. Hence

$$\Psi_{k,m}\langle f_n\rangle = h\big(j(k, m), n\big). \tag{2}$$

Putting

$$h^*_{n,m}(k) = h\big(j(k, m), n\big) \tag{3}$$

we obtain $\Psi'_m\langle f_n\rangle(k) = h^*_{n,m}(k)$, whence

$$\Omega_m\langle f_n\rangle = \Phi(\Psi'_m\langle f_n\rangle) = \Phi\langle h^*_{n,m}\rangle. \tag{4}$$

Putting $h'_z(k) = h^*_{Qz,Rz}(k)$, we infer from (3) that $\{h'_z\} \in \mathbf{R}$ (**C**) and

$$h'_{P(n,m)}(k) = h^*_{n,m}(k). \tag{5}$$

From 3.7 it follows that there exists such a sequence $\{a_k\} \in \mathbf{R}$ (**C**) that

$$\Phi\langle h_z\rangle = a_z. \tag{6}$$

From (4), (5), (6) we obtain

$$\Omega_m\langle f_n\rangle = a_{P(n,m)}.$$

Hence, by 3.9, $\{\Omega_m\} \in \mathfrak{F}_R$ ($\mathfrak{F}_C$).

As a special case of 3.14 it follows that

3.15. *If* $\{\Psi_k\} \in \mathfrak{F}_R^* \ (\mathfrak{F}_C^*)$ *and* $\Phi \in \mathfrak{F}_R \ (\mathfrak{F}_C)$, *where* $\Psi'\langle f\rangle(k) = \Psi_k\langle f\rangle$ *and* $\Omega\langle f\rangle = \Phi\langle\Psi'\langle f\rangle\rangle$ *for each* $f \in \mathbf{R}$ (**C**), *then* $\Omega \in \mathfrak{F}_R \ (\mathfrak{F}_C)$.

3.16. *If* $\Phi \in \mathfrak{F}_R \ (\mathfrak{F}_C)$, *then there exists* $\{\Psi_n\} \in \mathfrak{F}_R \ (\mathfrak{F}_C)$ *such that if* $\{f_n\} \in \mathbf{R}$ (**C**) *and* $F(z) = f_{Qz}(Rz)$, *then*

$$\Phi\langle f_n\rangle = \Psi_n\langle F\rangle. \tag{0}$$

For each $F \in \mathbf{R}$ (**C**) we put

$$\Omega_x\langle F\rangle(y) = F\big(P(x, y)\big) = \Omega_{x,y}\langle F\rangle, \tag{1}$$

$$\Psi_n\langle F\rangle = \Phi\langle\Omega_n\langle F\rangle\rangle. \tag{2}$$

The double sequence $\{\Omega_{x,y}\}$ of functionals is recursive (computable) according to 3.8, because there exists a single sequence $\{\theta_n\} \in \mathfrak{F}_R \ (\mathfrak{F}_C)$ such that $\Omega_{x,y}\langle F\rangle = \theta_{P(x,y)}\langle F\rangle$. Namely $\theta_n\langle F\rangle = F(n)$. Hence $\{\Psi_n\} \in \mathfrak{F}_R \ (\mathfrak{F}_C)$ by 3.14. From the fact that $F(z) = f_{Qz}(Rz)$ it follows that

$$F\big(P(x, y)\big) = f_x(y). \tag{3}$$

Hence from (1) and (3)

$$\Omega_x\langle F\rangle(y) = f_x(y). \tag{4}$$

(4) and (2) involve formula (0).

3.17. *If* $\Phi \in \mathfrak{F}_R \ (\mathfrak{F}_C)$ *and* $f \in \mathbf{R}$ (**C**), *then* $\Phi\langle f\rangle \in \mathscr{C} \ (\mathscr{R})$.

From 2.4 and 3.2.

3.18. *A sequence* $\{f_n\}$ *of natural functions is said to be convergent to the function* f (*in symbols*: $\{f_n\} \to f$) *if for each* $x \in \mathscr{N}$ *there exists such an* $n_0 \in \mathscr{N}$ *that for all* $n \geqslant n_0$ *we have* $f_n(x) = f(x)$.

This definition is evidently equivalent to the following:

3.19. $\{f_n\} \to f$ *if and only if there exists such a function* $g(x)$ *that for all* $x, n \in \mathscr{N}$

$$\text{if} \quad n \geqslant g(x), \quad \text{then} \quad f_n(x) = f(x). \tag{1}$$

3.20. *If for the function* g *of* 3.19 *we have* $g \in \mathbf{R}$ (**C**), *then we shall say that* $\{f_n\}$ *is convergent computably* (*recursively*) *to* f ($\{f_n\} \underset{\mathbf{R}\,(\mathbf{C})}{\longrightarrow} f$).

3.21. $\{f_n\} \to f$ ($\{f_n\} \underset{\mathbf{R}\,(\mathbf{C})}{\longrightarrow} f$) *if and only if there exists a non-decreasing function* h ($h \in \mathbf{R}$ (**C**)) *such that for all* $x, y, n \in \mathscr{N}$:

$$\text{if} \quad n \geqslant h(x) \quad \text{and} \quad y \leqslant x, \quad \text{then} \quad f_n(y) = f(y). \tag{2}$$

The first implication is evident. To prove the second we put

$$h(0) = g(0), \qquad h(x+1) = \max\big(h(x),\ g(x+1)\big).$$

Hence for $y \leqslant x$, $h(x) \geqslant g(y)$, and if $n > h(x)$, then $n > g(y)$, and, by (1), $f_n(y) = f(y)$.

3.22. $\{f_n\} \to f$ ($\{f_n\} \xrightarrow[\mathbf{R}\,(\mathbf{C})]{} f$) *if and only if there exists an increasing function* j ($j \in \mathbf{R}$ ($\mathbf{C}$)) *such that for all* $x, y, n \in \mathcal{N}$:

(3) $\quad$ *if* $\quad n > j(x), \quad$ *and* $\quad y \leqslant x, \quad$ *then* $\quad f_n(y) = f(y)$.

From 3.21 it follows that there exists a non-decreasing function h which satisfies condition (2). We put

$$j(0) = h(0),$$

$$j(n+1) = \begin{cases} j(n)+1, & \text{when} \quad h(n+1) \leqslant j(n), \\ h(n+1), & \text{when} \quad h(n+1) > j(n). \end{cases}$$

Hence j is strictly increasing and $j(n) \geqslant h(n)$; thus condition (2) involves (3).

The inverse implication is evident.

3.23. *If* Ω *assumes only natural values and* $\Omega\langle f\rangle = 0$ *if and only if, for each* $n \in \mathcal{N}$, $f(n) = 0$, *then* $\Omega \notin \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$).

From theorem 1.28 we infer that there exists a function $h \in \mathbf{R}$ of two variables $h(k, n) = h_k(n)$ such that function δ defined by

(1) $$\delta(k) = \begin{cases} 0 & \text{if } h(k, n) = 0 \text{ for each } n \in \mathcal{N}, \\ 1 & \text{if } h(k, n) \neq 0 \text{ for some } n \in \mathcal{N} \end{cases}$$

is not computable.

If $\Omega\langle f\rangle = 0$ if and only if, for each $n \in \mathcal{N}$, $f(n) = 0$, then, by (1),

(2) $$\operatorname{sgn}(\Omega\langle h_k\rangle) = \delta(k);$$

hence, by 3.3, $\Omega \notin \mathfrak{F}_R^*$ ($\mathfrak{F}_C^*$) because $\{h_k\} \in \mathbf{C}$ ($\mathbf{R}$) and $\delta \notin \mathbf{C}$.

3.24. *Let us put*

$$a(n, m) = \begin{cases} 0 & \textit{when} \quad m < n, \\ 1 & \textit{when} \quad m \geqslant n, \end{cases}$$

$$a_n(m) = a(n, m).$$

3.25. *A function* f *is said to be normal provided that* $f(x) = 0$ *for all* $x \in \mathcal{N}$, *or for some* $y : f(x) = a_y(x)$ *for all* $x \in \mathcal{N}$.

A sequence $\{f_n\}$ *is said to be normal if, for each* $n \in \mathcal{N}$, f_n *is normal.*

3.26. *If* $\Phi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$) *and* $\Psi\langle f\rangle = \min\left(\frac{1}{k_0+1}, \Phi\langle f\rangle\right)$, *then* $\Psi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$).

$$\min\left(\frac{1}{k_0+1}, \Phi\langle f\rangle\right) = \frac{\min\big(1, (k_0+1)\cdot\Phi\langle f\rangle\big)}{k_0+1}$$

$$= \frac{-1\cdot\max\big(-1, -(k_0+1)\cdot\Phi\langle f\rangle\big)}{k_0+1} = -\frac{\max\big(0, \big(-(k_0+1)\cdot\Phi\langle f\rangle\big)+1\big)-1}{k_0+1}.$$

Hence, if $\{f_n\} \in \mathbf{R}$ (C), then, according to 3.2, $\{\Phi\langle f_n\rangle\} \in \boldsymbol{R}$ $(\boldsymbol{C})$; thus the sequence

$$\Psi\langle f_n\rangle = -\frac{\max\big(0, \big(-(k_0+1)\cdot\Phi\langle f\rangle\big)+1\big)-1}{k_0+1}$$

belongs to $\boldsymbol{R}$ $(\boldsymbol{C})$ according to 2.53, 2.10, 2.8 and 2.11.

3.27. *If* $\Phi \in \mathfrak{F}_R^*$ $(\mathfrak{F}_C^*)$ *and* $\Phi\langle 0\rangle = 0$, $\{f_n\}$ *is normal and* $\{f_n\} \to 0$, *then there exists such an* $n_0 \in \mathcal{N}$ *that for* $n > n_0$

$$\Phi\langle f_n\rangle = 0.$$

3.27 is a special case of 3.28 having a simpler proof.

From 3.3 it follows that there exists such a function $j \in \mathbf{R}$ (C) that

(1) $$\Phi\langle a_n\rangle = j(n) \quad \text{for all} \quad n \in \mathcal{N}.$$

Putting

(2) $$\Psi_k\langle f\rangle = \operatorname{sgn}\Big(\sum_{y\leqslant k}\Big\{j(y)\cdot\sum_{z\leqslant y} f(z)\Big\}\Big),$$

(3) $$\Psi\langle f\rangle(k) = \Psi_k\langle f\rangle,$$

(4) $$\Omega\langle f\rangle = \operatorname{sgn}\big(\Phi\langle\Psi\langle f\rangle\rangle\big)$$

we infer from 3.10-3.15 that

(5) $$\Omega \in \mathfrak{F}_R^* \ (\mathfrak{F}_C^*).$$

Indeed, the sequence of functionals $\theta_z\langle h\rangle = h(z)$ is computable (recursive) by 3.11 (if we put $f(x) = g(x) = x$). Hence the sequence of functionals

$$\theta'_y\langle h\rangle = \sum_{z\leqslant y}\theta_z\langle h\rangle$$

is computable (recursive) by 3.10 because θ'_y can be defined recursively:

$$\theta'_0\langle h\rangle = \theta_0\langle h\rangle, \qquad \theta'_{n+1}\langle h\rangle = \theta_{n+1}\langle h\rangle + \theta'_n\langle h\rangle.$$

The sequence $\theta''_k\langle h\rangle = \sum_{y\leqslant k}\{j(y)\cdot\theta'_y\langle h\rangle\}$ can also be defined recursively. Hence $\{\Psi_k\} \in \mathfrak{F}_R^*$ $(\mathfrak{F}_C^*)$, by 3.10 and 3.13. And $\Omega \in \mathfrak{F}_R^*$ $(\mathfrak{F}_C^*)$ by 3.15, from (2), (3), (4). Notice that $\theta'_y\langle 0\rangle = 0$ for each $y \in \mathcal{N}$, whence

(6) $$\Psi\langle 0\rangle(k) = 0(k) = 0,$$

(7) $$\Omega\langle 0\rangle = 0 \quad \text{(from (4) and (6))}.$$

Now let us suppose that our theorem is false; this means that

(8) For each $n_0 \in \mathcal{N}$ there exists such an $n \in \mathcal{N}$ that $n > n_0$ and $\Phi\langle f_n\rangle \neq 0$.

On the other hand, the formula $\{f_n\} \to 0$ implies (by 3.21) that for each x there exists a number $n_0 = h(x)$ such that for each $n, y \in \mathcal{N}$:

(9) $$\text{If } n \geqslant n_0 \text{ and } y < x, \text{ then } f_n(y) = 0.$$

Combining (8) and (9) we infer that for each x there exists a number $n \in \mathcal{N}$ such that

(10) $$\Phi\langle f_n\rangle \neq 0 \quad \text{and} \quad f_n(y) = 0 \quad \text{for} \quad y < x.$$

Hence, for some x, $f_n(x) \neq 0$, because $\Phi\langle 0\rangle = 0$. Let $v = (\mu z)[f_n(z) \neq 0]$. From 3.25 it follows that

(11) $$f_n(n) = a_v(u) \quad \text{for all } u \in \mathcal{N},$$

because f_n is normal. From (10) and (11) it follows that $a_v(y) = 0$ for $y < x$, whence, by 3.24, $x \leqslant v$. From (10) and (11) we infer that:

(12) For each $x \in \mathcal{N}$ there exists such a $v \in \mathcal{N}$ that $v \geqslant x$ and $\Phi\langle a_v\rangle \neq 0$.

Hence from (1) it follows that:

(13) For $x \in \mathcal{N}$ there exists such a $v \in \mathcal{N}$ that $v \geqslant x$ and $j(v) \neq 0$.

Now we shall prove that condition (8) implies the following:

(14) $$\text{If, for some } x, f(x) \neq 0, \text{ then } \Omega\langle f\rangle = 1.$$

Put

(15) $$x_0 = (\mu x)[f(x) \neq 0],$$

(16) $$y_0 = (\mu v)[v \geqslant x_0 \text{ and } j(v) \neq 0].$$

From (14), (15) and (16) it follows that

(17) $$y_0 \geqslant x_0 \quad \text{and} \quad j(y_0) \neq 0,$$

(18) $$\sum_{z \leqslant y_0} f(z) \neq 0 \quad \text{(from (15) and (17))}.$$

Hence

(19) $$\sum_{y \leqslant x} \Big\{ j(y) \cdot \sum_{z \leqslant z} f(z) \Big\} \neq 0 \quad \text{for} \quad x \geqslant y_0.$$

On the other hand, if $x < y_0$, then, by (16), $x < x_0$ or $j(x) = 0$. But if $x < x_0$, then, by (15), $\sum_{z \leqslant x} f(z) = 0$.

Hence for $x < y_0$, $\{j(x) \cdot \sum_{z \leqslant x} f(z)\} = 0$ and

(20) $$\sum_{y \leqslant x} \Big\{ j(y) \cdot \sum_{z \leqslant y} f(z) \Big\} = 0 \quad \text{for} \quad x < y_0.$$

From (2), (3), (20), (19) it follows that

$$\Psi\langle f\rangle(x) = \begin{cases} 0 & \text{for} \quad x < y_0, \\ 1 & \text{for} \quad x \geqslant y_0. \end{cases} \tag{21}$$

(21) means that $\Psi\langle f\rangle(x) = a_{y_0}(x)$ (by 3.24). Hence from (4) and (1) it follows that

$$\Omega\langle f\rangle = \Phi\langle a_{y_0}\rangle = j(y_0).$$

From (17) we have $j(y_0) \neq 0$, whence, by (4), $\Omega\langle f\rangle = 1$. Thus condition (14) is proved. From (14) and (7) it follows that, if supposition (8) is true, then

$$\Omega\langle f\rangle = 0 \tag{22}$$

if and only if $f(x) = 0$ for each x. But this is impossible by 3.23 because $\Omega \in \mathfrak{F}_R^* \ (\mathfrak{F}_C^*)$.

3.28. *If* $\Phi \in \mathfrak{F}_R \ (\mathfrak{F}_C)$, $\Phi\langle 0\rangle = 0$, $\{f_n\}$ *is normal and* $\{f_n\} \to 0$, *then* $\Phi\langle f_n\rangle \to 0$.

From 3.2 it follows that there exists a sequence $\{a_n\} \in \boldsymbol{R}\ (\boldsymbol{C})$ such that

$$\Phi\langle a_n\rangle = a_n. \tag{1}$$

By 2.1 there exist two functions $g, g' \in \mathbf{R}$ (C) such that

$$\left| a_n - \frac{g(n,k) - g'(n,k)}{k+1} \right| < \frac{1}{k+1}. \tag{2}$$

Hence

$$\left| a_n \cdot (k+1) - \big(g(n,k) - g'(n,k)\big) \right| < 1. \tag{3}$$

Let us put

$$\Psi_{x,k}\langle f\rangle = \operatorname{sgn} \sum_{y \leqslant x} \left\{ \left[\frac{|g(y,k) - g'(y,k)|}{2} \right] \cdot \sum_{z \leqslant y} f(z) \right\}, \tag{4}$$

where $|x - y| = (x \dot{-} y) + (y \dot{-} x)$,

$$\Psi_k\langle f\rangle(x) = \Psi_{x,k}\langle f\rangle, \tag{5}$$

$$\Omega_k\langle f\rangle = \Phi\big\langle \Psi_k\langle f\rangle \big\rangle, \tag{6}$$

$$\theta_k\langle f\rangle = (k+1)\min\left(\frac{1}{k+1}, \Omega_k\langle f\rangle\right). \tag{7}$$

From 3.10-3.15 it follows, as in the preceding proof, that

$$\{\Omega_k\} \in \mathfrak{F}_R \ (\mathfrak{F}_C). \tag{8}$$

From (4), (5), (6), using the fact that $\Phi\langle 0\rangle = 0$, we obtain

$$\Omega_k\langle 0\rangle = 0 \quad \text{for each } k \in \mathcal{N}. \tag{9}$$

Let us suppose that $0 \neq \lim \Phi\langle f_n\rangle$. This means that there exists a number k_0 such that:

(10) For each $n_0 \in \mathcal{N}$ there exists such an $n > n_0$ that $|\Phi\langle f_n\rangle| \geqslant \dfrac{3}{k_0+1}$.

On the other hand, the formula $\{f_n\} \to 0$ implies by 3.21 that, for each x, there exists a number $n_0 = h(x)$ such that for each $n, y \in \mathcal{N}$:

(11) If $n \geqslant n_0$ and $y < x$, then $f_n(y) = 0$.

Combining (10) and (11) we infer that:

(12) For each x there exists an $n \in \mathcal{N}$ such that

$$|\Phi\langle f_n\rangle| \geqslant \frac{3}{k_0+1} \text{ and } f_n(y) = 0 \text{ for } y < x.$$

Hence, for some z, $f_n(z) \neq 0$ because $\Phi\langle 0\rangle = 0$. Let $v = \mu z[f_n(z) \neq 0]$. From 3.25 it follows that

(13) $$f_n(u) = a_v(u) \quad \text{for all } u \in \mathcal{N},$$

because f_n is normal. From (12) and (13) it follows that $a_v(y) = 0$ for $y < x$; hence, by 3.24, $x \leqslant v$. From (12) and (13) it follows that

(14) For each $x \in \mathcal{N}$ there exists such a $v \in \mathcal{N}$ that $v \geqslant x$ and

$$|\Phi\langle a_v\rangle| \geqslant \frac{3}{k_0+1}.$$

Hence from (1) and (14) we infer that

(15) $$|a_v|\cdot(k_0+1) \geqslant 3 \quad \text{and} \quad |g(v, k_0) - g'(v, k_0)| \geqslant 2.$$

From (14), (15) and (3) it follows that:

(16) For each $x \in \mathcal{N}$ there exists such a $v \in \mathcal{N}$ that $v \geqslant x$ and $|g(v, k_0) - g'(v, k_0)| \geqslant 2$.

Now we shall prove that condition (10) implies the following:

(17) If $f(x) \neq 0$ for some x, then $\theta_{k_0}\langle f\rangle = 1$.

Put

(18) $$x_0 = \mu x[f(x) \neq 0],$$

(19) $$y_0 = \mu v[v \geqslant x_0 \text{ and } |g(v, k_0) - g'(v, k_0)| \geqslant 2].$$

From (16), (18), (19) it follows that

(20) $$y_0 \geqslant x_0 \quad \text{and} \quad \left[\frac{|g(y_0, k_0) - g'(y_0, k_0)|}{2}\right] \neq 0.$$

Hence $\sum_{z\leqslant y_0} f(z) \neq 0$ (from (18 and (20)) and

$$(21)\qquad \sum_{y\leqslant x}\left\{\left[\frac{|g(y,k_0)-g'(y,k_0)|}{2}\right]\cdot\sum_{z\leqslant y} f(z)\right\} \neq 0 \quad \text{for} \quad x \geqslant y_0.$$

On the contrary, if $x < y_0$, then, by (19),

$$(22)\qquad x < x_0 \quad \text{or} \quad |g(x,k_0)-g'(x,k_0)| < 2.$$

Hence

$$\sum_{z\leqslant x} f(z) = 0 \quad \text{or} \quad \left[\frac{|g(x,k_0)-g'(x,k_0)|}{2}\right] = 0.$$

Thus

$$(23)\qquad \text{if } x < y_0, \text{ then } \sum_{y\leqslant x}\left\{\left[\frac{|g(x,k_0)-g'(x,k_0)|}{2}\right]\cdot\sum_{z\leqslant x} f(z)\right\} = 0.$$

From (21) and (23) it follows that

$$(24)\qquad \Psi_{k_0}\langle f\rangle(x) = \begin{cases} 0 & \text{for} \quad x < y_0, \\ 1 & \text{for} \quad x \geqslant y_0. \end{cases}$$

Hence

$$(25)\qquad \Psi_{k_0}\langle f\rangle(x) = a_{y_0}(x) \quad \text{for all } x \in \mathcal{N}.$$

From (25), (1), (6) it follows that

$$(26)\qquad \Omega_{k_0}\langle f\rangle = \Phi\langle a_{y_0}\rangle = a_{y_0}.$$

From (19), (3), (26) we deduce that

$$(27)\qquad |a_{y_0}\cdot(k_0+1)| \geqslant 1.$$

Hence $\min\left(\frac{1}{k_0+1},\ \Omega_{k_0}\langle f\rangle\right) = \frac{1}{k_0+1}$, and, from (7),

$$(28)\qquad \theta_{k_0}\langle f\rangle = 1.$$

Condition (17) is also proved. The functional θ_{k_0} assumes only 0, 1 as values according to (7), (9) and (17), and

$$(29)\qquad \theta_{k_0}\langle f\rangle = 0 \text{ if and only if } f(x) = 0 \text{ for each } x \in \mathcal{N}.$$

From (29) and 3.26 we obtain

$$(30)\qquad \theta_{k_0} \in \mathfrak{F}_R^* \ (\mathfrak{F}_C^*).$$

On the other hand, we can prove the formula inconsistent with (30). Indeed, let us notice that from (7), (8) and 3.26 it follows that

$$(31)\qquad \theta_{k_0} \in \mathfrak{F}_R \ (\mathfrak{F}_C).$$

31) implies by 3.3 that

(32) $$\theta_{k_0} \in \mathfrak{F}_R^* \ (\mathfrak{F}_C^*)$$

because θ_{k_0} assumes only 0, 1 as values by (7), (9) and (17). Also supposition (10) leads to a contradiction.

3.29. *If* $\Phi \in \mathfrak{F}_R$ $(\mathfrak{F}_C)$ *and* $\{f_n\} \underset{\mathbf{R}\,(\mathbf{C})}{\longrightarrow} 0$ *and* $\{f_n\} \in \mathbf{R}$ $(\mathbf{C})$, *then* $\Phi\langle f_n\rangle \to \Phi\langle 0\rangle$.

From 3.22 it follows that there exists a function $j \in \mathbf{R}$ $(\mathbf{C})$ such that for all $x, y, n \in \mathcal{N}$:

(1) $$\text{If } n \geqslant j(x) \text{ and } y \leqslant x, \text{ then } f_n(y) = 0.$$

Let us put

(2) $$\Psi_x\langle g\rangle = \sum_{n \leqslant j(x)} \Big\{ f_n(x) \cdot g(n) \cdot \varrho\Big(\sum_{u \leqslant n} g(u)\Big)\Big\},$$

where

(3) $$\varrho(0) = \varrho(1), \quad \varrho(n+2) = 2,$$

and

(4) $$\Psi\langle g\rangle(x) = \Psi_x\langle g\rangle,$$

(5) $$x\langle g\rangle = \Phi\big\langle \Psi\langle g\rangle\big\rangle,$$

(6) $$\Omega\langle g\rangle = |x\langle g\rangle - \Phi\langle 0\rangle|.$$

From 3.10-3.15 we obtain

(7) $$\{\Psi_x\},\ x,\ \Omega \in \mathfrak{F}_R\ (\mathfrak{F}_C).$$

Now we shall prove that $\Psi_x\langle a_k\rangle = f_k(x)$. Indeed, by (3) it is evident that

(8) $$a_k(n) \cdot \varrho\Big(\sum_{u \leqslant n} a_k(n)\Big) = \begin{cases} 1 & \text{for} \quad n = k, \\ 0 & \text{for} \quad n \neq k. \end{cases}$$

From (8) and (2) it follows that

(9) $$\Psi\langle a_k\rangle = \begin{cases} f_k(x) & \text{if} \quad k \leqslant j(x), \\ 0 & \text{if} \quad k > j(x). \end{cases}$$

But from (1) we infer that

(10) $$\text{if} \quad k > j(x), \text{ then } f_k(x) = 0.$$

Hence

(11) $$\Psi_x\langle a_k\rangle = f_k(x) \quad \text{(by (9) and (10))},$$

(12) $$x\langle a_n\rangle = \Phi\langle f_n\rangle \quad \text{(by (11), (4) and (5))}.$$

From (2) it is evident that

(13) $$\Psi_x\langle 0\rangle = 0.$$

Hence

(14) $$\Omega\langle 0\rangle = |\Phi\langle 0\rangle - \Phi\langle 0\rangle| = 0 \quad \text{(by (5), (6), (13))}.$$

It follows from 3.28 that

(15) $$\Omega\langle a_n\rangle \to 0.$$

This means that

(16) $$x\langle a_n\rangle \to 0 \quad \text{(by (6) and (15))}$$

and

$$\Phi\langle f_n\rangle \to \Phi(0) \quad \text{(by (12) and (16))}.$$

3.30. *If* $\Phi \in \mathfrak{F}_R$ $(\mathfrak{F}_C)$ *and* $\{f_n\} \underset{\mathbf{R}\,(\mathbf{C})}{\longrightarrow} f$ *and* $\{f_n\} \in \mathbf{R}$ $(\mathbf{C})$, *then* $\Phi\langle f_n\rangle \to \Phi\langle f\rangle$.

We put

(1) $$g_n(x) = 2^{f_n(x) \dot{-} f(x)} \cdot 3^{f(x) \dot{-} f_n(x)} \dot{-} 1,$$

(2) $$\tau_x\langle h\rangle = \big(f(x) + \exp\big(h(x)+1, 2\big)\big) \dot{-} \exp\big(h(x)+1, 3\big),$$

where $\exp(m, k) = (\mu x)[m$ is not divisible by $k^{x+1}]$. From 3.22 we infer that there exists a function $j \in \mathbf{R}$ $(\mathbf{C})$ such that for all $x, y, n \in \mathcal{N}$:

(3) $$\text{If } n \geqslant j(x) \text{ and } y \leqslant x, \text{ then } f_n(y) = f(x).$$

Hence

(4) $$f(x) = f_{j(x)}(x).$$

It follows from (3) that $f \in \mathbf{R}$ $(\mathbf{C})$, and hence

$$\{g_n\} \in \mathbf{R}\ (\mathbf{C}) \quad \text{and} \quad \{\tau_x\} \in \mathfrak{F}_R\ (\mathfrak{F}_C).$$

From (1) and (3) it we infer that:

(5) If $n \geqslant j(x)$, and $y \leqslant x$, then $g_n(y) = 0$. This means that

(6) $$\{g_n\} \underset{\mathbf{R}\,(\mathbf{C})}{\longrightarrow} 0 \quad \text{(by (5) and 2.25)}.$$

Let us put

(7) $$\tau\langle h\rangle(x) = \tau_x\langle h\rangle,$$

(8) $$\Phi'\langle h\rangle = \Phi\big\langle \tau\langle h\rangle\big\rangle,$$

$\Phi' \in \mathfrak{F}_R$ $(\mathfrak{F}_C)$ by 3.15, whence from (6) it follows by 3.29 that

(9) $$\Phi'\langle g_n\rangle \to \Phi'\langle 0\rangle.$$

On the other hand, we infer from (1), (2) and (7) that

(10) $$\tau\langle 0\rangle(x) = f(x),$$

(11) $$\tau\langle g_n\rangle(x) = f_n(x).$$

Thus

(12) $$\Phi'\langle g_n\rangle = \Phi\langle f_n\rangle \quad \text{(by (8) and (11))},$$

(13) $$\Phi'\langle 0\rangle = \Phi\langle f\rangle \quad \text{(by (8) and (10))},$$

$$\Phi\langle f_n\rangle \to \Phi\langle f\rangle \quad \text{(by (9), (12), (13))}.$$

Let us put

3.31. $\Gamma_n\langle f\rangle(x) = \begin{cases} f(x) & \textit{for} \quad x \leqslant n, \\ 0 & \textit{for} \quad x > n, \end{cases}$

3.32. $\{\Gamma_n\langle f\rangle\} \underset{\mathbf{R}\,(C)}{\dashrightarrow} f$.

From 3.31 it follows that if $n \geqslant x$ and $y \leqslant x$, then $\Gamma_n\langle f\rangle(y) = f(y)$, whence by 3.21 we obtain $\{\Gamma_n\langle f\rangle\} \underset{\mathbf{R}\,(C)}{\longrightarrow} f$.

3.33. *If* $f \in \mathbf{R}$ (C) *and* $\Phi \in \mathfrak{F}_R$ ($\mathfrak{F}_C$), *then* $\Phi\langle\Gamma_n\langle f\rangle\rangle \to \Phi\langle f\rangle$.

From 3.31. it follows that $\{\Gamma_n\langle f\rangle\} \in \mathbf{R}$ (C), whence by 3.30 and 3.32 we obtain our theorem. Let us put

3.34. $\Delta_m(x) = \exp(m+1, p_x)$ *where* $p_x =$ *the* x*-th prime number* ($p_0 = 2$).

3.35. *For each natural function* f *and* $n \in \mathcal{N}$ *there exists an* $m \in \mathcal{N}$ *such that for all* $x \in \mathcal{N}$:

$$\Gamma_n\langle f\rangle(x) = \Delta_m(x).$$

Namely let us put

(1) $$m = 2^{f(0)} \cdot 3^{f(1)} \cdot 5^{f(2)} \cdot \ldots \cdot p_n^{f(n)} - 1;$$

it is evident by 3.34 that

(2) $$\Delta_m(x) = \begin{cases} f(x) & \text{for} \quad x \leqslant n, \\ 0 & \text{for} \quad x > n. \end{cases}$$

But (2) implies by 3.31 that $\Gamma_n\langle f\rangle(x) = \Delta_m(x)$.

3.36. *If* $\Phi \in \mathfrak{F}_C$, $f \in \mathbf{C}$, $\{f_n\} \in \mathbf{C}$ *for all* $n \in \mathcal{N}$, *and* $\{f_n\} \to f$, *then* $\Phi\langle f_n\rangle \to \Phi\langle f\rangle$.

Suppose that $\Phi\langle f_n\rangle \to \Phi\langle f\rangle$. This means that there exist an $\varepsilon = 4/(r_0+1)$ and an increasing function $s(n)$ such that

(1) $$|\Phi\langle f_{s(n)}\rangle - \Phi\langle f\rangle| \geqslant \varepsilon.$$

From 3.33 we infer that for all $n \in \mathcal{N}$

(2) $$\Phi\langle\Gamma_m\langle f_{s(n)}\rangle\rangle \to \Phi\langle f_{s(n)}\rangle.$$

This means that for all $n \in \mathcal{N}$ there exists such a $k_0 \in \mathcal{N}$ that for all $m \geqslant k_0$

(3) $$|\Phi\langle\Gamma_m\langle f_{s(n)}\rangle\rangle - \Phi\langle f_{s(n)}\rangle| < \frac{\varepsilon}{2}.$$

And from (1) and (3) we have

$$(4)\qquad \left|\Phi\langle\Gamma_m\langle f_{s(n)}\rangle\rangle-\Phi\langle f\rangle\right| \geqslant \frac{\varepsilon}{2} = \frac{2}{r_0+1}.$$

The number k_0 depends on n. Hence let us consider the function $k_0(n)$. $k_0(n)$ can be understood as an increasing function such that for all n, $m \in \mathcal{N}$

$$(5)\qquad \text{if } m \geqslant k_0(n), \text{ then } \left|\Phi\langle\Gamma_m\langle f_{s(n)}\rangle\rangle-\Phi\langle f\rangle\right| \geqslant \frac{2}{r_0+1}.$$

(5) implies that for all $n \in \mathcal{N}$

$$(6)\qquad \left|\Phi\langle\Gamma_{k_0(n)}\langle f_{s(n)}\rangle\rangle-\Phi\langle f\rangle\right| \geqslant \frac{2}{r_0+1}.$$

By theorem 3.35 it follows from (6) that there exists a function $m(n)$ such that for all $n \in \mathcal{N}$

$$(7)\qquad \left|\Phi\langle\Delta_{m(n)}\rangle-\Phi\langle f\rangle\right| \geqslant \frac{2}{r_0+1},$$

where

$$(8)\qquad m(n) = 2^{fs(n)(0)}\cdot 3^{fs(n)(1)}\cdot\ldots\cdot p_{k_0(n)}^{fs(n)(k_0(n))}-1.$$

On the other hand, from the fact that $\{f_n\} \to f$ it follows by 3.22 that there exists an increasing function $j(x)$ such that for all $x, y, n \in \mathcal{N}$:

$$(9)\qquad \text{If } n \geqslant j(x) \text{ and } y \leqslant x, \text{ then } f_n(y) = f(y).$$

From (9) we infer that $f_{s(n)}(y) = f(y)$ for $y \leqslant x$ if $n \geqslant j(x)$. From (8) it follows that $f_{s(n)}(y) = \Delta_{m(n)}(y)$ for $n \geqslant k_0(x)$ and $y \leqslant x$. Hence (8) and (9) imply

$$(10)\qquad \text{if } n \geqslant j\big(k_0(x)\big), \text{ then } \Delta_{m(n)}(y) = f(y) \text{ for } y \leqslant x.$$

Now let us consider two cases:

(α) The set M of values of the function $m(n)$ is finite: $M = \{m_1, \ldots, \ldots, m_r\}$. Then putting $m_0 = \max[m_1, \ldots, m_r]$ we infer from (8) that

$$(11)\qquad \text{if } n \geqslant m_0 \text{ and } y \geqslant m_0, \text{ then } f_{s(n)}(y) = 0.$$

Conditions (9) and (11) imply that

$$(12)\qquad \text{if } y \geqslant m_0, \text{ then } f(y) = 0.$$

Because the function j is an increasing one, then $y \geqslant m_0$ implies $j(y) \geqslant m_0$ and by (9) we have $f(y) = f_{s(j(y))}(y)$. From (9), (11) and (12) it follows that

$$(13)\qquad \text{if } n \geqslant j(m_0), \text{ then } f_{s(n)}(y) = f(y) \text{ for all } y \in \mathcal{N}.$$

The number $j(m_0)$ is constant, whence, by (13), $\{f_{s(n)}\} \in \mathbf{R}$ and $\{f_{s(n)}\} \underset{\mathbf{R}\,(C)}{\longrightarrow} f$, hence $\Phi\langle f_{s(n)}\rangle \to \Phi\langle f\rangle$ by 3.30, which contradicts supposition (1). The second possibility remains:

(β) The set of values of the function $m(n)$ is infinite. By 3.34 it is true that $\{\Delta_n\} \in \mathbf{R}$; then by 3.2 there exists a sequence $\{a_n\} \in C$ such that

$$|\Phi\langle\Delta_n\rangle - \Phi\langle f\rangle| = a_n. \tag{14}$$

By 2.1 it follows from (14) that there exists a function $g \in \mathbf{C}$ such that

$$\left||\Phi\langle\Delta_n\rangle - \Phi\langle f\rangle| - \frac{g(n,k)}{k+1}\right| < \frac{1}{k+1}. \tag{15}$$

From (15) and (7) it follows that

$$\frac{g\big(m(n),k\big)}{k+1} \geqslant \frac{1}{r_0+1} \quad \text{for} \quad k \geqslant r_0. \tag{16}$$

From (β) it follows by (10) and (16) that for all $x, u \in \mathcal{N}$ there exists such a $k \in \mathcal{N}$ that

$$k \geqslant u, \quad \frac{g(k,k)}{k+1} \geqslant \frac{1}{r_0+1}, \quad \text{and} \quad \Delta_k(y) = f(y) \quad \text{for} \quad y \leqslant x. \tag{17}$$

Hence putting

$$\begin{aligned} h(0) &= (\mu k)\left(\frac{g(k,k)}{k+1} \geqslant \frac{1}{r_0+1} \text{ and } \Delta_k(0) = f(0)\right), \\ h(x+1) &= (\mu k)\left(k > h(x), \frac{g(k,k)}{k+1} \geqslant \frac{1}{r_0+1} \text{ and } \Delta_k(y) = f(y), \right. \\ &\qquad\qquad \left. \text{for } y \leqslant x+1\right), \end{aligned} \tag{18}$$

we infer by (17) that $h \in \mathbf{C}$. From (18) h is an increasing function and, for $n \geqslant x$, $\Delta_{h(n)}(y) = f(y)$ for $y \leqslant x$, whence $\{\Delta_{h(n)}\} \underset{\mathbf{R}\,(C)}{\longrightarrow} f$ and $\{\Delta_{h(n)}\} \in \mathbf{C}$. From this we infer by 3.30 that

$$\Phi\langle\Delta_{h(n)}\rangle \to \Phi\langle f\rangle. \tag{19}$$

From (19) it follows that

$$|\Phi\langle\Delta_{h(n)}\rangle - \Phi\langle f\rangle| \to 0. \tag{20}$$

Hence there exists such an n_0 that for $n > n_0$

$$|\Phi\langle\Delta_{h(n)}\rangle - \Phi\langle f\rangle| < \frac{1}{2r_0+2}. \tag{21}$$

From (15) and (21) it follows that, for $k > 2r_0+1$,

$$\frac{g\big(h(n),k\big)}{k+1} < \frac{1}{k+1} + \frac{1}{2r_0+1} \leqslant \frac{1}{r_0+1}. \tag{22}$$

Since h is increasing, then there exists such an $n_1 > n_0$ that $h(n_1) > 2r_0+1$, and hence from (23) we have

$$\frac{g\big(h(n_1), h(n_1)\big)}{h(n_1)+1} < \frac{1}{r_0+1}. \tag{23}$$

But from (18) it follows that

$$\frac{g\big(h(n_1), h(n_1)\big)}{h(n_1)+1} \geqslant \frac{1}{r_0+1}. \tag{24}$$

Thus the supposition $\Phi\langle f_n\rangle \to \Phi\langle f\rangle$ leads to a contradiction.

3.37. *The analogon of* 3.36 *for recursive functionals* ($\mathfrak{F}_R$) *is not true.*

Let $h(x, y)$ be a recursive function and $j(x)$ a computable function such that h and j satisfy the following conditions:

(1) $$h(x, y) = 0 \text{ or } 1,$$

(2) $$\text{for all } x, \quad h(x, 0) = 0,$$

(3) for all x, there exists such a y that $h(x, y) = 1$,

(4) $$j(x) = (\mu y)\big(h(x, y) = 1\big),$$

(5) the function j increases faster than any recursive function.

The construction of a function h satisfying these conditions presents no difficulties. Let $j'(x)$ be the computable function of Ackermann, or of Robinson, which increases faster than any recursive function. The function $j'(x)+1$ can be presented in the canonical form:

$$j'(x)+1 = K\Big((\mu y)\big(h'(x, y) = 0\big)\Big), \tag{6}$$

where h is a recursive function, K is the pairing function and h satisfies the condition of effectivity, i. e. for all $x \in \mathcal{N}$ there exists such a y that $h'(x, y) = 0$. The function h defined as

$$h(x, y) = 1 \dot{-} h'(x, y) \tag{7}$$

satisfies conditions (1)-(5). (1) and (3) are evident by (7). In order to prove (2) and (5) let us notice that $Kz \leqslant z$ for all z. Hence, by (4),

$$j(x) = (\mu y)[h(x, y) = 1] \geqslant j'(x)+1 > 0. \tag{8}$$

From (8) it follows that $h(x, 0) = 0$ for all $x \in \mathcal{N}$. If $f \in \mathbf{R}$, then there exists such an $n_0 \in \mathcal{N}$ that

$$j'(x)+1 > f(x) \quad \text{for} \quad x > n_0 \tag{9}$$

because $j'(x)+1$ increases faster than any recursive function. From (9) and (8) it follows that

$$j(x) > f(x) \quad \text{for} \quad x > n_0; \tag{10}$$

condition (5) is also satisfied. Put

$$\Phi\langle f\rangle = \sum_{x=0}^{\infty} h\big(x, f(x)\big) \quad \text{for} \quad f \in \mathbf{R}. \tag{11}$$

We shall prove that the functional Φ_0 defined over the set of recursive functions is recursive. First notice that $\Phi_0\langle f\rangle$ is a well-defined number if $f \in \mathbf{R}$. Namely from (4) it follows that $h(x, y) = 0$ for $y < j(x)$. Hence from (10) and (4) it follows that for $x > n_0$ we have $h\big(x, f(x)\big) = 0$. Thus for each $f \in \mathbf{R}$ there exists such an n_0 that

$$\Phi_0\langle f\rangle = \sum_{x=0}^{n_0} h\big(x, f(x)\big). \tag{12}$$

Let $\{f_k\} \in \mathbf{R}$; putting

$$g(x) = \max[f_k(x)] \quad \text{for} \quad k \leqslant x, \tag{13}$$

we infer from (10) that there exists an $n_0 \in \mathcal{N}$ such that

$$g(x) < j(x) \quad \text{for} \quad x \geqslant n_0. \tag{14}$$

From (13) and (14) it follows that

$$f_k(x) < j(x) \quad \text{for} \quad x \geqslant n_0 \text{ and } x \geqslant k. \tag{15}$$

Hence from (4) we have $h\big(x, f_k(x)\big) = 0$ for $x \geqslant \max[n_0, k]$. Thus from (4) and (15) it follows that

$$\Phi_0\langle f_k\rangle = \sum_{x=0}^{\max[n_0,k]} h\big(x, f_k(x)\big), \tag{16}$$

whence $\Phi_0\langle f_k\rangle$ is a recursive function (a recursive sequence of natural numbers) because the operation of limited summation does not lead out of the class $\mathbf{R}$. Also $\Phi_0 \in \mathfrak{F}_R$.

Let f_k be the following sequence of recursive functions:

$$f_k(x) = \begin{cases} j(k) & \text{for} \quad k = x, \\ 0 & \text{for} \quad k \neq x. \end{cases} \tag{17}$$

It is evident that $\{f_k\} \underset{\mathbf{R}}{\to} 0$. $\Phi_0\langle 0\rangle = 0$ according to condition (2). From (17), (2), (3) and (4) it follows that

$$h\big(x, f_k(x)\big) = \begin{cases} 1 & \text{for} \quad x = k, \\ 0 & \text{for} \quad x \neq k. \end{cases} \tag{18}$$

From (18) and (11) we infer that

$$\Phi_0\langle f_k\rangle = 1 \quad \text{for all} \quad k \in \mathcal{N}. \tag{19}$$

Hence $\Phi_0\langle f_k\rangle \to 1$. Thus it is not true that $\Phi_0\langle f_k\rangle \to \Phi\langle 0\rangle$, and the functional Φ_0 cannot be extended to a computable functional.

3.38. *If $\Phi \in \mathfrak{F}_C^*$ and $f \in \mathbf{C}$, then there exists a number $n \in \mathcal{N}$ such that, if, for all $g \in \mathbf{C}$, $g(u) = f(u)$ for $u \leqslant n$, then*

$$\Phi\langle f\rangle = \Phi\langle g\rangle.$$

This theorem means that the value of Φ depends on a finite number of values of function f. (The class $\mathfrak{F}_R^*$ does not have this property. The functional Φ_0 of 3.37 is a counter-example.)

Suppose that the theorem is not true. Hence for each number n there exists a $g \in \mathbf{C}$ such that

$$(1) \qquad g(u) = f(u) \quad \text{for} \quad u \leqslant n \text{ and } \Phi\langle f\rangle = \Phi\langle g\rangle.$$

Let $G_n(x)$ be the universal function for the class of computable functions of one argument. We set $G(n, x) = G_n(x)$ and

$$(2) \quad g_n(x) = G\Big((\mu z)\big(G_z(u) = f(u) \text{ for } u \leqslant n, \text{ and } \Phi\langle f\rangle \neq \Phi\langle G_z\rangle\big), x\Big).$$

From (2) it follows that $g_n \in \mathbf{C}$ for each n. From (1) and (2) by 3.22 we infer that $\{g_n\} \to f$. Hence using theorem 3.36 we find that

$$(3) \qquad \Phi\langle g_n\rangle \to \Phi\langle f\rangle.$$

But this means that there exists such a z that $\Phi\langle g_y\rangle = \Phi\langle f\rangle$ for $y \geqslant z$. On the other hand, from (1) and (2) it follows that $\Phi\langle g_y\rangle \neq \Phi\langle f\rangle$ for each y. We obtain a contradiction.

Let us set

3.39. $\Lambda_n\langle f\rangle = \big(\prod_{k\leqslant n} p_k^{f(k)}\big) \dot{-} 1.$

3.40. *For each $\Phi \in \mathfrak{F}_C$ ($\mathfrak{F}_R$) there exists a sequence $\{a_k\} \in C$ ($\boldsymbol{R}$) such that*

$$(0) \qquad \Phi\langle f\rangle = \lim a_{\Lambda_n\langle f\rangle}.$$

If $\Phi \in \mathfrak{F}_C$ ($\mathfrak{F}_R$), then by 3.34 and 3.2 there exists a sequence $\{a_k\} \in C(\boldsymbol{R})$ such that

$$(1) \qquad \Phi\langle \Delta_m\rangle = a_m.$$

From 3.31, 3.34 and 3.39 it follows that for any $x \in \mathcal{N}$

$$(2) \qquad \Delta_{\Lambda_n\langle f\rangle}(x) = \Gamma_n\langle f\rangle(x).$$

From (1) and (2) we obtain

$$(3) \qquad \Phi\langle \Gamma_n\langle f\rangle\rangle = a_{\Lambda_n\langle f\rangle}.$$

From (3) and 3.33 it follows that (0) is true.

Theorem 3.40 establishes a mapping of a subclass of the class of computable sequences into the class $\mathfrak{F}_R$ ($\mathfrak{F}_C$). Hence the class $\mathfrak{F}_R$ ($\mathfrak{F}_C$) is enumerable, because the class of computable sequences as well as the class of computable functions is also enumerable. This mapping is a one-to-one correspondence. Suppose that $a_k \neq b_k$ and

$$\Phi\langle f\rangle = \lim a_{\Lambda_n\langle f\rangle}, \qquad \Psi\langle f\rangle = \lim b_{\Lambda_n\langle f\rangle}. \tag{4}$$

From 3.34 and 3.39 it follows that, for $n > k$,

$$\Lambda_n\langle\Delta_k\rangle = k. \tag{5}$$

(4) and (5) imply that $\Phi\langle\Delta_k\rangle = a_k \neq b_k = \Psi\langle\Delta_k\rangle$, whence $\Phi \neq \Psi$.

From 3.40 it follows that each computable functional can be extended to a functional defined by means of the quantifiers bounding only the number variables, i. e. to an elementarily definable functional(1).

(1) For other notions of recursive (computable) functionals see S. C. Kleene [24]–[26], G. Kreisel [28], K. Gödel [15] and A. Grzegorczyk [17], [20]. The main problem concerning the relation between the Mazur functionals and the others was solved by Friedberg in [5].

IV. RECURSIVE AND COMPUTABLE REAL FUNCTIONS

This section is devoted to the study of the properties of recursive and computable real functions. A real function, defined on a set of recursive (computable) numbers, is said to be *recursive* (*computable*) if it transforms any recursive (computable) sequence into a recursive (computable) one.

The mains result are as follows. The four arithmetical operation performed on computable functions do not lead out of the class of computable functions (4.9, 4.10). Likewise, the operations of addition, multiplication and subtraction do not lead out of the class of recursive functions (4.9). However, the function $1/x$ is not recursive on the set of all positive recursive numbers (4.15). A sufficient condition under which the operation of division does not lead out of the set of recursive functions if formulated in 4.16.

The functions "signum x" and "integer-part of x" considered even on the set of all recursive numbers of the closed interval $\langle 0, 1\rangle$ are not computable (4.18, 4.19). On the other hand, "signum x" is computable (but not recursive) on the set of all computable (recursive) numbers different from 0 (4.20, 4.23). Likewise, the function "integer-part of x" is computable (but not recursive) on the set of all computable (recursive) non-integers (4.21, 4.22).

A function defined by a power-series with a recursive (computable) sequence of coefficients and with a recursive (computable) centre, considered on the set of all recursive (computable) numbers from the interval of convergence of this series, is not necessarily recursive (computable) (4.24). On the other hand, it is recursive (computable) on a set of all recursive (computable) numbers from an arbitrary closed interval contained in the interval of convergence (4.25, 4.6).

The most interesting results concern computable functions. Namely, it is proved that a computable function on a set of all computable numbers of an interval is continuous (4.33) and has the property of Darboux (4.32).

Finally, the reciprocal dependence of the notion of recursive real function and the same notion introduced by Specker is investigated, it

is shown that a function is recursive in the sense of Specker if and only if it is recursive and recursively uniformly continuous (4.38, 4.39). Consequently, recursive functions do not necessarily have the property of Darboux.

We shall consider functions whose arguments run over some subset of the set $\mathscr{R}$ ($\mathscr{C}$) of all recursive (computable) numbers.

The letter Δ will exclusively denote an interval and the symbol $\Delta\mathscr{R}$ ($\mathscr{C}$) the set of all recursive (computable) numbers of an interval Δ.

The letters $\mathscr{R}_0$ ($\mathscr{C}_0$), $\mathscr{R}_1$ ($\mathscr{C}_1$), ... will always denote arbitrary subsets of the set $\mathscr{R}$ ($\mathscr{C}$).

4.1. *Let $\varphi(x)$ be a function defined on an arbitrary subset $\mathscr{R}_0$ ($\mathscr{C}_0$) of the set $\mathscr{R}$ ($\mathscr{C}$). The function $\varphi(x)$ is called recursive (computable) if for every recursive (computable) sequence $\{a_k\}$ such that $a_k \in \mathscr{R}_0$ ($\mathscr{C}_0$) the sequence $\{\varphi(a_k)\}$ is also recursive (computable), i. e. the conditions*

(i) $\{a_k\} \in \boldsymbol{R}$ $(\boldsymbol{C})$,

(ii) $a_k \in \mathscr{R}_0$ ($\mathscr{C}_0$) *for* $k \in \mathscr{N}$,

imply $\{\varphi(a_k)\} \in \boldsymbol{R}$ $(\boldsymbol{C})$.

Then we shall write

$$\varphi \in \mathfrak{R}\ (\mathfrak{C}).$$

4.2. *The values of a recursive (computable) function are recursive (computable) numbers.*

This follows immediately from definition 4.1 and from theorems 2.6 and 2.4.

The following two remarks are direct consequences of definition 4.1.

4.3. *The superposition of two recursive (computable) functions is also a recursive (computable) one.*

4.4. *If $\varphi(x)$ is a recursive (computable) function defined on $\mathscr{R}_0$ ($\mathscr{C}_0$) and $\mathscr{N} \subset \mathscr{R}_0$ ($\mathscr{C}_0$), then the sequence $\{a_k\} = \{\varphi(k)\} \in \boldsymbol{R}$ $(\boldsymbol{C})$.*

4.5. *If $\{a_k\} \in \boldsymbol{R}$ $(\boldsymbol{C})$, then the function $\varphi(k) = a_k$ is recursive (computable) on $\mathscr{N}$.*

This follows from 4.1 and 2.44.

4.6. *Let $\varphi(x)$ be a function defined on a set $\Delta\mathscr{R}$ ($\mathscr{C}$). Let $a \in \Delta\mathscr{R}$ ($\mathscr{C}$), and let $\mathscr{X}_1 = \underset{x}{E}\left[(x \in \Delta\mathscr{R}\ (\mathscr{C})) \wedge x \leqslant a\right]$ and $\mathscr{X}_2 = \underset{x}{E}\left[(x \in \Delta\mathscr{R}\ (\mathscr{C})) \wedge x \geqslant a\right]$. If $\varphi(x)$ is recursive (computable) on $\mathscr{X}_1$ and $\mathscr{X}_2$, then $\varphi(x)$ is recursive (computable) on $\Delta\mathscr{R}$ ($\mathscr{C}$).*

Let $\{a_k\} \in \boldsymbol{R}$ $(\boldsymbol{C})$ and $a_k \in \Delta\mathscr{R}$ ($\mathscr{C}$) for $k \in \mathscr{N}$.

Obviously,

$$\varphi(a_k) = \varphi\big(\max(a_k, a)\big) + \varphi\big(\min(a_k, a)\big) - \varphi(a). \tag{1}$$

On the other hand,

$$\max(a_k, a) = \frac{a_k + a + |a_k - a|}{2} \tag{2}$$

and

$$\min(a_k, a) = \frac{a_k + a - |a_k - a|}{2}. \tag{3}$$

Since $\{a_k\} \in \boldsymbol{R}\ (\boldsymbol{C})$, we infer by (2), (3) that

$$\{\max(a_k, a)\} \in \boldsymbol{R}\ (\boldsymbol{C}) \quad \text{and} \quad \{\min(a_k, a)\} \in \boldsymbol{R}\ (\boldsymbol{C}). \tag{4}$$

Clearly

$$\max(a_k, a) \in \mathscr{Z}_2, \tag{5}$$

$$\min(a_k, a) \in \mathscr{Z}_1. \tag{6}$$

Since $\varphi(x)$ is recursive (computable) on $\mathscr{Z}_1$ and $\mathscr{Z}_2$, we have on account of (4), (5), (6)

$$\{\varphi(\max(a_k, a))\} \in \boldsymbol{R}\ (\boldsymbol{C}), \tag{7}$$

$$\{\varphi(\min(a_k, a))\} \in \boldsymbol{R}\ (\boldsymbol{C}), \tag{8}$$

$$\{\varphi(a)\} \in \boldsymbol{R}\ (\boldsymbol{C}). \tag{9}$$

Thus, by (1), (7), (8), (9) and 2.7, 2.11

$$\{\varphi(a_k)\} \in \boldsymbol{R}\ (\boldsymbol{C}),$$

which completes the proof.

4.7. *If $c \in \mathscr{R}$ ($\mathscr{C}$), then the function $\varphi(x) = c$ is recursive (computable).*

This follows from 4.1 and 2.6. By 4.1 we obtain

4.8. *Let $\varphi(x) = x$ for $x \in \mathscr{R}$ ($\mathscr{C}$). Then $\varphi(x) \in \mathfrak{R}$ ($\mathfrak{C}$).*

4.9. *Let $\varphi(x)$ and $\psi(x)$ be two recursive (computable) functions on $\mathscr{R}_0$ ($\mathscr{C}_0$). Then, on $\mathscr{R}_0$ ($\mathscr{C}_0$)*

(i) $\varphi(x) + \psi(x) \in \mathfrak{R}$ ($\mathfrak{C}$),

(ii) $\varphi(x) \cdot \psi(x) \in \mathfrak{R}$ ($\mathfrak{C}$),

(iii) $\varphi(x) - \psi(x) \in \mathfrak{R}$ ($\mathfrak{C}$),

(iv) $|\varphi(x)| \in \mathfrak{R}$ ($\mathfrak{C}$).

Theorem 4.9 follows from 4.1 and 2.7, 2.10, 2.11, 2.8. By 4.1 and 2.18 we obtain

4.10. *Let $\varphi(x)$ and $\psi(x)$ be two computable functions on $\mathscr{C}_0$ and $\psi(x) \neq \neq 0$. Then $\varphi(x)/\psi(x)$ is also computable on $\mathscr{C}_0$.*

The following results immediately from 4.7, 4.8 and 4.9:

4.11. *Let $\varphi(x)$ be a polynomial whose coefficients are recursive (computable) numbers. Then*

$$\varphi(x) \in \mathfrak{R}\ (\mathfrak{C}) \quad \textit{in} \quad \mathscr{R}\ (\mathscr{C}).$$

4.12. *Let $x_0, x_1, y_0, y_1 \in \mathscr{R}$ ($\mathscr{C}$) and $x_0 < x_1$. Then the straight line passing through the points $(x_0 y_0)$ and $(x_1 y_1)$ is a recursive (computable) function on the set $\mathscr{R}$ ($\mathscr{C}$).*

Indeed, on account of 1.3 and 4.11 the function

$$\varphi(x) = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) + y_0$$

is recursive (computable) on $\mathscr{R}$ ($\mathscr{C}$).

4.13. *A polygonal line formed from a finite number of segments whose points have recursive (computable) coordinates is a recursive (computable) function.*

This follows from 4.12 and 4.1.

4.14. *Let $\varphi(x) = \psi(x)/\theta(x)$, where $\psi(x)$ and $\theta(x)$ are polynomials with computable coefficients and $\theta(x) \neq 0$ for $x \in \mathscr{C}_0$. Then $\varphi(x)$ is a computable function on $\mathscr{C}_0$.*

By 4.11 and 4.10.

4.15. *The function $\varphi(x) = 1/x$ is not recursive on the set $\mathscr{R}_p$ of all positive recursive numbers.*

In fact, it follows by 2.12 and 2.13 that there exists a sequence $\{a_k\} \in \boldsymbol{R}$ such that $a_k \in \boldsymbol{R}_p$ and the sequence $\{1/a_k\} \notin \boldsymbol{R}$. This proves the theorem.

4.16. *Let $\varphi(x) = \psi(x)/\theta(x)$, where $\psi(x)$ and $\theta(x)$ are recursive functions on $\mathscr{R}_0$, $\theta(x) \neq 0$ and $\inf\limits_{\mathscr{R}_0} |\theta(x)| > 0$. Then $\varphi(x) \in \mathfrak{R}$ on $\mathscr{R}_0$.*

Indeed, let $\{a_k\} \in \boldsymbol{R}$ and $a_k \in \mathscr{R}_0$. By 4.1 $\{\psi(a_k)\}, \{\theta(a_k)\} \in \boldsymbol{R}$. Obviously, $|\theta(a_k)| \geqslant \inf\limits_{\mathscr{R}_0} |\theta(x)| > 0$, Hence, by 2.16, $\{1/\theta(a_k)\} \in \boldsymbol{R}$ and, by 2.10, $\{\psi(a_k)/\theta(a_k)\} \in \boldsymbol{R}$. This completes the proof.

The following theorem is an immediate consequence of 4.16:

4.17. *Let $\varphi(x) = \psi(x)/\theta(x)$, where $\psi(x)$ and $\theta(x)$ are polynomials with recursive coefficients, and $\inf\limits_{\mathscr{R}_0} |\theta(x)| > 0$. Then $\varphi(x) \in \mathbf{R}$ on $\mathscr{R}_0$.*

4.18. *The function $\operatorname{sgn} x \notin \mathfrak{C}$ even on the set $\langle 0, 1 \rangle \mathscr{R}$.*

In fact, on account of 2.28 there exists a sequence $\{a_k\} \in \boldsymbol{R}$ such that $a_k \in \langle 0, 1 \rangle \mathscr{R}$ for $k \in \mathscr{N}$ and $\{\operatorname{sgn} a_k\} \notin \boldsymbol{C}$.

4.19. *The function $[x] \notin \mathfrak{C}$, even on the set $\langle 0, 1 \rangle \mathscr{R}$.*

Indeed, by 2.30 there exists a sequence $\{a_k\} \in \boldsymbol{R}$ such that $a_k \in \langle 0, 1 \rangle \mathscr{R}$ for $k \in \mathscr{N}$, and $\{[a_k]\} \notin \boldsymbol{C}$.

4.20. *The function $\operatorname{sgn} x$ is computable on the set of all computable numbers $x \neq 0$.*

In fact, we have $\operatorname{sgn} x = |x|/x$ for $x \neq 0$. Thus 4.20 follows from 4.8, 4.9 and 4.10.

4.21. *The function* $[x]$ *is computable on the set* $\mathscr{C}_p$ *of all computable numbers* x *which are not integers.*

Indeed, we have for every $x \in \mathscr{C}_0$

$$[|x|] = (\mu m)\big(\operatorname{sgn}(m+1-|x|) = 1\big), \quad \text{where} \quad m \in \mathscr{N}. \tag{1}$$

Simultaneously, for every number $x \in \mathscr{C}_0$,

$$[x] = \operatorname{sgn} x\,[|x|] + \frac{\operatorname{sgn} x - 1}{2}. \tag{2}$$

Let $\{a_k\}$ be an arbitrary computable sequence such that $a_k \in \mathscr{C}_0$. We shall show that the relation

$$R \overset{\mathrm{df}}{=} (\hat{m}, \hat{k})\big(\operatorname{sgn}(m+1-|a_k|) = 1\big) \tag{3}$$

is computable.

Since $\{a_k\} \in C$, by 2.9 and 2.2 (ii), there exists a computable function $f(k, n)$ such that

$$\left| |a_k| - \frac{f(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for every} \quad k, n \in \mathscr{N}. \tag{4}$$

Thus

$$\frac{f(k,n)-1}{n+1} < |a_k| < \frac{f(k,n)+1}{n+1} \quad \text{for every} \quad k, n \in \mathscr{N}. \tag{5}$$

It is clear, on account of $a_k \in \mathscr{C}$, that for every k there exists such an $n(k)$ that

$$\left[\frac{f(k,n)-1}{n+1}\right] = \left[\frac{f(k,n)+1}{n+1}\right].$$

Let

$$n_0(k) = (\mu n)\left\{\left[\frac{f(k,n)-1}{n+1}\right] = \left[\frac{f(k,n)+1}{n+1}\right]\right\}. \tag{6}$$

Thus for every k we have

$$[|a_k|] = \left[\frac{f\big(k, n_0(k)\big)}{n_0(k)+1}\right]. \tag{7}$$

It is evident by (3) that $mRk \equiv m+1 > [|a_k|]$. Consequently, by (7),

$$mRk \equiv m+1 > \left[\frac{f\big(k_1, n_0(k)\big)}{n_0(k)+1}\right]. \tag{8}$$

This equivalence proves that the relation R is computable. Hence, the function

$$g(k) = (\mu m)\, mRk = (\mu m)\big(\operatorname{sgn}(m+1-|a_k|) = 1\big).$$

Consequently, the sequence $\{[|a_k|]\} = \{(\mu m)(\operatorname{sgn}(m+1-|a_k|) = 1)\}$ is computable. Hence, by 4.1 the function $[|x|]$ is computable on $\mathscr{C}_0$. Thus, by (2), 4.20, 4.7, 4.9, the function $[x]$ is computable on the set $\mathscr{C}_0$.

4.22. *The function* $[x]$ *is not recursive on the set* $\mathscr{R}_0$ *of all recursive numbers which are not integers.*

In fact, on account of 1.13 there exists such a number $a \in \mathscr{R}$ that the function $[10^k \cdot a]$ is not recursive. By 2.5 and 2.10 the sequence $\{a_k\} = \{10^k \cdot a\}$ is recursive and $a_k \in \mathscr{R}_0$. Since the function $[10^k \cdot a]$ is not recursive, it follows from 2.5 that the sequence $\{[a_k]\}$ is not recursive. This completes the proof.

4.23. *The function* $\operatorname{sgn} x$ *is not recursive on the set* $\mathscr{R}_1$ *of all recursive numbers* $x \neq 0$.

Suppose that $\operatorname{sgn} x$ is a recursive function on $\mathscr{R}_1$. Consider the sequence $\{a_k\} = \{10^k \cdot a\}$ of 4.22. Obviously $a_k \in \mathscr{R}_1$. Moreover, $\{a_k\} \in \boldsymbol{R}$, the sequence $\{[a_k]\}$ is nevertheless not recursive. Let

$$R = (\hat{m}, \hat{k})(\operatorname{sgn}(m+1-a_k) = 1). \tag{1}$$

We shall show that R is a recursive relation. Let $k(n)$, $L(n)$ and $J(m, k)$ be the pairing functions of Robinson. Since $\{a_k\} \in \boldsymbol{R}$, the sequence $\{a_{L(n)}\} \in \boldsymbol{R}$ and consequently the sequence $\{K(n)+1-a_{L(n)}\} \in \boldsymbol{R}$. Moreover, $k(n)+1-a_{L(n)} \in \mathscr{R}_1$. The function $\operatorname{sgn} x$ being recursive on $\mathscr{R}_1$, we infer that

$$\{g(n)\} = \{\operatorname{sgn}(K(n)+1-a_{L(n)})\} \in \boldsymbol{R}. \tag{2}$$

Let $g^*(n) = \operatorname{sgn}(K(n)+1 \dot{-} a_{L(n)})$. Evidently

$$g^*(n) = g(n) - \frac{g(n)-1}{2}. \tag{3}$$

Hence $\{g^*(n)\} \in \boldsymbol{R}$. Since the arguments of $g^*(n)$ run over $\mathscr{N}$ and the values belong to $\mathscr{N}$, it follows from 2.5 that $g^*(n) \in \mathbf{R}$. In consequence $g^*(J(m, k)) \in \mathbf{R}$, that is

$$g^*(J(m, k)) = \operatorname{sgn}(K(J(m, k))+1 \dot{-} a_{L(J(m,k))}) = \operatorname{sgn}(m+1 \dot{-} a_k) \in \mathbf{R}. \tag{4}$$

It is easy to see that

$$mRk \equiv (\operatorname{sgn}(m+1 \dot{-} a_k) = 1). \tag{5}$$

On account of (4) and (5) the relation R is recursive. On the other hand, we have

$$[a_k] = (\mu m)((\operatorname{sgn}(m+1-a_k) = 1) \wedge (m < k)).$$

Thus the function $f(k) = [a_k]$ is recursive and $\{[a_k]\} \in \boldsymbol{R}$, contrary to our supposition.

The next two theorems concern functions defined by power-series with a recursive (computable) sequence of coefficients and with a recursive (computable) centre.

4.24. *There exists a function*

$$\varphi(x) = a_0 + a_1(x-a) + a_1(x-a)^2 + \ldots$$

such that $a \in \mathscr{R}$, $\{a_k\} \in \boldsymbol{R}$ *and* $\varphi(x) \notin \mathfrak{R}$ *on the set* $(-R, R)\mathscr{R}$, where

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}}.$$

Consider the function

$$\varphi(x) = 1 + x + x^2 + x^3 + \ldots$$

In this case $R = 1$ and $\varphi(x) = 1/(1-x)$ for x satysfying the conditions $|x| < 1 = R$ and $x \in \mathscr{R}$. We shall prove that the function $\varphi(x) = 1/(1-x)$ is not recursive for $|x| < 1$ and $x \in \mathscr{R}$. It is easy to show in a similar way to the used in 2.12 that there exists a recursive function $g(m, n)$ which satisfies conditions (i)-(v) of 2.12 and, moreover, the condition

(vi) $\qquad g(m, 1) = 1 \quad$ for every $\quad m \in \mathscr{N}$.

It follows from (vi) and (v) that the function $f(m) = (\mu x)(g(m, x) = 0)$ is not recursive, $f(m) \in \mathbf{C}$ and $f(m) > 1$. By 2.5 $\{f(k)\} \notin \boldsymbol{R}$. In the same way to that used in 2.12 it can be proved that $\{1/f(k)\} \in \boldsymbol{R}$. Obviously, $0 < 1/f(k) < 1$. Putting $\{a_k\} = \{1 - 1/f(k)\}$ we have

(i) $\{a_k\} \in \boldsymbol{R}$,

(ii) $0 < a_k < 1$,

(iii) $\{\varphi(a_k)\} = \{1/(1-a_k)\} = \{f(k)\} \notin \boldsymbol{R}$

which completes the proof.

4.25. *If* $\{a_k\} \in \boldsymbol{R}$ (C) *and* $a \in \mathscr{R}$ $(\mathscr{C})$, *then the function*

$$\varphi(x) = a_0 + a_1(x-a) + a_2(x-a) + \ldots$$

is recursive (computable) on the set of all recursive (computable) numbers x *of an interval* $|x-a| \leqslant r$, *where* r *are arbitrary numbers satisfying the condition*

$$0 < r < \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}}.$$

Obviously we have

(1) $$\left|\varphi(x) - \left(a_0 + a_1(x-a) + \ldots + a_n(x-a)^n\right)\right| \leqslant |a_{n+1}| \cdot r^{n+1} + |a_{n+2}| r^{n+2} + \ldots$$

Let $\varepsilon > 0$ be such a number that $r+\varepsilon < R$. The series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $|x-a| < R$. Hence the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$, in particular, for $x = r+\varepsilon$. Thus the sequence $|a_k|(r+\varepsilon)^k$ is limited. Let

$$|a_k|(r+\varepsilon)^k \leqslant n_0 \quad \text{for} \quad k \in \mathcal{N}, \tag{2}$$

that is

$$|a_n| \leqslant \frac{n_0}{(r+\varepsilon)^n}. \tag{3}$$

By (1) and (3)

$$\begin{aligned} &\left|\varphi(x)-\left(a_0+a_1(x-a)+\ldots+a_n(x-a)^n\right)\right| \\ &\leqslant n_0\left(\frac{1}{1-r/(r+\varepsilon)}-\frac{1-\left(r/(r+\varepsilon)\right)^{n+1}}{1-r/(r+\varepsilon)}\right) = n_0\cdot\frac{r+\varepsilon}{\varepsilon}\cdot\frac{1}{(1+\varepsilon/r)^{n+1}}. \end{aligned} \tag{4}$$

Making use of the inequality of Bernoulli we obtain

$$\begin{aligned} \left|\varphi(x)-\left(a_0+a_1(x-a)+\ldots+a_n(x-a)^n\right)\right| &\leqslant n_0\cdot\frac{r+\varepsilon}{\varepsilon}\cdot\frac{1}{1+(n+1)\dfrac{\varepsilon}{r}} \\ &< n_0\cdot\frac{r+\varepsilon}{\varepsilon}\cdot\frac{r}{(n+1)\varepsilon} < \frac{q}{n+1}, \end{aligned} \tag{5}$$

where q is a non-negative integer. Let $\{a_k\}$ be an arbitrary recursive (computable) sequence such that

$$|a_k-a| \leqslant r. \tag{6}$$

Then by (5)

$$\left|\varphi(a_k)-\sum_{p=0}^{n} a_p(a_k-a)^p\right| < \frac{q}{n+1}. \tag{7}$$

Now we shall show that there exist three functions f_1^*, f_2^* and $g \in \mathbf{R}$ (C) such that

$$\left|\sum_{p=0}^{n} a_p(a_k-a)^p-\frac{f_1^*(k,n)-f_2^*(k,n)}{g(k,n)+1}\right| < \frac{1}{n+1}$$

for $k, n \in \mathcal{N}$. Since $\{a_k\} \in \boldsymbol{R}$ (C), there exist three functions $f', f'', g' \in \mathbf{R}$ (C) such that

$$\left|a_k-\frac{f'(k,n)-f''(k,n)}{g'(k,n)+1}\right| < \frac{1}{n+1} \quad \text{for every } k, n \in \mathcal{N}. \tag{8}$$

Since $\{a_k\} \in \boldsymbol{R}$ (C), the sequence $\{a_k - a\}$ is also a recursive (computable) one. Thus there exist three functions $f_1, f_2, g_1 \in \mathbf{R}$ $(\mathbf{C})$ such that

$$(9) \qquad \left| a_k - a - \frac{f_1(k,n) - f_2(k,n)}{g_1(k,n)+1} \right| < \frac{1}{n+1} \quad \text{for } k, n \in \mathcal{N}.$$

It follows from (8) that

$$(10) \qquad |a_p| < 1 + |f'(p,0) - f''(p,0)|.$$

On the other hand, (8) implies that

$$(11) \qquad \left| \frac{f'(p,n) - f''(p,n)}{g'(p,n)+1} \right| < |a_p| + \frac{1}{n+1} \quad \text{for every } p, n \in \mathcal{N}.$$

By (10) and (11) we have

$$(12) \qquad \left| \frac{f'(p,n) - f''(p,n)}{g'(p,n)+1} \right| < 2 + |f'(p,0) - f''(p,0)|.$$

It follows from (9) that

$$(13) \qquad |a_k - a| < 1 + |f_1(k,0) - f_2(k,0)|$$

and consequently

$$(14) \qquad \left| \frac{f_1(k,n) - f_2(k,n)}{g_1(k,n)+1} \right| < 2 + |f_1(k,0) - f_2(k,0)|.$$

For every real number a, x, $\bar{a}$, $\bar{x}$, and a non-negative integer p we have

$$(15) \qquad |ax^p - \bar{a}\bar{x}^p| \leqslant |ax^p - \bar{a}x^p| + |\bar{a}x^p - \bar{a}\bar{x}^p| \leqslant |a - \bar{a}| \cdot |x|^p + |\bar{a}| \cdot |x^p - \bar{x}^p|.$$

Making use of the theorem on the mean value we obtain from (15)

$$(16) \qquad |ax^p - \bar{a}\bar{x}^p| \leqslant |a - \bar{a}| \cdot |x|^p + p\,|\bar{a}| \cdot |x - \bar{x}| \cdot |\bar{\bar{x}}|^{p-1}$$

where $\bar{\bar{x}}$ is a number between x and $\bar{x}$.

Putting $a = a_p$, $\bar{a} = \dfrac{f'(p,n) - f''(p,n)}{g'(p,n)+1}$, $x = a_k - a$ and $\bar{x} = \dfrac{f_1(k,n) - f_2(k,n)}{g_1(k,n)+1}$ we have by (16), (8), (13), (12), (9), (13) and (14)

$$(17) \qquad \left| a_p(a_k - a)^p - \frac{f'(p,n) - f''(p,n)}{g'(p,n)+1} \cdot \left(\frac{f_1(k,n) - f_2(k,n)}{g_1(k,n)+1} \right)^p \right|$$

$$\leqslant \frac{1}{n+1} \cdot \big(1 + |f_1(k,0) - f_2(k,0)|\big)^p +$$

$$+ \big(2 + |f'(p,0) - f''(p,0)|\big) \cdot \frac{p}{n+1} \cdot \big(2 + |f_1(k,0) - f_2(k,0)|\big).$$

Let

$$(18)\qquad o(k,p) = \big(1+|f_1(k,0)-f_2(k,0)|\big)^p + \\ +p\big(2+|f'(p,0)-f''(p,0)|\big)\big(2+|f_1(k,0)-f_2(k,0)|\big).$$

For every $p \in \mathcal{N}$ let O_p and E_p denote the sets of all odd non-negative integers $i \leqslant p$ and the set of all even non-negative integers $j \leqslant p$, respectively.

We shall define three functions $\bar{f}_1, \bar{f}_2$ and $\bar{g}$ as follows:

$$(19)\qquad \bar{f}_1(p,k,n) = f'(p,n)\cdot \sum_{i\in E_p}\binom{p}{i} f_1(k,n)^{p-i} f_2(k,n)^i + \\ + f''(p,n)\cdot \sum_{i\in O_p}\binom{p}{i} f_1(k,n)^{p-i} f_2(k,n)^i,$$

$$(20)\qquad \bar{f}_2(p,k,n) = f'(p,n)\cdot \sum_{i\in O_p}\binom{p}{i} f_1(k,n)^{p-i}\cdot f_2(k,n)^i + \\ + f''(p,n) \sum_{i\in E_p}\binom{p}{i} f_1(k,n)^{p-i} f_2(k,n)^i,$$

$$(21)\qquad \bar{g}(p,k,n) = g'(p,n)\cdot g_1(k,n) + g'(p,n) + g_1(k,n).$$

Obviously, $o, \bar{f}_1, \bar{f}_2, \bar{g} \in \mathbf{R}$ (C) and we have by (17), (18), (19,) (20), (21)

$$(22)\qquad \left| a_p(a_k-a)^p - \frac{\bar{f}_1(p,k,n)-\bar{f}_2(p,k,n)}{\bar{g}(p,k,n)+1}\right| < \frac{o(k,p)}{n+1}.$$

Putting

$$\bar{\bar{f}}_1(p,k,n) = \bar{f}_1\big(p,k,o(k,p)\cdot(n+1)^2\big),$$
$$\bar{\bar{f}}_2(p,k,n) = \bar{f}_2\big(p,k,o(k,p)\cdot(n+1)^2\big),$$
$$\bar{\bar{g}}(p,k,n) = \bar{g}\big(p,k,o(k,p)\cdot(n+1)^2\big)$$

we obtain by (22)

$$(23)\qquad \left| a_p(a_k-a)^p - \frac{\bar{\bar{f}}_1(p,k,n)-\bar{\bar{f}}(p,k,n)}{\bar{\bar{g}}(p,k,n)+1}\right| < \frac{1}{(n+1)^2},$$

where $\bar{\bar{f}}_1, \bar{\bar{f}}_2, \bar{\bar{g}} \in \mathbf{R}$ (C). Hence

$$(24)\qquad \left| \sum_{p=0}^{n} a_p(a_k-a)^p - \sum_{p=0}^{n} \frac{\bar{\bar{f}}_1(p,k,n)-\bar{\bar{f}}(p,k,n)}{\bar{\bar{g}}(p,k,n)+1}\right| < \frac{1}{n+1}.$$

Let

$$g(k, n) = \prod_{p=0}^{n} \left(\bar{g}(p, k, n)+1\right) \dot{-} 1,$$

$$f_1^*(k, n) = \sum_{p=0}^{n} \left(\bar{f}_1(p, k, n) \prod_{\substack{i=1 \\ i \neq p}} \left(\bar{g}(i, k, n)+1\right)\right),$$

$$f_2^*(k, n) = \sum_{p=0}^{n} \left(\bar{f}_2(p, k, n) \prod_{\substack{i=1 \\ i \neq p}}^{n} \left(\bar{g}(i, k, n)+1\right)\right).$$

Clearly $g, f_1^*, f_2^* \in \mathbf{R}$ (C). It follows from (24) that

$$\left| \sum_{p=0}^{n} a_p (a_k - a)^p - \frac{f_1^*(k, n) - f_2^*(k, n)}{g(k, n)+1} \right| < \frac{1}{n+1}. \tag{25}$$

On account of (7) and (25) we have

$$\left| \varphi(a_k) - \frac{f_1^*(k, n) - f_2^*(k, n)}{g(k, n)+1} \right| < \frac{q+1}{n+1} \quad \text{for} \quad n \in \mathcal{N}. \tag{26}$$

This completes the proof.

In the next part of this section we intend to show that computable functions are continuous and that they possess the property of Darboux. We begin with the following lemma:

4.26. *Let $\varphi(x)$ be a computable function on a set $\mathscr{C}_0$ of computable numbers including* 0. *Let $\{a_k\}$ be an arbitrary sequence of real numbers satysfying the following conditions:*

(i) $a_k \in \mathscr{C}_0$ *for* $k \in \mathcal{N}$,

(ii) $\{a_k\} \in C$,

(iii) $\{a_k\} \underset{C}{\to} 0$.

Then $\{\varphi(a_k)\} \to \varphi(0)$.

To prove this theorem, consider the functional Φ defined by the formula

$$\Phi\langle f\rangle = \varphi\left(\sum_{k=0}^{\infty} a_k \operatorname{sgn} f(k) \cdot g\left(\operatorname{sgn} f(0) + \ldots + \operatorname{sgn} f(k)\right)\right), \tag{1}$$

where $g(0) = g(1) = 1$ and $g(n) = 0$ for $n \in \mathcal{N}$ and $n > 1$. We shall show that $\Phi \in \mathfrak{F}_C$. Let $f_\lambda(k) = f(\lambda, k) \in \mathbf{C}$. We introduce the following functions:

$$\bar{f}(\lambda, k) = \operatorname{sgn} f(\lambda, k) \cdot g\left(\operatorname{sgn} f(\lambda, 0) + \ldots + \operatorname{sgn} f(\lambda, k)\right), \tag{2}$$

$$\bar{g}(\lambda) = \sum_{k=0}^{\infty} a_k \cdot \bar{f}(\lambda, k). \tag{3}$$

Obviously $\bar{f}(\lambda, k) \in \mathbf{C}$. We shall establish that $\bar{g}(\lambda) \in \mathbf{C}$. Let

$$\bar{g}(\lambda, p) = \sum_{k=0}^{p} a_k \cdot \bar{f}(\lambda, k). \tag{4}$$

It follows from the definition of $\bar{g}(\lambda)$ that

$$\bar{g}(\lambda) = \begin{cases} a_k & \text{if there exists a } k = (\mu m)\big(\bar{f}(\lambda, m) \neq 0\big), \\ 0 & \text{in the case where } f(\lambda, m) = 0 \text{ for every } m \in \mathcal{N}. \end{cases}$$

Hence it is easy to see that

$$|\bar{g}(\lambda) - \bar{g}(\lambda, p)| \leqslant \sup_{k>p} |a_k|. \tag{5}$$

Since, by (iii), $\{a_k\} \underset{C}{\to} 0$, there exists a strictly increasing function $\bar{h}(n)$ such that

$$|a_k| < \frac{1}{n+1} \quad \text{for} \quad k > \bar{h}(n). \tag{6}$$

Putting $h(n) = \bar{h}(n+1)$ we have

$$|a_k| < \frac{1}{n+2} \quad \text{for} \quad k > h(n),$$

where $h(n) \in \mathbf{C}$ and $h(n)$ is a strictly increasing function.

If $p > h(n)$, then

$$|a_k| < \frac{1}{n+2} \quad \text{for} \quad k > p.$$

Hence

$$\sup_{k>p} |a_k| \leqslant \frac{1}{n+2} < \frac{1}{n+1}. \tag{7}$$

By (5) and (7) we have

$$|\bar{g}(\lambda) - \bar{g}(\lambda, p)| < \frac{1}{n+1} \quad \text{for} \quad p > h(n). \tag{8}$$

Since $\{a_k\} \in C$, there exist three functions $f', f'', g' \in \mathbf{C}$ such that $g' > 0$ and

$$\left| a_k - \frac{f'(k, n) - f''(k, n)}{g'(k, n)} \right| < \frac{1}{n+1} \quad \text{for} \quad k, n \in \mathcal{N}. \tag{9}$$

Hence, putting $(p+1)^2 - 1$ for n, we obtain

$$\left| a_k - \frac{f'\big(k(p+1)^2 - 1\big) - f''\big(k(p+1)^2 - 1\big)}{g'\big(k, (p+1)^2 - 1\big)} \right| < \frac{1}{(p+1)^2}. \tag{10}$$

The function $\bar{f}(\lambda, k)$ assumes only two values: 0 and 1. Hence, by (10)

$$(11)\qquad \left| a_k \cdot \bar{f}(\lambda, k) - \frac{f'\big(k, (p+1)^2-1\big) - f''\big(k, (p+1)^2-1\big)}{g'\big(k, (p+1)^2-1\big)} \cdot \bar{f}(\lambda, k) \right| < \frac{1}{(p+1)^2}.$$

It follows from (11) that

$$(12)\qquad \left| \bar{g}(\lambda, p) - \sum_{k=0}^{p} \frac{f'\big(k, (p+1)^2-1\big) - f''\big(k, (p+1)^2-1\big)}{g'\big(k, (p+1)^2-1\big)} \bar{f}(\lambda, k) \right| < \frac{1}{p+1}.$$

Replacing in (12) p by $h^*(n) = h(2n+1)+2n+1$ we obtain

$$(13)\qquad \left| \bar{g}\big(\lambda, h^*(n)\big) - \right.$$

$$\left. - \sum_{k=0}^{h^*(n)} \frac{f'\big(k, (h^*(n)+1)^2-1\big) - f''\big(k, (h^*(n)+1)^2-1\big)}{g'\big(k, (h^*(n)+1)^2-1\big)} \cdot \bar{f}(\lambda, k) \right| < \frac{1}{2(n+1)}.$$

It follows from (8), on account of $h^*(n) = h(2n+1)+2n+1 > h(2n+1)$, that

$$(14)\qquad \big|\bar{g}(\lambda) - \bar{g}\big(\lambda, h^*(n)\big)\big| < \frac{1}{2(n+1)}.$$

By (13) and (14) we have

$$(15)\qquad \left| \bar{g}(\lambda) - \sum_{k=0}^{h^*(n)} \frac{f'\big(k, (h^*(n)+1)^2-1\big) - f''\big(k, (h^*(n)+1)^2-1\big)}{g'\big(k, (h^*(n)+1)^2-1\big)} \cdot \bar{f}(\lambda, k) \right| < \frac{1}{n+1}.$$

Let

$$g^*(\lambda, n) = \prod_{k=0}^{h^*(n)} g'\big(k, (h^*(n)+1)^2-1\big) f_1^*(\lambda, n)$$

$$= \sum_{k=0}^{h^*(n)} \left(f'\big(k, (h^*(n)+1)^2-1\big) \cdot \bar{f}(\lambda, k) \cdot \prod_{\substack{i=1 \\ i \neq k}}^{h^*(n)} g'\big(i, (h^*(n)+1)^2-1\big) \right),$$

$$f_2^*(\lambda, n) = \sum_{k=0}^{h^*(n)} \left(f''\big(k, (h^*(n)+1)^2-1\big) \bar{f}(\lambda, k) \cdot \prod_{\substack{i=1 \\ i \neq k}}^{h^*(n)} g'\big(i, (h^*(n)+1)^2-1\big) \right).$$

Since $h^*, g', f', f'', \bar{f} \in \mathbf{C}$ and $g' > 0$, we have $f_1^*, f_2^*, g^* \in \mathbf{C}$ and $g^* > 0$. Moreover,

$$\left| \bar{g}(\lambda) - \frac{f_1^*(\lambda, n) - f_2^*(\lambda, n)}{g^*(n)} \right| < \frac{1}{n+1}.$$

Hence $\bar{g}(\lambda) \in \mathbf{C}$. On account of 2.5 $\{\bar{g}(\lambda)\} \in \boldsymbol{C}$. Moreover, $\bar{g}(\lambda) \in \mathscr{C}_0$ for every $\lambda \in \mathscr{N}$. Since the function $\varphi(x)$ is computable on $\mathscr{C}_0$, it follows from 4.1 that $\{\varphi(\bar{g}(\lambda))\} = \{\Phi\langle f_\lambda\rangle\} \in \boldsymbol{C}$. Hence $\Phi \in \mathfrak{F}_c$. It is easy to see that

$$\Phi\langle 0\rangle = \varphi(0) \tag{16}$$

and

$$\Phi\langle a_k\rangle = \varphi(a_k), \tag{17}$$

where a_k (for $k = 1, 2, 3, \ldots$) are the functions defined in 3.27. Hence, using 3.31, we obtain

$$\{\Phi\langle a_k\rangle\} \to \Phi(0);$$

that is

$$\{\varphi(a_k)\} \to \varphi(0).$$

This completes the proof.

The condition that 0 is the limit of a sequence $\{a_k\}$ is not essential. In fact, we have the following lemma:

4.27. *Let $\psi(x)$ be a computable function on a set $\mathscr{C}_1 \subset \mathscr{C}$. Let $\{b_k\}$ be an arbitrary sequence satisfying the following conditions:*

1° $b_k \in \mathscr{C}_1$ *for every* $k \in \mathscr{N}$,

2° $\{b_k\} \in \boldsymbol{C}$,

3° $\{b_k\} \underset{\boldsymbol{C}}{\to} b$,

4° $b \in \mathscr{C}_1$.

Then $\{\psi(b_k)\} \to \psi(b)$.

Let $\mathscr{C}_0$ be the set of all computable numbers x such that $x + b \in \mathscr{C}_1$. Obviously $\mathscr{C}_0$ is the set of computable numbers, $0 \in \mathscr{C}_0$ and $(b_k - b) \in \mathscr{C}_0$ for every $k \in \mathscr{N}$. The sequence $\{b_k - b\} \underset{\boldsymbol{C}}{\to} 0$ and $\{b_k - b\} \in \boldsymbol{C}$. The function $\varphi(x) = \psi(x + b)$ is computable on the set $\mathscr{C}_0$. Then, by 4.26, $\{\varphi(b_k - b)\} \to \varphi(0)$. Hence $\{\psi(b_k)\} \to \psi(b)$.

Lemma 4.27 can be formulated in a still stronger form, namely, we shall prove the following theorem:

4.28. *Let $\varphi(x)$ be a computable function on a set $\mathscr{C}_1 \subset \mathscr{C}$ and let $\{a_k\}$ be an arbitrary sequence satysfying the following conditions:*

1° $a_k \in \mathscr{C}_1$,

2° $\{a_k\} \in \boldsymbol{C}$,

3° $\{a_k\} \to a$,

4° $a \in \mathscr{C}_1$.

Then $\{\varphi(a_k)\} \to \varphi(a)$.

To prove 4.28 suppose that the sequence $\{\varphi(a_k)\}$ does not converge to $\varphi(a)$. Consequently, it follows from 2.45 that there exist an $m_0 \in \mathcal{N}$ and an increasing computable function $h(k)$ such that

$$(1) \qquad |\varphi(a_{h(k)}) - \varphi(a)| > \frac{1}{m_0+1} \quad \text{for every } k \in \mathcal{N}.$$

Evidently

$$\{a_{h(k)}\} \to a.$$

By 2.44 and 2.5 we have $\{a_{h(k)}\} \in C$. By 2.43 there exists an increasing computable function $g(k)$ such that

$$\{a_{h(g(k))}\} \underset{C}{\to} a.$$

Hence, by 4.27,

$$(2) \qquad \{\varphi(a_{h(g(k))})\} \to \varphi(a),$$

contrary to (1).

In order to show the property of Darboux for computable functions we shall also need lemma 4.31, which results easily from the following one:

4.29. *There is no computable function φ on the set $\langle 0, 1\rangle\mathscr{C}$, assuming only two values, 0 and 1, and such that $\varphi(0) = 0$, $\varphi(1) = 1$.*

Indeed, suppose that a function φ fulfils the conditions of 4.29, i. e. that $\varphi(x)$ is computable on the set $\langle 0, 1\rangle\mathscr{C}$ assuming only the values 0 and 1, and $\varphi(0) = 0$, $\varphi(1) = 1$.

To prove 4.29 we shall consider the sequence of intervals $\langle a_n, b_n\rangle$ defined as follows:

$$a_0 = 0, \quad b_0 = 1,$$

$$(1) \qquad a_{n+1} = \frac{a_n+b_n}{2} + \frac{a_n-b_n}{2}\varphi\left(\frac{a_n+b_n}{2}\right),$$

$$b_{n+1} = b_n + \frac{a_n-b_n}{2}\varphi\left(\frac{a_n+b_n}{2}\right).$$

Clearly, $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$ in the case of $\varphi\,\frac{(a_n+b_n)}{2} = 1$ and $a_{n+1} = \frac{a_n+b_n}{2}$, $b_{n+1} = b_n$ in the case of $\varphi\frac{(an+bn)}{2} = 0$. Thus we have

$$(2) \qquad a_0 \leqslant a_1 \leqslant \ldots \leqslant b_1 \leqslant b_0$$

and

$$(3) \qquad b_n - a_n = \frac{1}{2^n} \quad \text{for every } n \in \mathcal{N}.$$

Hence, on account of the theorem of Ascoli, there exists a number c such that

(4) $$a_n \leqslant c \leqslant b_n \quad \text{for every } n \in \mathcal{N}.$$

By (3) and (4) we have

(5) $$|c - a_n| \leqslant \frac{1}{2^n} \leqslant \frac{1}{m+1} \quad \text{for} \quad m > n$$

and

(6) $$|c - b_n| \leqslant \frac{1}{2^n} \leqslant \frac{1}{m+1} \quad \text{for} \quad m > n,$$

that is

(7) $$\{a_n\} \underset{\mathcal{C}}{\to} c \quad \text{and} \quad \{b_n\} \underset{\mathcal{C}}{\to} c.$$

We shall prove that the sequences $\{a_n\}$ and $\{b_n\}$ are computable. Let $f(p, q) = \dfrac{2^q - (2^2 \dot{-} p)}{2^q}$ for every $p, q \in \mathcal{N}$. Clearly

(8) $$0 \leqslant f(p, q) \leqslant 1.$$

Let J, K, L be the pairing functions. Then

(9) $$KJ(x, y) = x, \quad LJ(x, y) = y.$$

Since $J, K, L \in \mathbf{C}$, it is easy to see that the sequence

$$\{a_u^*\} = \Big\{f\Big(K(u) + K\big(L(u)\big), \quad L\big(L(u)\big) + 1\Big)\Big\} \in \boldsymbol{C}.$$

Moreover, on account of (8), $a_n^* \in \langle 0, 1\rangle \mathcal{C}$. Since the function $\varphi(x)$ is computable on the set $\langle 0, 1\rangle \mathcal{C}$, the sequence

$$\{b_u^*\} = \Big\{\varphi\Big(f\big(K(u) + K\big(L(u)\big), L\big(L(u)\big) + 1\Big)\Big\} \in \boldsymbol{C}.$$

Hence, by 2.5 the function

$$j(u) = \varphi\Big(f\Big(K(u) + K\big(L(u)\big), L\big(L(u)\big) + 1\Big)\Big) \in \mathbf{C}.$$

Let $J^*(x, y, z) = J\big(x, J(y, z)\big)$, and $h^*(x, y, n) = \varphi\big(f(x+y, n+1)\big)$. Obviously $j(J^*(x, y, z) \in \mathbf{C}$. On the other hand, we have

(10) $$j\big(J^*(x, y, z)\big)$$
$$= \varphi\Big(f\Big(K\big(J^*(x, y, z)\big) + K\Big(L\big(J^*(x, y, z)\big)\Big), L\Big(L\big(J^*(x, y, z)\big)\Big) + 1\Big)\Big).$$

It is easy to verify that

(11) $$K\big(J^*(x, y, z)\big) = x, \quad K\Big(L\big(J^*(x, y, z)\big)\Big) = y, \quad L\Big(L\big(J^*(x, y, z)\big)\Big) = z.$$

By (10) and (11) we have

$$j\big(J^*(x, y, z)\big) = \varphi\big(f(x+y, z+1)\big) = h^*(x, y, z). \tag{12}$$

Hence $h^* \in \mathbf{C}$.

We shall now consider two functions, $o_1(n)$ and $o_2(n)$, defined by means of simultaneous recursions:

$$\begin{aligned} &o_1(0) = 0, \quad o_2(0) = 1, \\ &o_1(n+1) = o_1(n)+o_2(n) \dot{-} \big(o_2(n) \dot{-} o_1(n)\big)\cdot h^*\big(o_1(n), o_2(n), n\big), \\ &o_2(n+1) = 2o_2(n) \dot{-} \big(o_2(n) \dot{-} o_1(n)\big)\cdot h^*\big(o_1(n), o_2(n), n\big). \end{aligned} \tag{13}$$

Obviously, $o_1, o_2 \in \mathbf{C}$. Moreover,

$$\begin{aligned} h^*\big(o_1(n), o_2(n), n\big) &= \varphi\big(f(o_1(n)+o_2(n), n+1)\big) \\ &= \varphi\left(\frac{2^{n+1} - \big(2^{n+1} \dot{-} (o_1(n)+o_2(n))\big)}{2^{n+1}}\right). \end{aligned} \tag{14}$$

It is easy to prove that $o_2(n) = o_1(n)+1$ for $n \in \mathcal{N}$. In fact, $o_2(0) = o_1(0)+1$. Suppose $o_2(n) = o_1(n)+1$. Then

$$\begin{aligned} o_1(n+1) &= o_1(n)+o_1(n)+1 \dot{-} h^*\big(o_1(n), o_2(n), n\big), \\ o_2(n+1) &= 2o_1(n)+2 \dot{-} h^*\big(o_1(n), o_2(n), n\big). \end{aligned}$$

Hence $o_2(n+1) - o_1(n+1) = 1$. Moreover, $2^{n+1} > o_1(n)+o_2(n)$ for every $n \in \mathcal{N}$. Indeed, $2 > o_1(0)+o_2(0)$. Suppose that $2^{n+1} > o_1(n)+o_2(n)$. Thus $o_1(n+1)+o_2(n+1) = 2o_1(n+1)+1 = \leqslant 2[o_1(n)+o_2(n)]+1 < 2^{n+2}$. Hence, on account of (14) we have

$$h^*\big(o_1(n), o_2(n), n\big) = \varphi\left(\frac{o_1(n)+o_2(n)}{2^{n+1}}\right), \tag{15}$$

that is

$$\begin{aligned} &o_1(0) = 0, \quad o_2(0) = 1, \\ &o_1(n+1) = o_1(n)+o_2(n)+\big(o_1(n)-o_2(n)\big)\cdot \varphi\left(\frac{o_1(n)+o_2(n)}{2^{n+1}}\right), \\ &o_2(n+1) = 2o_2(n)+\big(o_1(n)-v_2(n)\big)\cdot \varphi\left(\frac{o_1(n)+o_2(n)}{2^{n+1}}\right). \end{aligned} \tag{16}$$

Hence, by (1) we obtain

$$a_n = \frac{o_1(n)}{2^n}, \quad b_n = \frac{o_2(n)}{2^n}. \tag{17}$$

It follows from 2.5 and 2.18 that $\{a_n\}, \{b_n\} \in C$. Consequently by (7), 2.38 and 4.28 we infer that

$$\varphi(a_n) \to \varphi(c), \quad \varphi(b_n) \to \varphi(c). \tag{18}$$

On the other hand, $\varphi(a_n) = 0$ and $\varphi(b_n) = 1$ for every $n \in \mathcal{N}$. In fact, $\varphi(a_0) = \varphi(0) = 0$ and $\varphi(b_0) = \varphi(1) = 1$. Suppose that $\varphi(a_n) = 0$, $\varphi(b_n) = 1$. If $\varphi\left(\frac{a_n+b_n}{2}\right) = 0$, then $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$, whence $\varphi(a_{n+1}) = 0$ and $\varphi(b_{n+1}) = \varphi(b_n) = 1$. In the case of $\varphi\left(\frac{a_n+b_n}{2}\right) = 1$, $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$, that is also $\varphi(a_{n+1}) = 0$ and $\varphi(b_{n+1}) = 1$. Hence $\varphi(a_n) \to 0$ and $\varphi(b_n) \to 1$, contrary to (18).

4.30. *If $\varphi(x)$ is a computable function on a set $\Delta\mathscr{C}$, assuming at the most two values* 0 *and* 1, *then* $\varphi(x) = \text{const}$.

Suppose that there exist $a, b \in \Delta\mathscr{C}$ such that $\varphi(a) = 0$ and $\varphi(b) = 1$. Consider the case of $a < b$. Let $\psi(t) = \varphi\big(a + t(b-a)\big)$ for $t \in \langle 0, 1 \rangle \mathscr{C}$. Then $\psi(t)$ is a computable function on $\langle 0, 1 \rangle \mathscr{C}$, $\psi(0) = \varphi(a) = 0$ and $\Psi(1) = \varphi(b) = 1$, h ich contradicts 4.29. In the case of $b < a$, let $\varphi_1(x) = 1 - \varphi(x)$. The function $\varphi_1(x)$ is computable on $\Delta\mathscr{C}$ and assumes at the most two values, 0 and 1. Moreover, $\varphi_1(b) = 0$, $\varphi_1(a) = 1$. By the similar method to that used in the case of $a < b$, we infer that our supposition contradicts 4.29. Hence $\varphi(a) = \varphi(b)$.

4.31. *If $\varphi(x)$ is a computable function on a set $\Delta\mathscr{C}$ assuming positive and negative values, then the function $\varphi(x)$ assumes also the value* 0.

To prove this theorem suppose that $\varphi(x) \neq 0$ for every $x \in \Delta\mathscr{C}$. Then by 4.9 and 4.10 the function

$$\psi(x) = \frac{1 + |\varphi(x)|}{\varphi(x)} \in \mathfrak{C}. \tag{1}$$

Moreover,

$$\psi(x) \neq 0 \quad \text{for} \quad x \in \Delta\mathscr{C}. \tag{2}$$

By 4.20 the function $\operatorname{sgn} x$ is computable on the set of all computable numbers $x \neq 0$. Hence, on account of 4.3, (1) and (2), the function $\operatorname{sgn} \psi(x)$ is computable. This implies by 4.9 that the function

$$\theta(x) = \frac{1 + \operatorname{sgn}\psi(x)}{2}$$

is computable. Since $\theta(x)$ assumes at the most two values, 0 and 1, it follows from 4.30 that $\theta(x) = \text{const}$. In consequence $\operatorname{sgn}\psi(x) = \text{const}$ and the function $\varphi(x)$ does not change the sign, contrary to our suppo-

sition. It follows in a familiar way from 4.31 that the property of Darboux holds for computable functions. Indeed, we have the following theorem:

4.32. *Let $\varphi(x)$ be a computable function on a set $\Delta\mathscr{C}$. Let a and b be values of $\varphi(x)$, and let $a < b$. Then every computable number C such that $a < c < b$ is also the value of $\varphi(x)$.*

4.33. *If $\varphi(x)$ is a computable function on the set $\langle 0, 1\rangle\mathscr{C}$, then $\varphi(x)$ is continuous in this set.*

To prove this suppose that $\varphi(x)$ is not continuous at a point $a \in \langle 0,1\rangle\mathscr{C}$. Then there exists such a rational number $1/r > 0$ that for every $p \in \mathscr{N}$ there exists a rational number $x \in \langle 0, 1\rangle$ for which the following conditions are fulfilled:

$$|x-a| < \frac{1}{p+1}, \tag{1}$$

$$|\varphi(x)-\varphi(a)| > \frac{1}{r}. \tag{2}$$

Indeed, since $\varphi(x)$ is not continuous in a, there exists such a number $2/r$ that for every $p \in \mathscr{N}$ there exists a $y \in \langle 0, 1\rangle\mathscr{C}$ such that $|y-a| < 1/2(p+1)$ and $|\varphi(y) - \varphi(a)| > 2/r$. Since $y \in \mathscr{C}$, there exists a function $o(x) \in \mathbf{C}$ such that $\left|y - \frac{o(n)}{n+1}\right| < \frac{1}{n+1}$. Obviously, there exists an $n_0 \in \mathscr{N}$ such that $\frac{o(n)}{n+1} \in \langle 0, 1\rangle\mathscr{C}$ for $n \geqslant n_0$. Let $x_n = \frac{o(n_0+n)}{n_0+n+1}$. Obviously $\{x_n\} \in \boldsymbol{C}$ and $\{x_n\} \to y$. Hence, by 4.28, $\{\varphi(x_n)\} \to \varphi(y)$. Let n_1, n_2, n_3 be such positive integers that $|\varphi(y)-\varphi(x_n)| < \frac{1}{r}$ for $n \geqslant n_1$, $|x_n - y| < \frac{1}{2(p+1)}$ for $n \geqslant n_2$ and $n_3 = \max(n_1, n_2)$. Then $x_{n_3} = x$ is a rational number and

$$|\varphi(x)-\varphi(a)| \geqslant |\varphi(y)-\varphi(a)| - |\varphi(y)-\varphi(x)| > \frac{2}{r} - \frac{1}{r} = \frac{1}{r},$$

$$|x-a| \leqslant |x-y| + |y-a| < \frac{1}{2(p+1)} + \frac{1}{2(p+1)} = \frac{1}{p+1}.$$

Let f^* and g^* be functions defined as follows:

$$f^*(0) = 0, \quad g^*(0) = 1,$$

$$f^*(n+1) = \begin{cases} f^*(n)+1 & \text{in the case of } g^*(n) \dot{-} f^*(n) \neq 0, \\ 1 & \text{in the case of } g^*(n) \dot{-} f^*(n) = 0, \end{cases}$$

$$g^*(n+1) = \begin{cases} g^*(n)+1 & \text{in the case of } g^*(n) \dot{-} f^*(n) = 0, \\ g^*(n) & \text{in the case of } g^*(n) \dot{-} f^*(n) \neq 0. \end{cases}$$

Obviously, the sequence $\left\{\frac{f^*(k)}{g^*(k)}\right\}$ is computable and contains every rational number of the interval $\langle 0, 1\rangle$. Moreover, for every $k_0 \in \mathcal{N}$ there exists a $k > k_0$ and $k \in \mathcal{N}$ such that

$$\frac{f^*(k_0)}{g^*(k_0)} = \frac{f^*(k)}{g^*(k)}.$$

Thus, for every $p \in \mathcal{N}$ there exists arbitrarily great $k \in \mathcal{N}$ such that

$$\left|\frac{f^*(k)}{g^*(k)} - a\right| < \frac{1}{p+1}, \tag{3}$$

$$\left|\varphi\left(\frac{f^*(k)}{g^*(k)}\right) - \varphi(a)\right| > \frac{1}{r}. \tag{4}$$

Since $a \in \mathscr{C}$, there exists a function $h^* \in \mathbf{C}$ such that

$$\left|a - \frac{h^*(k)}{k+1}\right| < \frac{1}{k+1}\Bigg| \quad \text{for} \quad k \in \mathcal{N}. \tag{5}$$

The values of a computable function are computable numbers. Hence, there exist two functions $g', g'' \in \mathbf{C}$ such that

$$\left|\varphi(a) - \frac{g'(k) - g''(k)}{k+1}\right| < \frac{1}{k+1} \quad \text{for} \quad k \in \mathcal{N}. \tag{6}$$

Since the sequence $\left\{\varphi\left(\frac{f^*(k)}{g^*(k)}\right)\right\}$ is computable, there exist two functions f' and $f'' \in \mathbf{C}$ such that

$$\left|\varphi\left(\frac{f^*(k)}{g^*(k)}\right) - \frac{f'(k,n) - f''(k,n)}{n+1}\right| < \frac{1}{n+1} \quad \text{for} \quad k, n \in \mathcal{N}. \tag{7}$$

Given p, let $k > \sup(4/r, p)$ be such a positive integer that (3) and (4) hold. Then by (3) and (5)

$$\left|\frac{f^*(k)}{g^*(k)} - \frac{h^*(k)}{k+1}\right| \leqslant \frac{1}{p+1} + \frac{1}{p+1} < \frac{2}{p+1} \tag{8}$$

and by (4), (6), (7)

$$(9)\qquad \left|\frac{f'(k,k)-f''(k,k)}{k+1}-\frac{g'(k)-g''(k)}{k+1}\right|$$

$$\geqslant \frac{1}{r}-\frac{1}{k+1}-\frac{1}{k+1}=\frac{1}{r}-\frac{2}{k+1}>\frac{1}{r}-\frac{1}{2r}=\frac{1}{2r}.$$

Let R and S be the relations between p and k defined by means of inequalities (8) and (9), respectively. It is easy to show that R and S are computable. Then we put

$$h_1(p)=(\mu k)\big((kRp)\wedge(kSp)\big),\qquad h_2(p,n)=(\mu k)\big((kRp)\wedge(kSp)\wedge(k>n)\big).$$

Obviously, $h_1, h_2 \in \mathbf{C}$. Let $h(p)$ be the function defined as follows:

$$h(0)=h_1(0),\qquad h(p+1)=h_2\big(p+1,h(p)\big).$$

Then h is a computable increasing function, and we have

$$(10)\qquad \left|\frac{f^*(h(p))}{g^*(h(p))}-\frac{h^*(h(p))}{h(p)+1}\right|<\frac{2}{p+1}$$

forevery p and

$$(11)\qquad \left|\frac{f'(h(p),h(p))-f''(h(p),h(p))}{h(p)+1}-\frac{g'(h(p))-g''(h(p))}{h(p)+1}\right|\geqslant\frac{1}{2r}.$$

By (5) and (10), $\left\{\frac{f^*(h(p))}{g^*(h(p))}\right\}\to a$. Hence, by 4.28, $\varphi\left(\frac{f^*(h(p))}{g^*(h(p))}\right)\to\varphi(a)$, contrary to (11), (7) and (6). The following easily results from 4.33:

4.34. *A computable function on a set $\Delta\mathscr{C}$ is continuous in that set.*

Indeed, let $\varphi(x)$ be a computable function on a set $\Delta\mathscr{C}$ and let a', $a''\in\Delta\mathscr{C}$, $a'<a''$. Consider the function

$$\psi(t)=\varphi\big(a'+t(a''-a')\big)\qquad\text{on the set } \langle 0,1\rangle\mathscr{C}.$$

$\psi(t)$ is computable on $\langle 0,1\rangle\mathscr{C}$, on account of 4.3 and 4.11. Hence, by 4.33, $\psi(t)$ is continuous at every point $t\in\langle 0,1\rangle\mathscr{C}$. Thus $\psi\left(\frac{x-a'}{a''-a'}\right)$ is continuous in $\langle a',a''\rangle\mathscr{C}$. Consequently, $\varphi(x)$ is continuous in $\langle a',a''\rangle\mathscr{C}$.

We now turn to the investigation of recursive functions. Our aim is to establish a connection between the notion of recursive functions and the same notion in the sense of the definition of Specker (4.37). It is convenient to introduce the following definition:

4.35. *Let $\varphi(x)$ be an arbitrary function defined on a set $\mathscr{X}$. $\varphi(x)$ is said to be recursively uniformly continuous if for every $n\in\mathscr{N}$ there exists a recur-*

sive function $h(n) \neq 0$ *such that the conditions* $x, y \in \mathscr{X}$ *and* $|x-y| < 1/h(n)$ *imply*

(i) $|\varphi(x)-\varphi(y)| \leqslant \dfrac{1}{n+1}$.

4.36. *The function* h *of* 4.35 *can be regarded as satisfying the condition* $h(n) > n$.

Indeed, suppose that conditions of 4.35 are satisfied. Let h^* be the function defined as follows:

$$h^*(0) = h(0)+1, \quad h^*(n+1) = \max\big(h(n+1), h^*(n)+1\big).$$

Then $h^* \in \mathbf{R}$ and

$$h^*(0) > 0, \quad h^*(n+1) \geqslant h^*(n)+1 > n+1.$$

Hence $h^*(n) > n$ for every $n \in \mathscr{N}$. Moreover,

$$h^*(0) > h(0), \quad h^*(n+1) \geqslant h(n+1),$$

that is $h^*(n) \geqslant h(n)$ for every $n \in \mathscr{N}$. Hence

$$|\varphi(x)-\varphi(y)| < \frac{1}{n+1} \quad \text{for} \quad |x-y| < \frac{1}{h^*(n)} \quad \text{and} \quad x, y \in \mathscr{X}.$$

4.37. *Let* $\varphi(x)$ *be a continuous function in the set* $\mathscr{X}_0$ *of all non-negative real number. Let* $\varphi(x)$ *satisfy the following condition: there exists a sequence of functions* $F_k(x)$ *defined on the set of all non-negative rational numbers such that*

(i) $$F_k\left(\frac{p}{q+1}\right) = \frac{f'(p, q, k)-f''(p, q, k)}{f'''(p, q, k)} \quad \textit{for } p, q, k \in \mathscr{N},$$

where $f', f'', f''' \in \mathbf{R}$ *and* $f'''(p, q, k) \neq 0$,

(ii) *there exists a recursive function* $g_1(n) \neq 0$ *such that the condition* $|x-y| < \dfrac{1}{g_1(n)}$ *implies* $|F_k(x)-F_k(y)| < \dfrac{1}{n+1}$,

(iii) *there exists a recursive function* $g_2(n)$ *such that*

$$|F_p(x)-F_q(x)| < \frac{1}{n+1} \quad \textit{for} \quad p, q > g_2(n),$$

(iv) $\lim\limits_{k\to\infty} F_k(x) = \varphi(x)$.

Then $\varphi(x)$ *is said to be a recursive function in the sense of Specker.*

4.38. *Let* $\varphi(x)$ *be a recursive function in the sense of Specker, then* $\big(\varphi(x)\,|\,\mathscr{X}_0 \cdot \mathscr{R}\big) \in \mathfrak{R}$ *and* $\varphi(x)$ *is a recursively uniformly continuous function.*

Suppose that $\varphi(x)$ fulfils the conditions of 4.37. It follows from (ii) that for every $k \in \mathscr{N}$ the function F_k is uniformly continuous. Hence

every function F_k can be prolonged to a function F_k^* continuous in the set of all non-negative real numbers. The functions F_k^* fulfil the following conditions:

(ii*) $|F_k^*(x)-F_k^*(y)| \leqslant \frac{1}{n+1}$, when $|x-y| < \frac{1}{g_1(n)}$,

(iii*) $|F_p^*(x)-F_q^*(x)| \leqslant \frac{1}{n+1}$ for $p, q > g_2(n)$,

(iv*) $\lim\limits_{k\to\infty} F_k^*(x) = \varphi(x)$.

It follows from (ii*) and (iv*) that

$$|\varphi(x)-\varphi(y)| \leqslant \frac{1}{n+1}, \quad \text{when} \quad |x-y| < \frac{1}{g_1(n)}. \tag{1}$$

Thus $\varphi(x)$ is recursively uniformly continuous. By (iii*) and (iv*) we have

$$|F_k^*(x)-\varphi(x)| \leqslant \frac{1}{n+1} \quad \text{for} \quad k > g_2(n). \tag{2}$$

Consider the function $\varphi(x)$ on the set $\mathscr{Z}_0\cdot\mathscr{R}$ of all recursive non-negative numbers. Let $\{a_m\}$ be a recursive sequence (where $a_m \in \mathscr{Z}_0\mathscr{R}$). By (2) we have

$$|\varphi(a_m)-F_k^*(a_m)| \leqslant \frac{1}{m+1} \quad \text{for} \quad k > g_2(n). \tag{3}$$

Since $\{a_m\} \in \mathbf{R}$, there exists a function $g'(m, n) \in \mathbf{R}$ such that

$$\left|a_m - \frac{g'(m, n)}{n+1}\right| < \frac{1}{n+1}. \tag{4}$$

Hence

$$\left|a_m - \frac{g'(m, g_1(n))}{g_1(n)+1}\right| < \frac{1}{g_1(n)+1}. \tag{5}$$

Consequently, on account of (ii*), we obtain

$$\left|F_k^*(a_m) - F_k^*\left(\frac{g'(m, g_1(n))}{g_1(n)+1}\right)\right| \leqslant \frac{1}{n+1}. \tag{6}$$

Hence, by (3),

$$\left|\varphi(a_m) - F_k^*\left(\frac{g'(m, g_1(n))}{g_1(n)+1}\right)\right| \leqslant \frac{1}{n+1} \quad \text{for} \quad k > g_2(n). \tag{7}$$

On account of (7), 4.37 (i) and 2.3 the sequence $\{\varphi(a_m)\}$ is recursive. Thus the function $|\varphi(x)|\mathscr{Z}_0\mathscr{R}$ is recursive in the sense of 4.1. This completes the proof.

4.39. *Let $\varphi(x)$ be a recursive function on the set $\mathscr{Z}_0\mathscr{R}$ of all non-negative recursive numbers, and let $\varphi(x)$ be recursively uniformly continuous in this set. Then the only uniformly continuous extension $\bar{\varphi}(x)$ of $\varphi(x)$ on the set $\mathscr{Z}_0$ of all non-negative real numbers is a recursive function in the sense of the definition of Specker.*

Suppose that $\varphi(x)$ fulfils the conditions of this theorem. Let $K(m)$, $L(m)$, $J(p, k)$ be the pairing functions of Robinson. The sequence $\left\{\varphi\left(\frac{K(m)}{L(m)+1}\right)\right\}$ is recursive by 2.5 and 4.1. Hence, there exist two functions $g_1(m, n), g_2(m, n) \in \mathbf{R}$ such that

$$(1) \qquad \left|\varphi\left(\frac{K(m)}{L(m)+1}\right) - \frac{g_1(m, n) - g_2(m, n)}{n+1}\right| < \frac{1}{n+1} \quad \text{for} \quad m, n \in \mathscr{N}.$$

Let

$$g'(p, k, n) = g_1\big(J(p, k), n\big),$$

$$g''(p, k, n) = g_2\big(J(p, k), n\big).$$

Obviously, $g', g'' \in \mathbf{R}$ and we have

$$(2) \qquad \left|\varphi\left(\frac{p}{k+1}\right) - \frac{g'(p, k, n) - g''(p, k, n)}{n+1}\right| < \frac{1}{n+1}.$$

Now we shall define a sequence of functions F_k on the set of all non-negative rational numbers. Given $k \in \mathscr{N}$, let

$$(3) \qquad F_k\left(\frac{p}{k+1}\right) = \frac{g'(p, k, k) - g''(p, k, k)}{k+1} \quad \text{for every } p \in \mathscr{N},$$

$$(4) \qquad F_k\left(\frac{r}{q+1}\right) = \frac{F_k\left(\frac{p+1}{k+1}\right) - F_k\left(\frac{p}{k+1}\right)}{\frac{1}{k+1}} \cdot \left(\frac{r}{q+1} - \frac{p}{k+1}\right) + F_k\left(\frac{p}{k+1}\right)$$

for every $\frac{r}{q+1}$ such that

$$(5) \qquad \frac{p}{k+1} \leqslant \frac{r}{q+1} < \frac{p+1}{k+1}.$$

It is easy to see that (4) holds also for $\frac{r}{q+1} = \frac{p+1}{k+1}$. We shall show that the sequence of functions F_k satisfies conditions (i)-(iv) of 4.37.

(i) Let $\frac{r}{q+1}$ fulfil inequality (5). Then $p = \left[(k+1)\cdot\frac{r}{q+1}\right]$. Hence, on account of (4),

$$(5') \quad F_k\left(\frac{r}{q+1}\right) = F_k\left(\frac{\left[(k+1)\frac{r}{q+1}\right]}{k+1}\right) + (k+1)\left(\frac{r}{q+1} - \frac{\left[(k+1)\frac{r}{q+1}\right]}{k+1}\right)\cdot$$
$$\cdot\left\{F_k\left(\frac{\left[(k+1)\frac{r}{q+1}\right]+1}{k+1}\right) - F_k\left(\frac{\left[(k+1)\frac{r}{q+1}\right]}{k+1}\right)\right\}.$$

It is easy to show, making use of (3) and (5′), that there exist three functions $f', f'', f''' \in \mathbf{R}$ such that $f'''(r, q, k) \neq 0$ and

$$(6) \qquad F_k\left(\frac{r}{q+1}\right) = \frac{f'(r, q, k) - f''(r, q, k)}{f'''(r, q, k)} \quad \text{for} \quad r, q, k \in \mathcal{N}.$$

(ii) Since $\varphi(x)$ is a recursively uniformly continuous function, there exists a function $h(n) \in \mathbf{R}$ such that $h(n) > n$, and

$$(7) \qquad |\varphi(x) - \varphi(y)| < \frac{1}{n+1} \quad \text{for} \quad |x-y| \leqslant \frac{1}{h(n)}.$$

Suppose that x_0, y_0 are arbitrary elements of the set $\mathscr{Z}_0\mathscr{R}$, and $x_0 < y_0$. Let $h^*(n) = h(26n+25)$. Obviously

$$[h^*(n)(y_0 - x_0)] \leqslant h^*(n)\cdot(y_0 - x_0) < [h^*(n)(y_0 - x_0)] + 1.$$

Hence

$$x_0 + \frac{[h^*(n)(y_0 - x_0)]}{h^*(n)} \leqslant y_0 < x_0 + \frac{[h^*(n)(y_0 - x_0)] + 1}{h^*(n)}.$$

It is evident that

$$\left|\varphi(x_0) - \varphi\left(x_0 + \frac{1}{h^*(n)}\right)\right| < \frac{1}{26(n+1)},$$

$$(8) \qquad \left|\varphi\left(x_0 + \frac{1}{h^*(n)}\right) - \varphi\left(x_0 + \frac{2}{h^*(n)}\right)\right| < \frac{1}{26(n+1)},$$

$$\left|\varphi\left(x_0 + \frac{[h^*(n)(y_0 - x_0)]}{h^*(n)}\right) - \varphi(y_0)\right| < \frac{1}{26(n+1)}.$$

Hence

$$(9) \qquad |\varphi(x_0) - \varphi(y_0)| < \frac{[h^*(n)\cdot(y_0 - x_0)] + 1}{26(n+1)}.$$

On the other hand, by (2) and (3) we obtain

$$\left|F_k\left(\frac{p}{k+1}\right)-\varphi\left(\frac{p}{k+1}\right)\right| < \frac{1}{k+1}. \tag{10}$$

Obviously

$$\left|F_k\left(\frac{p}{k+1}\right)-F_k\left(\frac{q}{k+1}\right)\right| \tag{11}$$

$$\leqslant \left|F_k\left(\frac{p}{k+1}\right)-\varphi\left(\frac{p}{k+1}\right)\right|+\left|\varphi\left(\frac{p}{k+1}\right)-\varphi\left(\frac{q}{k+1}\right)\right|+\left|\varphi\left(\frac{q}{k+1}\right)-F_k\left(\frac{q}{k+1}\right)\right|.$$

(11), (10) and (9) give

$$\left|F_k\left(\frac{p}{k+1}\right)-F_k\left(\frac{q}{k+1}\right)\right| < \frac{2}{k+1}+\frac{\left[h^*(n)\cdot\frac{|p-q|}{k+1}\right]+1}{26(n+1)} \tag{12}$$

for every $p, k, q \in \mathcal{N}$.

To prove (ii) we shall consider three cases.

(a) Suppose that x, y are arbitrary non-negative rational numbers such that $|x-y| < 1/h^*(n)$ and

$$\frac{p}{k+1} \leqslant x \leqslant \frac{p+1}{k+1}, \qquad \frac{p}{k+1} \leqslant y \leqslant \frac{p+1}{k+1}. \tag{13}$$

Then, by (4),

$$|F_k(x)-F_k(y)| = \left|F_k\left(\frac{p+1}{k+1}\right)-F_k\left(\frac{p}{k+1}\right)\right|\cdot|x-y|\cdot(k+1). \tag{14}$$

Hence, on account of (12), we have

$$|F_k(x)-F_k(y)| \leqslant \left(\frac{2}{k+1}+\frac{\left[h^*(n)\cdot\frac{1}{k+1}\right]+1}{26(n+1)}\right)\cdot(k+1)|x-y| \tag{15}$$

$$\leqslant 2|x-y|+\frac{k+1}{26(n+1)}|x-y|+\frac{h^*(n)}{26(n+1)}|x-y|$$

$$< \frac{2}{h^*(n)}+\frac{k+1}{26(n+1)(k+1)}+\frac{h^*(n)}{26(n+1)\cdot h^*(n)}.$$

Since $h^*(n) = h(26n+25) \geqslant 26n+26$, it follows from (15) that

$$|F_k(x)-F_k(y)| < \frac{2}{26(n+1)}+\frac{2}{26(n+1)} < \frac{1}{n+1}. \tag{16}$$

(b) Suppose that $|x-y| < 1/h^*(n)$ and

$$\frac{p}{k+1} \leqslant x < \frac{p+1}{k+1} < y < \frac{p+2}{k+1}.$$

Obviously

$$(17) \quad |F_k(x)-F_k(y)| \leqslant \left|F_k(x)-F_k\left(\frac{p+1}{k+1}\right)\right| + \left|F_k\left(\frac{p+1}{k+1}\right)-F_k(y)\right|.$$

By (16)

$$(18) \qquad \left|F_k\left(\frac{p+1}{k+1}\right)-F_k(y)\right| < \frac{4}{26(n+1)}$$

and

$$(19) \qquad \left|F_k(x)-F_k\left(\frac{p+1}{k+1}\right)\right| < \frac{4}{26(n+1)}.$$

By (17), (18) and (19) we obtain

$$(20) \qquad |F_k(x)-F_k(y)| < \frac{8}{26(n+1)} < \frac{1}{n+1}.$$

(c) Suppose that $|x-y| < 1/h^*(n)$ and

$$\frac{p}{k+1} \leqslant x < \frac{p+1}{k+1} < \frac{p+2}{k+1} \leqslant \frac{q}{k+1} \leqslant y < \frac{q+1}{k+1}.$$

Obviously by (16) and (12)

$$(21) \quad |F_k(x)-F_k(y)|$$

$$\leqslant \left|F_k(x)-F_k\left(\frac{p+1}{k+1}\right)\right| + \left|F_k\left(\frac{p+1}{k+1}\right)-F_k\left(\frac{q}{k+1}\right)\right| + \left|F_k\left(\frac{q}{k+1}\right)-F_k(y)\right|$$

$$< \frac{4}{26(n+1)} + \frac{2}{k+1} + \frac{\left[h^*(n)\frac{|p-q|}{k+1}\right]+1}{26(n+1)} + \frac{4}{26(n+1)}.$$

Since $|x-y| < 1/h^*(n) \leqslant 1/26(n+1)$ and $1/(k+1) < |x-y|$, we have

$$(22) \qquad \frac{1}{k+1} < \frac{1}{26(n+1)}.$$

On the other hand,

$$\left|\frac{p}{k+1} - \frac{q}{k+1}\right| < |x-y| < \frac{1}{h^*(n)}.$$

Hence

$$(23) \qquad \left[h^*(n)\frac{|p-q|}{k+1}\right] = 0.$$

By (21), (22), (23), we obtain

(24) $|F_k(x)-F_k(y)|$

$$< \frac{8}{26(n+1)}+\frac{2}{26(n+1)}+\frac{1}{26(n+1)} < \frac{11}{26(n+1)} < \frac{1}{n+1}.$$

Thus, (ii) is proved.

(iii) and (iv). Suppose that $k > h^*(n)$. Then

$$\frac{1}{k+1} < \frac{1}{h^*(n)} \leqslant \frac{1}{26(n+1)},$$

that is

$$x-\frac{[(k+1)\cdot x]}{k+1} = \frac{(k+1)\cdot x-[(k+1)\cdot x]}{k+1} < \frac{1}{k+1} < \frac{1}{h^*(n)}. \tag{25}$$

Hence, by the proof of (ii) we have

$$\left| F_k(x)-F_k\left(\frac{[(k+1)\cdot x]}{k+1}\right)\right| < \frac{11}{26(n+1)}. \tag{26}$$

By (10), we obtain

$$\left| F_k\left(\frac{[(k+1)\cdot x]}{k+1}\right)-\varphi\left(\frac{[(k+1)\cdot x]}{k+1}\right)\right| < \frac{1}{k+1} < \frac{1}{26(n+1)}. \tag{27}$$

It follows from (7), (25) and the definition of $h^*(n)$ that

$$\left|\varphi\left(\frac{[(k+1)\cdot x]}{k+1}\right)-\varphi(x)\right| < \frac{1}{26(n+1)}. \tag{28}$$

(26), (27), (28) imply that

$$|F_k(x)-\varphi(x)| < \frac{11}{26(n+1)}+\frac{1}{26(n+1)}+\frac{1}{26(n+1)} \tag{29}$$

$$= \frac{13}{26(n+1)} < \frac{1}{n+1}$$

for $k > h^*(n)$. Hence

$$\lim_{k\to\infty} F_k(x) = \varphi(x) = \bar{\varphi}(x).$$

On the other hand, it follows from (29) that

$$|F_k(x)-F_m(x)| < \frac{1}{n+1} \quad \text{for every } k, m \in \mathcal{N} \text{ and } k > h^*(n),\ m > h^*(n).$$

Thus, conditions (iii) and (iv) are proved.

Theorems 4.38 and 4.39 completely explain the reciprocal dependence of the notions of recursive function in the sense of the definition of Specker and of definition 4.1. Since Specker gave an example of a recursive function for which the property of Darboux does not hold, it follows from 4.38 and 4.39 that the recursive functions in the sense of 4.1 do not always possess the property of Darboux(1).

(1) For other notions of recursive real functions see E. Specker [40], D. Lacombe [31] and A. Grzegorczyk [16].

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CONTENTS

Foreword 3

I. Recursive and computable numbers 4

1. Recursive numbers 5

2. Computable numbers 17

II. Recursive and computable sequences 22

III. Recursive and computable functionals 60

IV. Recursive and computable real functions 80

Bibliography 109