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ON CATEGORY-RAISING AND DIMENSION-RAISING OPEN MAPPINGS WITH DISCRETE FIBERS

BY

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1. Introduction. In this note by a *space* we mean (unless the other is explicitly stated) a metrizable space, and by the *dimension* we understand the covering dimension (dim). Our terminology and notation follow those of [5]; in particular, $w(T)$ stands for the weight of a space T, $|A|$ for the cardinality of a set A, and N for the set of natural numbers.

Kuratowski [11] proved that if $f: S \to T$ is an open mapping of a separable space S of first category onto a space T and each fiber of f is a space of second category, then T is of first category. Aleksandrov proved that if $f: S \to T$ is an open mapping of a separable space S onto a space T and each fiber of f is not dense in itself, then $\dim T \leqslant \dim S$ (for the references and some related results see [2], Section 3, and also [21], [7], [4], [1], and [15]).

In both theorems the assumption of separability of S can be relaxed to the assumption of separability of the fibers of f (the proofs are simple combinations of those of Kuratowski and Aleksandrov with an idea of Hansell [6])(1). One can also show (Section 4.2) that if $w(T) = \aleph_1$, then both theorems remain valid if we assume, instead of separability of S, that f takes σ-discrete sets to σ-discrete sets. In another direction, the condition that the fibers of an open mapping f are uniformly complete guarantees that f does not raise the category (see Section 4.3); this condition together with the assumption that the fibers of f are scattered yields that f does not raise the dimension.

However, in general, it is easy to show that none of the assertions remains true in the non-separable case (2), even if f is "very scattered"; in fact, every first-countable T_1-topological space is an open image of

(1) A similar situation occurs if one considers the classical theorem of Mazurkiewicz [12] on the extension of open mappings defined on separable spaces over G_δ-sets (see the Remark in Section 4.1).

(2) This is also the case of the theorem of Mazurkiewicz mentioned in footnote (1).

a σ-discrete metrizable space under a mapping with discrete fibers (Section 2.1). Here $w(S) = |T|$, so in many interesting situations one cannot exclude the case (without the generalized continuum hypothesis) in which $w(S)$ is much greater than $w(T)$. To avoid this defect we give in Section 2.2 another general construction (for metrizable spaces only) which yields, in particular, an example of an open mapping $f: S \to T$ with discrete fibers, which maps a space S of weight $\aleph_1$ onto a separable 1-dimensional (even connected) space T. By this construction we can also obtain a category-raising mapping of this kind of a space of weight $\aleph_1$ (by Martin's Axiom, in such an example the range of the mapping should be also of weight $\aleph_1$). In Section 3 we give another, "essentially non-separable" construction which, if we do not require T to be separable, gives more exact results. In particular, it allows to show that every space T which is non-separable at each point is an image of a first-category space $S \subset T \times T$ under an open mapping with discrete fibers; in fact, the mapping is the projection.

Roughly speaking, we give in this note three different constructions which allow, under relatively weak assumptions, to represent a given space T as an image of a space S under an open mapping $f: S \to T$ with discrete fibers. The space S is loosely built from small (in the sense of category or dimension) pieces of the space T and, moreover, we require S to be, if possible, closely related to T.

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2. In this section we give two methods of construction of category-raising and dimension-raising open mappings with discrete fibers (cf. [20] and [14], p. 337).

2.1. THEOREM 1. *Every first-countable T_1-topological space T is an open image of a σ-discrete metrizable space S under a mapping with discrete fibers.*

Proof. Let $B(T)$ be the countable product $\prod_{n \in N} T_n$, where T_n is the set T considered with the discrete topology; given a finite sequence $(t_1, \ldots, t_n) \in T^n$ we put

$$B_{t_1,\ldots,t_n} = \{(u_i) \in B(T)\colon u_i = t_i \text{ for } i \leqslant n\}.$$

For every point $t \in T$ choose an open descending base $V_1(t) \supset V_2(t) \supset \ldots$ at t. We define the space $S \subset B(T)$ as follows:

A sequence $(t_1, t_2, \ldots) \in B(T)$ belongs to S if

(i) $t_m \in V_1(t_1) \cap \ldots \cap V_{m-1}(t_{m-1})$ for every $m \in N$;

(ii) if $t_i = t_{i+m}$, then $t_i = t_{i+j}$ for every $j \leqslant m$;

(iii) there exists an i such that $t_i = t_{i+m}$ for every $m \in N$.

The mapping $f: S \to T$ is defined by $f(t_1, t_2, \ldots) = t_i$, where t_i is as in (iii).

The space S is σ-discrete (by (iii)) and the mapping f is onto (since $(t, t, \ldots) \in S$ for every $t \in T$).

Fix a point $p \in S$. By (iii) and (ii) we can write $p = (t_1, \ldots, t_k, t, t, \ldots)$, where $t_i \neq t$. Let us fix the sequence $(t_1, \ldots, t_k, t, \ldots, t)$ of length $m > k$. We claim that the following two inclusions hold:

$$V = V_1(t_1) \cap \ldots \cap V_k(t_k) \cap V_m(t) \setminus \{t_1, \ldots, t_k\}$$

$$\subset f(B_{t_1, \ldots, t_k, t, \ldots, t} \cap S) \subset V_m(t).$$

The second inclusion follows from (i). To prove the first one it is enough to observe that for a point $v \in V$ we have

$$q = (t_1, \ldots, t_k, t, \ldots, t, v, v, \ldots) \in B_{t_1, \ldots, t_k, t, \ldots, t} \cap S \quad \text{and} \quad f(q) = v.$$

Thus we have shown that the mapping f is open.

Finally, by (iii), $B_{t_1, \ldots, t_k, t} \cap f^{-1}(t) = \{p\}$, which proves that the space $f^{-1}(t)$ is discrete.

In particular, *the unit real interval is an open image of a σ-discrete space S under a mapping with discrete fibers.* Observe that in this case S is of first category and $\dim S = 0$.

Although the above-given construction is very simple, it is not quite satisfactory: here $w(S) = |T|$, and hence in many cases we have $w(S) > w(T)$. For example, without any additional hypothesis from the set theory it does not answer the question whether there exists a category-raising or dimension-raising open mapping with discrete fibers which is defined on a space of weight $\aleph_1$. The desired examples can be obtained by another general construction described in the next section.

2.2. THEOREM 2. *For every space Y and a cardinal $\mathfrak{m} \geqslant w(Y)$ the following conditions are equivalent:*

(a) *there exists an open mapping $f: X \to Y$ with discrete fibers, which maps a 0-dimensional space X of weight less than or equal to $\mathfrak{m}$ onto Y;*

(b) *$Y = \bigcup \mathscr{K}$, where $\mathscr{K}$ is a family of closed 0-dimensional subsets of Y of cardinality $|\mathscr{K}| \leqslant \mathfrak{m}$* (3).

Proof. The implication (a) $\Rightarrow$ (b) follows immediately from the argument of Aleksandrov (see [2], the proof of 3.7), where a base of X of cardinality $\mathfrak{m}$ is to be considered instead of a countable base.

Assuming (b), we shall describe a construction of a space X and a mapping $f: X \to Y$ as in (a).

(3) Note that if a separable space T is the union of less than $2^{\aleph_0}$ compact subspaces of dimension not greater than n, then (under Martin's Axiom) $\dim T \leqslant n$. This follows easily from the topological form of the axiom (see [9]) and the classical embedding theorems of Hurewicz [8], Chapter V,6 (see also [3], [16] and [8], Chapter VII); cf. Corollary 1.

Let $B(\mathfrak{m})$ be the countable product of discrete spaces of cardinality $\mathfrak{m}$ (cf. [5]). Fix a metric on Y and on $B(\mathfrak{m})$; by $B(M, \varepsilon)$ we denote the ε-ball around a set M with respect to a given metric.

Let U be an open set in $B(\mathfrak{m})$, $A \subset U$ a nowhere dense set, $F \subset Y$ a closed 0-dimensional set, and W an open neighbourhood of F. Finally, let $h: A \to F$ be a homeomorphism. Any so defined system (U, A, h, F, W) will be called an ***admissible quintuple***.

Given an admissible quintuple (U, A, h, F, W) and a natural number n, let us proceed as follows:

(a) Split the space F into disjoint closed-and-open subspaces $\{F_s: s \in S\}$ such that $\operatorname{diam} F_s < 2^{-n}$ and, for $A_s = h^{-1}(F_s)$, $\operatorname{diam} A_s < 2^{-n}$.

(b) Choose pairwise disjoint open sets $W_s \subset W$ with $F_s \subset W_s \subset B(F_s, 2^{-n})$.

(c) Choose pairwise disjoint open sets $U_s \subset U$ with $A_s \subset U_s \subset B(A_s, 2^{-n})$ (cf. [5], 4.5.1).

(d) For every $s \in S$ choose a disjoint family $\mathscr{U}_s$ of cardinality $\mathfrak{m}$ of open subsets of U_s such that A_s is contained in a member V_s of $\mathscr{U}_s$ (this can be done as A is nowhere dense).

(e) Write

$$(K \setminus F) \cap W_s = \bigcup_i L_i \quad \text{with } \bar{L}_i = L_i,$$

where K runs over $\mathscr{K}$ and s over S; let $\mathscr{L}$ be the family of all non-empty L_i and let $\mathscr{L}_s = \{L \in \mathscr{L}: L \subset W_s\}$.

(f) For every $s \in S$ fix an injection $L \to U_L$ of the set $\mathscr{L}_s$ to the set $\mathscr{U}_s \setminus \{V_s\}$.

(g) For every $L \in \mathscr{L}$ choose a nowhere dense subspace $A_L \subset U_L$ and a homeomorphism $h_L: A_L \to L$ (cf. [5], Theorem 7.3.15).

(h) Finally, put $W_L = W_s \setminus F$ for $L \in \mathscr{L}_s$.

In effect, we obtain admissible quintuples (V_L, A_L, h_L, L, W_L) and $(V_s, A_s, h|A_s, F_s, W_s)$, where $L \in \mathscr{L}$ and $s \in S$. Observe that the family $\{V_L: L \in \mathscr{L}\} \cup \{V_s: s \in S\}$ is disjoint (as the family $\bigcup_{s \in S} \mathscr{U}_s$ is; cf. (c) and (d)), and so one can define naturally a continuous mapping $f_E: E \to Y$ of the union

$$E = \bigcup \{A_L: L \in \mathscr{L}\} \cup \bigcup_{s \in S} A_s \supset A,$$

which agrees with h_L or h on their domains. Call E the *associated space* and f_E the *associated mapping*.

We can pass to the construction of the space X and of the mapping $f: X \to Y$. At first, choose an open disjoint family $\mathscr{U}$ in $B(\mathfrak{m})$ of cardinality $\mathfrak{m}$ and fix an injection $K \to U_K$ of $\mathscr{K}$ into $\mathscr{U}$. For every $K \in \mathscr{K}$ choose a nowhere dense subspace $A_K \subset U_K$ and a homeomorphism $h_K : A_K \to K$ (cf. [5], Theorem 7.3.15). Finally, put $W_K = Y$. Let $E_0 = \bigcup\{A_K : K \in \mathscr{K}\}$ and let $f_0 : E_0 \to Y$ agree with h_K on each A_K. Notice that $f(E_0) = Y$.

We have the family $\{(U_K, A_K, h_K, K, W_K) : K \in \mathscr{K}\}$ of admissible quintuples. Apply to each one the construction (a)-(h) with $n = 1$. Denote the union of the associated sets by E_1 and let $f_1 : E_1 \to Y$ agree with each of the associated mappings on its domain.

Now, we have a new family of admissible quintuples. Repeat the construction (a)-(h) for each of the quintuples with $n = 2$ and define $f_2 : E_2 \to Y$ as above. Repeating this again and again with $n = 2, 3, \ldots$ we obtain at each stage of the process a mapping $f_n : E_n \to Y$. Put

$$X = \bigcup_n E_n.$$

Since $E_0 \subset E_1 \subset \ldots$ and $f_i|E_{i-1} = f_{i-1}$, we have the natural mapping $f : X \to Y$. We claim that $f : X \to Y$ is open and that the fibers of f are discrete.

To this end, let us fix a point $x \in X$. Let $x \in E_{n+1}$ for some n; so there exists an admissible quintuple (U, A, h, F, W) defined at the n-th stage of the construction, such that $x \in A_s$ for $s \in S$, where we adopt the notation introduced in (a)-(h). Consider the neighbourhood U_s of A_s; we claim that $f(U_s \cap X) = W_s$. Indeed, by (e) and (g), $f(U_s \cap X) \supset f(U_s \cap E_{n+1}) \supset W_s$; on the other hand, the inclusion $f(U_s \cap X) \subset W_s$ follows from (h). Now, $\operatorname{diam} W_s \leqslant 2^{-(n-1)}$ by (a) and (b), and $\operatorname{diam} U_s \leqslant 2^{-(n-1)}$ by (a) and (c), and thus, since n can be taken here arbitrarily large, f is continuous and open at x. Finally, let us verify that x is an isolated point of the space $f^{-1}f(x)$. Indeed, by (h) we have

$$f(U_s \cap (X \setminus A_s)) \cap F_s = \emptyset,$$

and so $f^{-1}f(x) \cap U_s = \{x\}$. The proof is completed.

COROLLARY 1. *There exists an open mapping with discrete fibers, which maps a 0-dimensional space X of weight $\aleph_1$ onto a separable, connected, non-one-point space Y.*

Proof. To apply Theorem 2 we need only a separable, connected, non-one-point space Y which is the union of a family $\mathscr{K}$ of cardinality $\aleph_1$ of its closed 0-dimensional subspaces. Such a space Y can be obtained by a modification of the celebrated Knaster-Kuratowski fan: it is enough to replace in the construction given in [5], 6.3.23, the set P of the non-end--points of the Cantor set C by a subset of C of cardinality $\aleph_1$ which is not an F_σ-set at each point of C (one can take, for example, a subset of cardinality $\aleph_1$ of a totally imperfect set in C; see [10], § 40).

Since every space is the union of $2^{\aleph_0}$ closed 0-dimensional subspaces (cf. [15], Chapter 3, 13-15), we have also

Corollary 2. *Every space of weight greater than or equal to $2^{\aleph_0}$ is an open image of a 0-dimensional space of the same weight under a mapping with discrete fibers.*

By the argument of Aleksandrov, under Martin's Axiom (cf. [9]), if $f: S \to T$ is an open category-raising mapping with discrete fibers and T is separable, then $w(S) = 2^{\aleph_0}$. Also in Corollary 1 the space Y should be of first category (cf. Corollary 4 in Section 3.1). Notice that if every $K \in \mathscr{K}$ in Theorem 2 is in addition a boundary set, then the space X — constructed in the proof — is of first category. This allows one to use this construction to obtain category-raising open mappings with discrete fibers. However, the construction proposed in the next section gives, if we do not require Y to be separable, much stronger results.

3. We assume that symbols $\alpha, \xi, \eta, \lambda$ stand for ordinal numbers. This section is devoted mainly to the proof of the following result:

3.1. Proposition. *Let $X_1 \subset \ldots \subset X_\xi \subset \ldots \subset X$, where $\xi < \lambda$, be an increasing sequence of closed boundary subsets of a space X such that*

$$X = \bigcup_{\xi<\lambda} X_\xi.$$

Then there exists a subspace $E \subset X \times X$ satisfying the following conditions:

(i) *every space $E_x = \{y \in X : (x, y) \in E\}$ is discrete and non-empty;*

(ii) *the restriction to the space E of the projection onto the first axis is open;*

(iii) *every set $E^x = \{y \in X : (y, x) \in E\}$ is contained in a set X_ξ for some $\xi < \lambda$ and the set $\{x \in X : E^x \neq \emptyset\}$ is σ-discrete.*

From this proposition we derive two corollaries.

Corollary 3. *Every space T non-separable at each point is an image of a space $S \subset T \times T$ which is of first category under an open mapping with discrete fibers.*

Proof. By a result of Štěpánek and Vopěnka [22] (cf. [19] for a simple proof) there exists an increasing sequence $T_1 \subset \ldots \subset T_\xi \subset \ldots \subset T$ $(\xi < \omega_1)$ of closed boundary subsets of T such that

$$T = \bigcup_{\xi<\omega_1} T_\xi.$$

Now, putting in the Proposition $X = T$ and $X_\xi = T_\xi$ we obtain the space $S = E$ with the required properties.

Corollary 4. *There exists a connected space T of weight $\aleph_1$ which is an open image of a 0-dimensional space $S \subset T \times T$ under an open mapping with discrete fibers; moreover, T can be of second category at each point.*

Proof. There exists a connected space T, each non-empty open subspace of which has the weight $\aleph_1$ and every separable subspace of which is 0-dimensional (see [17], Example 2). Let $\{t_a : a < \omega_1\}$ be a set dense in T and let us write $T_\xi = \overline{\{t_a : a < \xi\}}$. Let us put in the Proposition $X = T$, $X_\xi = T_\xi$ and $S = E$. From (iii) and the Sum Theorem we infer that $\dim S = 0$, and so we are done by (i) and (ii). To prove that T can be of second category at each point it is enough to replace $M(S, C)$ by $T = M(C, S)$ in Example 2 of [17].

3.2. In the rest of Section 3, X is a fixed space as in the Proposition. Observe that without loss of generality we can assume that the sequence $\{X_\xi\}_{\xi<\lambda}$ satisfies also the following condition (cf. [18], p. 96):

(1) $X_\xi = \overline{\bigcup_{a<\xi} X_a}$ for every limit $\xi < \lambda$.

Indeed, one can put

$$X'_{\xi+1} = X_{\xi+1} \quad \text{and} \quad X'_\xi = \overline{\bigcup_{a<\xi} X'_a} \quad \text{for a limit } \xi,$$

and consider the family $\{X'_\xi\}$, as we have $X'_\xi \subset X_\xi$.

It is easy to define inductively families $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of subsets of X and choose at every stage of the construction a point $p(G) \in G$ for every $G \in \mathcal{G}_n$ in a way such that

(2) $\mathcal{G}_n$ is an open disjoint family the union of which is dense in X and $\operatorname{diam} G < 1/n$ for every $G \in \mathcal{G}_n$;

(3) $\mathcal{G}_{n+1}$ is a refinement of $\mathcal{G}_n$;

(4) $M_n = \{p(G) : G : \mathcal{G}_n\} \subset \bigcup \mathcal{G}_{n+1}$;

(5) if $G \in \mathcal{G}_n$, $H \in \mathcal{G}_{n+1}$ and $p(G) \in H$, then $p(G) = p(H)$.

Put

$$\mathcal{G} = \bigcup_n \mathcal{G}_n \quad \text{and} \quad M = \bigcup_n M_n = \{p(G) : G \in \mathcal{G}\}.$$

The following easily-verifiable properties of the construction will be used in the sequel:

(6) $\mathcal{G}$ is a non-archimedean family, i.e., every two members of $\mathcal{G}$ are either disjoint or one is contained in the other;

(7) $M_1 \subset M_2 \subset \ldots$, M is σ-discrete and dense in X;

(8) if $G, H \in \mathcal{G}$, $H \subset G$ and $p(G) \in H$, then $p(G) = p(H)$.

3.3. For every $\xi < \lambda$ put

$$\mathcal{G}_\xi = \{G \in \mathcal{G} : G \text{ is a maximal member of } \mathcal{G} \text{ disjoint with } X_\xi\}$$

and let

(9) $E_\xi = \{p(G) : G \in \mathcal{G}_\xi\} \subset M \setminus X_\xi$.

Notice that, by (6), every family $\mathcal{G}_\xi$ is disjoint, so

(10) E_ξ is a discrete space.

Observe also that (see (7), (2) and (4))

(11) $M\setminus X_\xi \subset \bigcup \mathcal{G}_\xi$.

3.4. For every $\xi < \lambda$ put $P_\xi = X_\xi \setminus \bigcup_{a<\xi} X_a$ and let

$$E = \bigcup_{\xi<\lambda} P_\xi \times E_\xi \subset X \times X.$$

Since for $x \in P_\xi$ we have $E_x = E_\xi$, property (i) follows from (10) and (11). Now, let $x \in X$, i.e., $x \in X_\xi$ for some $\xi < \lambda$. If $y \in E^x$, then, by (9), $(y, x) \in P_a \times E_a$ and $a < \xi$, as $E_a \cap X_a = \emptyset$; thus $E^x \subset X_\xi$ which proves the first part of (iii). To prove the second part observe that $\{x: E^x \neq \emptyset\} \subset M$, and so, by (7), this set is σ-discrete. It remains to prove (ii).

3.5. Now we check two easy lemmas; in the first one we put $\mathcal{G}_0 = \{X\}$.

LEMMA 1. *Let* $H \in \mathcal{G}_n$, $H \subset G \in \mathcal{G}_{n-1}$, $G \cap X_a \neq \emptyset$, *and* $p(H) \notin X_a$. *Then* $p(H) \in E_a$.

Proof. Since $p(H) \in M \setminus X_a$, by (11) there exists a set $U \in \mathcal{G}_a$ with $p(H) \in U$. Now either $U \supset G$ or $U \subset H$. The first possibility is excluded because $U \cap X_a = \emptyset$, so we have $U \subset H$ and, by (8), $p(H) = p(U) \in E_a$.

LEMMA 2. *If* $p \in M \cap X_a$ *and* V *is a neighbourhood of* p, *then* $V \cap E_a \neq \emptyset$.

Proof. By (7), (4) and (2), there is an $H \in \mathcal{G}$ such that $p \in H \subset V$. Using again these facts we infer from $M \cap H \neq \emptyset$ (cf. (7)) that there exists a $G \in \mathcal{G}$ with $G \cap X_a = \emptyset$ and $G \subset H$. Take a maximal member W of $\mathcal{G}$ such that $G \subset W \subset X \setminus X_a$. Of course, $W \in \mathcal{G}$ and $W \subset H$. We have $p(W) \in E_a \cap V$.

3.6. Now we are ready to prove (ii). Let $(x, y) \in E$ and let $U \times V$ be an open neighbourhood of the point in $X \times X$. Whenever we show that there exists a neighbourhood $W \subset U$ of the point x such that

(12) if $W \cap P_a \neq \emptyset$, then $E_a \cap V \neq \emptyset$,

the proof of (ii) will be completed, since (12) implies $E_y \cap V \neq \emptyset$ for every $y \in W$.

Assume that $x \in P_\xi$. Thus $y \in E_\xi$, and so $y = p(H)$ for some $H \in G_\xi$. Consequently, we have

(13) $y = p(H)$, $H \in \mathcal{G}_n$, $H \subset G \in \mathcal{G}_{n-1}$ and $H \cap X_\xi = \emptyset$, but $G \cap X_\xi \neq \emptyset$.

Let us choose a neighbourhood $W \subset U$ of the point x in such a way that, for every $a \geqslant \xi$,

(14) if $W \cap P_a \neq \emptyset$, then $G \cap X_a \neq \emptyset$.

This can be done as follows: if $\xi = \eta+1$, we take $W = U\setminus X_\eta$; if ξ is a limit ordinal, then, by (1) and (7), there exists an $\eta < \xi$ with $G\cap X_\eta \neq \emptyset$, and $W = X\setminus X_\eta$ is the required neighbourhood.

Now, let $W\cap P_a \neq \emptyset$. Then by (14) also $G\cap X_a \neq \emptyset$. If $y \notin X_a$, then, by (13) and Lemma 1, we have $y \in E_a\cap V$; if $y \in X_a$, then we use Lemma 2 to conclude (12).

4. In this section we give some conditions under which the assertions of the theorems of Kuratowski and Aleksandrov mentioned in the introduction hold in the non-separable case.

4.1. THEOREM 3. *Let $f: S \to T$ be an open mapping with separable fibers, which maps a space S onto a space T.*

(a) *If the fibers of f are spaces of second category and S is of first category, then T is of first category* (cf. [19]).

(b) *If each fiber of f is not dense in itself, then* $\dim T \leqslant \dim S$.

For the proof it is enough to repeat the arguments of Kuratowski and Aleksandrov, respectively, where a countable base is to be replaced by a σ-discrete base, and to use the following lemma due to Hansell ([6], Proposition 3.11).

LEMMA 3. *Let $f: S \to T$ be an open mapping with separable fibers. Then for every discrete family $\{K_a : a \in A\}$ of subsets of S there exists a family $\{L_{an} : a \in A,\ n \in N\}$ of subsets of T such that*

$$f(K_a) = \bigcup_n L_{an}$$

and for every n the family $\{L_{an} : a \in A\}$ is discrete.

Remark. This is a classical result of Mazurkiewicz [12] that *if $f: X \to Y$ is a continuous mapping of a completely metrizable* ([4]) *separable space X to a space Y, then for every set $A \subset X$ such that the restriction $f|A : A \to f(A)$ is open there exists a G_δ-set $B \subset X$ such that $A \subset B$ and the restriction $f|B : B \to f(B)$ is open.*

As in the case of the theorems of Kuratowski and Aleksandrov, the condition of separability of X can be replaced by the assumption of separability of the fibers of f; for the proof we need only to replace in the proof given by Engelking in [5], 4.5.14 (a), a countable base by a σ-discrete base, countable families $\{U_{i_1,\ldots,i_k}\}$ and $\{V_{i_1,\ldots,i_k}\}$ by point-countable (uncountable) families $\{U_{s_1,\ldots,s_k}\}$ and $\{V_{s_1,\ldots,s_k}\}$ (which is possible by Lemma 3) and, finally, to use Lemma 3 again in order to verify that the appropriate sets are F_σ-sets. Notice also that the assertion of the Mazurkiewicz theorem is not valid in the non-separable case. The following is a simple counterexample (cf. [10], § 30, X, Remark).

([4]) The condition that X is complete can be relaxed to the condition that the fibers of f are uniformly complete (see 4.3); the proof given in [5], 4.5.14 (a), works also in this case without any change.

Let I be the unit real interval, P and Q the sets of irrational and rational numbers from I, respectively, and let T be a discrete space of cardinality $2^{\aleph_0}$; let the family $\{C_t : t \in T\}$ consist of all countable dense subsets of P. Put

$$Y = I, \quad X = \{(t, x) \in T \times I : x \notin C_t\} \subset T \times I, \quad A = T \times Q,$$

and define $f : X \to Y$ by $f(t, x) = x$. Choose any G_δ-set B with $A \subset B \subset X$ and assume that $f|B : B \to f(B)$ is open. Then, by a theorem of Hausdorff ([5], 5.5.8 (d)), $f(B)$ is a G_δ-set in Y, and thus there exists a $C_t \subset f(B)$. Now, $f(\{t\} \times I \cap B) \subset f(B) \setminus C_t$, a contradiction with the openness of f.

A modification of this example yields a space X of weight $\aleph_1$. Notice also that from a theorem of Michael [14] and a theorem of Vainstein (see [5], 4.5.13) it follows that if f is as in the Mazurkiewicz theorem, but X is non-separable, then there exist G_δ-sets $B \supset A$ and $C \supset f(A)$ such that $f : B \to f(B) = C$ is hereditarily quotient (here it suffices to use the argument similar to that in the example in Section 4.3). A result analogous to the Mazurkiewicz theorem holds also for open-perfect mappings; this follows from a result of Engelking (cf. [5], 4.5.13) and the Remark.

4.2. We say ([18], p. 100) that a mapping $f : S \to T$ *preserves σ-discreteness* if f takes σ-discrete sets to σ-discrete sets. Of course, every mapping with separable range preserves σ-discreteness, and so does every open mapping with separable fibers (by Lemma 3). Thus the following is another improvement of the theorems of Kuratowski and Aleksandrov (cf. also Theorem 3).

THEOREM 4. *Let $f : S \to T$ be an open mapping which preserves σ-discreteness of a space S onto a space T of weight less than or equal to $\aleph_1$.*

(a) *If the fibers of f are spaces of second category and S is a space of first category, then T is of first category.*

(b) *If each fiber of f is not dense in itself, then* $\dim T \leqslant \dim S$.

The theorem follows immediately from Theorem 3 and from the following

LEMMA 4. *Let $f : S \to T$ be a mapping of a space S onto a space T of weight less than or equal to $\aleph_1$ which preserves σ-discreteness. Then the set $C = \{t \in T : w(f^{-1}(t)) \geqslant \aleph_1\}$ is σ-discrete; if f is in addition open, then C is open.*

Proof. Assume that the set C is not σ-discrete. Then there exists a set $E \subset C$ of cardinality $\aleph_1$ which is not σ-discrete, because either there exists a separable uncountable subset of C or else $|C| = \aleph_1$. Let

$$\mathscr{B} = \bigcup_n \mathscr{B}_n$$

be a base of S with each $\mathscr{B}_n$ discrete. Put

$$E_n = \{t \in E : |\{B \in \mathscr{B}_n : f^{-1}(t) \cap B \neq \emptyset\}| \geqslant \aleph_1\}. \tag{15}$$

Since

$$E = \bigcup_n E_n,$$

there exists an E_n which is not σ-discrete. Now, using (15) and the fact that $|E_n| = \aleph_1$, we can choose for every $t \in E_n$ a set $B_t \in \mathscr{B}_n$ and a point $s(t) \in f^{-1}(t) \cap B_t$ in such a way that different sets B_t are assigned to different points t. The set $A = \{s(T) : t \in E_n\}$ is discrete while its image $E_n = f(A)$ is not σ-discrete, a contradiction.

Now, let f be open. Take $t \in C$ and assume that there exists a sequence $(x_n) \subset T \setminus C$ with $x_n \to t$. The space

$$F = \overline{\bigcup_n f^{-1}(x_n)}$$

is separable, and so the set $W = S \setminus F$ intersects the fiber $f^{-1}(t)$. Thus $f(W)$ contains t but not any x_n, a contradiction with openness of f.

4.3. The following notion was introduced by Michael [13]: we say that the fibers of a mapping $f: S \to T$ of a space S to a space T are *uniformly complete* if there exists a metric ϱ, agreeing with the topology of S, such that every space $f^{-1}(t)$ is complete with respect to ϱ.

One can prove that open mappings with uniformly complete fibers do not raise the category (this was stated in [19], Remark 5 (b); a similar fact was observed independently by E. K. van Douwen). Using Michael's theorem [13] one can prove also that if the fibers of a mapping $f: S \xrightarrow{\text{onto}} T$ are in addition scattered, then $\dim T \leqslant \dim S$; this follows from results of Čoban [4], Theorems 5, 6 or 7 (one can use also our Theorem 3). The following counterexample shows that the condition that the fibers of the mapping f are scattered cannot be relaxed to the condition that each fiber of this mapping contains a dense discrete subspace.

Let in Theorem 1 the space T be the unit real interval and let $f: S \xrightarrow{\text{onto}} T$ with $S \subset B(T)$ (see the proof) be an open mapping with discrete fibers. There exists a G_δ-set $H \subset B(T)$ such that $H \supset S$ and f extends to a continuous mapping $\bar{f}: H \to T$ over H. Put $G = \bigcup \{\overline{f^{-1}(t)} : t \in T\}$, where the closure is taken in the space H, so $G \subset H$, and put $g = \bar{f}|G$. Now, $g: G \xrightarrow{\text{onto}} T$ is the required open mapping (cf. [2], § 6), since all fibers of g are closed in the space H which is completely metrizable.

Added in proof. A result similar to Theorem 1 in Section 2.1 was obtained by H. J. K. Junnila, *Stratifiable pre-images of topological spaces*, in: Colloquia Mathematica Societatis János Bolyai, 23. Topology, Vol. II (1980), p. 689-703.

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