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ON SPACES WHOSE PRODUCT WITH EVERY LINDELÖF SPACE IS LINDELÖF

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E. Michael asked whether the product X^ω is Lindelöf provided that X belongs to $\mathscr{L}$, where $\mathscr{L}$ is the class of all spaces whose Cartesian product with every Lindelöf space is Lindelöf. In this note we present some remarks concerning this problem.

One can view this note as an unsuccessful effort to solve Michael's problem in the negative way (see [1]–[3] for related problems).

In the sequel, I, Q, and N stand for the unit interval, the set of rational numbers of the unit interval, and the set of natural numbers, respectively. The symbols ω and ω_1 denote the first infinite ordinal number and the first uncountable ordinal number, respectively.

If X is a *Lindelöf P-space*, i.e., every G_δ-subset of X is open, then X belongs to $\mathscr{L}$. Noble proved that the product X^ω is Lindelöf (see [6]), so if Michael's problem has a negative solution, then one has to look for a counterexample in a class of spaces which are very different from Lindelöf P-spaces. The class of Lindelöf spaces in which every point is a G_δ-set is the desired one.

It is easy to prove

PROPOSITION 1. *Assume that the Continuum Hypothesis holds. Then if $Z \subset I^\Gamma$ and Z belongs to $\mathscr{L}$, then for every countable $S \subset \Gamma$ the space $p_S(Z)$ is σ-compact, where p_S denotes the projection from I^Γ onto I^S.*

Proof. Suppose not. Let $S \subset \Gamma$ be a countable subset of Γ such that $p_S(Z)$ is not σ-compact. Then $P = I^S \setminus p_S(Z)$ is not a G_δ-subset of I^S. Let $\{G_\alpha\colon \alpha < \omega_1\}$ be a strictly decreasing sequence of G_δ-subsets of I^S containing P such that for every open $P \subset U \subset I^S$ there is $\alpha < \omega_1$ such that $G_\alpha \subset U$. For $\alpha < \omega_1$ choose $x_\alpha \in G_\alpha \cap p_S(Z)$ and $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$. Then

$$Y = P \cup \{x_\alpha\colon \alpha < \omega_1\}$$

with the topology induced by sets of the form $U \cup K$, where U is open in I^S and $K \subset p_S(Z)$ is Lindelöf (see [5], Ex. 1.2). Notice that $Y \times p_S(Z)$ is not

Lindelöf since $\{(x_\alpha, x_\alpha)\colon \alpha < \omega_1\}$ is a discrete and closed subset of $Y \times p_S(Z)$, and this contradicts the fact that the continuous image of $\mathscr{L}$ belongs to $\mathscr{L}$.

The main result of this note is the example which shows that the class of Lindelöf spaces $X \subset I^{\omega_1}$ such that every point in X is a G_δ-set and, for every $\alpha < \omega_1$, the projection of X onto the first α-coordinates is σ-compact but X is not, is not empty.

EXAMPLE. *There exists an uncountable space $X \subset Q^{\omega_1}$ such that*

(a) *$p_\alpha(X)$ is countable for every $\alpha < \omega_1$, where p_α is the projection $p_\alpha\colon Q^{\omega_1} \to Q^\alpha$;*

(b) *if $x \in X$, then $\{x\}$ is a G_δ-subset of X;*

(c) *for every hereditarily Lindelöf space Y the Cartesian product $Y \times X^\omega$ is Lindelöf.*

Construction of X. There exists a family $\{A_\alpha\colon 1 \leqslant \alpha < \omega_1\}$ such that

(1) A_α is a countable set consisting of strictly increasing sequences of rational numbers of Q of length α for $1 \leqslant \alpha < \omega_1$;

(2) if $\alpha < \beta < \omega_1$, then $p_\alpha(A_\beta) = A_\alpha$;

(3) if $a \in A_\alpha$ for $1 \leqslant \alpha < \omega_1$, then for every limit ordinal number $\beta < \alpha$ the numbers

$$a(\beta) = \lim_{\lambda \to \beta} a(\lambda) \quad \text{and} \quad \sup\{a(\lambda)\colon \lambda < \alpha\}$$

are rational (see [4], p. 91, the construction of the Aronszajn tree).

Let us attach to $a \in A_\alpha$, for $1 \leqslant \alpha < \omega_1$, $x_a \in Q^{\omega_1}$ such that

$$x_a(\beta) = \begin{cases} a(\beta) & \text{if } \beta < \alpha, \\ \sup\{a(\lambda)\colon \lambda < \alpha\} & \text{if } \beta \geqslant \alpha. \end{cases}$$

Let $X = \bigcup\{X_\alpha\colon 1 \leqslant \alpha < \omega_1\}$, where $X_\alpha = \{x_a\colon a \in A_\alpha\}$, be a subspace of Q^{ω_1}. It is obvious that X is an uncountable space.

Proof of (a). If $1 \leqslant \alpha < \omega_1$, then from (2) we infer that

$$p_\alpha(X) \subset A_\alpha \cup p_\alpha\big(\bigcup\{X_\beta\colon \beta < \alpha\}\big),$$

so it is countable.

Proof of (b). If $x = x_a$, where $a \in A_\alpha$, then

$$\{x \in X\colon p_{\alpha+2}(x_a) = p_{\alpha+2}(x)\} = \{x_a\}.$$

By (a), $p_{\alpha+2}(X)$ is countable, so from the last equality it follows that x is a G_δ-set in X.

Proof of (c). Let Y be a hereditarily Lindelöf space and $\mathscr{V}$ an open covering of $Y \times X^\omega$. Let $\mathscr{B} = \bigcup\{\mathscr{B}_n\colon n \in N\}$ be an open basis of Q such that $\mathscr{B}_{n+1}$ is a pairwise disjoint open covering of Q which refines $\mathscr{B}_n$. We denote by S the set of finite sequences of natural numbers. For $s \in S$ the symbol $|s|$ stands for the length of s and its elements are denoted by s_i for $i \leqslant |s|$. If $x = x_a \in X$, where $a \in A_\alpha$, $x_a(\alpha) \in B \in \mathscr{B}$ and $\alpha < \lambda$, then put

$$F(x, B, \lambda) = \big(\mathop{\mathbf{P}}_{\beta < \omega_1} F_\beta\big) \cap X,$$

where

$$F_\beta = \begin{cases} \{x(\beta)\} & \text{if } \beta \leqslant \alpha, \\ B & \text{if } \beta = \lambda, \\ Q & \text{otherwise.} \end{cases}$$

For $x = (x_1, \ldots, x_n, \ldots) \in X^N$ and $s \in S$, put

$$A(x, s) = \{y \in Y\colon \text{ there are an open neighbourhood } H_y \text{ of } y,$$
$$\lambda(y) < \omega_1 \text{ and } V \in \mathscr{V} \text{ such that}$$
$$(y, x) \in H_y \times \mathop{\mathbf{P}}_{i \leqslant |s|} F\big(x_i, B_i, \lambda(y)\big) \times X \times X \times \ldots \subset V\},$$

where $x_i \in X_{\alpha(x_i)}$, $x_i\big(\alpha(x_i)\big) \in B_i \in \mathscr{B}_{s(i)}$ for $i \leqslant |s|$. We can assume, without loss of generality, that $\lambda(y)$ for $y \in A(x, s)$ is as small as possible. Notice that $\{y \in A(x, s)\colon \lambda(y) \leqslant \beta\}$ is an open subset of $A(x, s)$ since it is equal to

$$\bigcup \{H_y\colon y \in A(x, s) \text{ and } \lambda(y) \leqslant \beta\}$$

for $\beta < \omega_1$, and $A(x, s)$ is a Lindelöf space, so

$$\lambda\big(A(x, s)\big) = \sup\{\lambda(y)\colon y \in A(x, s)\} < \omega_1.$$

If $A(x, s) = \emptyset$, then put $\lambda\big(A(x, s)\big) = 0$. Put

$$\beta_1 = \sup\{\lambda\big(A(x, s)\big)\colon x \in X_1^N \text{ and } s \in S\}.$$

Since $X_1^{|s|}$ is countable, we have $\beta_1 < \omega_1$. If β_n is defined, then put

$$\beta_{n+1} = \sup\{\lambda\big(A(x, s)\big)\colon x \in X_\alpha^N, \alpha \leqslant \beta_n + 1 \text{ and } s \in S\}.$$

Then

$$\beta = \sup\{\beta_n\colon n \in N\} < \omega_1.$$

Put

$$\mathscr{X} = \{H_y \times \mathop{\mathbf{P}}_{i \leqslant |s|} F\big(x_i, B_i, \lambda(y)\big) \times X \times X \times \ldots\colon y \in A(x, s),\ s \in S,$$
$$x \in X_\alpha^N \text{ and } \alpha < \beta\}.$$

Claim 1. *$\mathscr{X}$ is a covering which refines $\mathscr{V}$.*

Proof of the Claim. By the definition, $\mathscr{X}$ refines $\mathscr{V}$, so it is enough to show that $\bigcup \mathscr{Z} = Y \times X^{\omega}$. If $x = x_a \in X$, then put

$$x' = \begin{cases} x & \text{if } a \in \bigcup \{A_\alpha: \alpha < \beta\}, \\ x_{a|\beta} & \text{otherwise,} \end{cases}$$

where $a|\beta$ stands for the projection onto the first β-coordinates. Let $y \in Y$, $x = (x_1, \ldots, x_n, \ldots) \in X^N$ and $x' = (x'_1, \ldots, x'_n, \ldots)$. There exist $k \in N$, an open neighbourhood H of y, $x'_i \in D_i \subset X$, $\alpha < \lambda_i < \omega_1$, $\alpha < \beta$, $B_i \in \mathscr{B}_{s(i)}$ for $i \leqslant k$, and $V \in \mathscr{V}$ such that

$$H \times D_1 \times D_2 \times \ldots \times D_k \times X \times X \times \ldots \subset V,$$

where

$$D_i = \Big(\mathop{\mathbf{P}}_{\lambda < \omega_1} D_i(\lambda)\Big) \cap X,$$

where

$$D_i(\lambda) = \begin{cases} \{x'_i(\lambda)\} & \text{if } \lambda \leqslant \alpha, \\ B_i & \text{if } \lambda = \lambda_i, \\ Q & \text{otherwise.} \end{cases}$$

Without loss of generality we can assume that $x'_i(\lambda) \in B_i$ for $\lambda \geqslant \alpha$ and $i \leqslant k$ (see (3) and (1)). Let z_i for $i \leqslant k$ be a point of $\bigcup \{X_\lambda: \lambda \leqslant \alpha + 1\}$ such that

$$z_i(\alpha) = \begin{cases} x'_i(\lambda) & \text{if } \lambda \leqslant \alpha, \\ x'_i(\alpha) & \text{if } \lambda > \alpha. \end{cases}$$

Put $z = (z_1, z_2, \ldots, z_k, z_k, z_k, \ldots)$. Then

$$(y, z) \in H \times D_1 \times \ldots \times D_k \times X \times X \times \ldots \subset V$$

because $\alpha < \lambda_i$ and $x'_i(\lambda) \in B_i$ for $\lambda \geqslant \alpha$ and $i \leqslant k$. Consequently, $y \in A(z, s)$. By the definition of β, we have $\lambda(y) < \beta$ for $i \leqslant k$ and an open neighbourhood H_y of y such that

$$Z(y, z) = H_y \times \mathop{\mathbf{P}}_{i \leqslant k} F\big(z_i, B_i, \lambda(y)\big) \times X \times X \times \ldots \subset V'$$

for some $V' \in \mathscr{V}$. Notice that from the fact that $x'_i(\lambda) \in B_i$ for $\lambda \geqslant \alpha$, the definitions of $F\big(z_i, B_i \lambda(y)\big)$ and z_i it follows that $(y, x') \in Z(y, z)$. In order to show that $(y, x) \in Z(y, z)$ it is enough to observe that $x'|\beta = x|\beta$ and that

$$Z(y, z) = H \times \mathop{\mathbf{P}}_{i \leqslant k} p_\beta^{-1} p_\beta\Big(F\big(z(i), B_i, \lambda(y)\big)\Big) \times X \times X \times \ldots$$

The proof will be completed if we show that

CLAIM 2. *$\mathscr{Z}$ has a countable subcover.*

Let us denote by p_Y and p_{X^N} projections of $Y \times X^N$ onto Y and X^N, respectively. Notice that the set

$$\mathscr{P} = \{p_{X^N}(Z): Z \in \mathscr{Z}\}$$

is countable. Put

$$\mathscr{Z}(P) = \{Z \in \mathscr{Z}\colon\ p_{X^N}(Z) = P\} \quad \text{for } P \in \mathscr{P}$$

and

$$\mathscr{G}(P) = \{H\colon \text{ there is } Z \in \mathscr{Z}(P) \text{ such that } p_Y(Z) = H\}.$$

Since Y is a hereditarily Lindelöf space, there is a countable subfamily $\mathscr{H}(P)$ of $\mathscr{G}(P)$ such that $\bigcup \mathscr{H}(P) = \bigcup \mathscr{G}(P)$. The family

$$\{H \times P\colon\ H \in \mathscr{H}(P) \text{ and } P \in \mathscr{P}\}$$

is a countable subcover of $\mathscr{Z}$.

Remark 1. In [1] it was proved that if Y is a hereditarily Lindelöf space and Z a Lindelöf C-scattered space, then $Y \times Z^\omega$ has the Lindelöf property. Notice that if a Lindelöf σ- (C-scattered) space Z is a subspace of X, then Z is countable. X does not also contain uncountable separable metric subspaces.

In order to show the first part of the remark it is enough to observe that every compact subset of X is scattered; otherwise, we could find a countable subset S of ω_1 such that the projection $p_S(X)$ would be uncountable and that every Lindelöf scattered space in which every point is of G_δ-type is countable.

If the second part of the remark does not hold, then we could find a countable subset S of ω_1 such that $p_S(X)$ would be uncountable.

Let us finish with a result which says that certain subspaces of I^{ω_1} with "good projections" do not belong to $\mathscr{L}$.

Proposition 2. *Let $Z \subset I^{\omega_1}$ be an uncountable space satisfying the following conditions*:

(a) *$p_\alpha(Z)$ is countable for $\alpha < \omega_1$*;

(b) *for every $z \in Z$ there exists $\alpha_z < \omega_1$ such that, for every $\alpha_z < \beta < \omega_1$, $z(\beta) = z(\alpha_z)$, and $z|\alpha_z$ is a strictly increasing sequence*;

(c) *$\{y \in Z\colon\ y|\alpha_z = z|\alpha_z\}$ is countable for every $z \in Z$*;

(d) *for every $z \in Z$ and every countable limit ordinal number α,*

$$z(\alpha) = \sup\{z(\beta)\colon\ \beta < \alpha\}.$$

Then Z does not belong to $\mathscr{L}$.

Proof. If Z is not Lindelöf, then there is nothing to prove, so let us assume that Z has the Lindelöf property. For $1 \leqslant \alpha < \omega_1$ put

$$A_\alpha = \{a \in p_\alpha(Z)\colon\ a \text{ is a strictly increasing sequence and there is no } z \in Z \text{ and } \lambda \leqslant \alpha \text{ such that } \alpha_z = \lambda \text{ and } z|\lambda = a|\lambda\}.$$

Let Y' be a subspace of I^{ω_1} such that

$$Y' = \bigcup\{Y'_\alpha\colon\ \alpha < \omega_1\},$$

where

$$Y'_\alpha = \{y_a \in I^{\omega_1}\colon a \in A_\alpha\} \quad \text{and} \quad y_a|\alpha = a \quad \text{for } a \in A_\alpha,$$

and

$$y_a(\beta) = \sup\{a(\lambda)\colon \lambda < \alpha\} \quad \text{for } \beta \geqslant \alpha.$$

Notice that Y' is disjoint with Z. Put $Y = Y' \cup Z$. The topology on Y is induced by sets $U \cup K$, where U is open in I^{ω_1} and $K \subset Z$.

The product $Y \times Z$ is not a Lindelöf space. In order to show this it is enough to notice that $\{(z, z)\colon z \in Z\}$ is an uncountable discrete closed subset of $Y \times Z$.

CLAIM. *Y is a Lindelöf space.*

We shall split the last claim into two claims:

CLAIM (i). *Y' is a Lindelöf space.*

CLAIM (ii). *If $Y' \subset U$ and U is open in Y, then $Y \setminus U$ is countable.*

Proof of Claim (i). The proof is similar to the proof of the Lindelöf property of X. Let $\mathscr{B}$ be a countable base of I and $\mathscr{V}$ an open covering of Y'. For $y \in Y'_\alpha$, $\alpha < \omega_1$, put

$$\mathscr{A}(y) = \{B \in \mathscr{B}\colon y(\alpha) \in B \text{ and there are } \alpha < \beta(y, B) < \omega_1 \text{ and } V \in \mathscr{V} \text{ such that } F(y, B, \beta(y, B)) = (\mathop{\mathbf{P}}_{\lambda < \omega_1} F_\lambda) \cap Y' \subset V\},$$

where

$$F_\lambda = \begin{cases} \{y(\lambda)\} & \text{if } \lambda \leqslant \alpha, \\ B & \text{if } \lambda = \beta(y, B), \\ I & \text{otherwise.} \end{cases}$$

Since Y' consists of increasing sequences, $\mathscr{A}(y) \neq \emptyset$ for every $y \in Y'_\alpha$ and $\alpha < \omega_1$. Put

$$\beta_1 = \sup\{\beta(y, B)\colon y \in Y'_1 \text{ and } B \in \mathscr{A}(y)\}.$$

Since Y'_1 and $\mathscr{A}(y)$ for $y \in Y'_1$ are countable sets, $\beta_1 < \omega_1$. If β_n is defined, then put

$$\beta_{n+1} = \sup\{\beta(y, B)\colon y \in \bigcup\{Y'_\lambda\colon \lambda \leqslant \beta_n + 1\} \text{ and } B \in \mathscr{A}(y)\}$$

and

$$\beta = \sup\{\beta_n\colon n \in N\}.$$

To complete the proof of the claim it is enough to show that

$$Y' = \bigcup\{F(y, B, \beta(y, B))\colon y \in \bigcup\{Y'_\lambda\colon \lambda < \beta\} \text{ and } B \in \mathscr{A}(y)\}.$$

Let y be an element of Y'_λ for $\lambda \geqslant \beta$. Then $y|\beta$ belongs to A_β. Let y' be a point of Y'_β such that $y'|\beta = y|\beta$. There exist $B \in \mathscr{B}$, α_1, α_2 and $V \in \mathscr{V}$ such that $y'(\beta) \in B$, $\alpha_1 < \beta < \alpha_2$, and

$$F = \big(\mathop{\mathbf{P}}_{\lambda < \omega_1} F_\lambda\big) \cap Y' \subset V,$$

where

$$F_\lambda = \begin{cases} \{y'(\lambda)\} & \text{if } \lambda < \alpha_1, \\ B & \text{if } \lambda = \alpha_2, \\ I & \text{otherwise.} \end{cases}$$

Without loss of generality we can assume that

$$\lim_{\lambda \to \alpha_1} y'(\lambda) \in B.$$

Let v be an element of Y'_{α_1} such that $v|\alpha_1 = y'|\alpha_1$. Then $B \in \mathscr{A}(v)$ and $\beta(v, B) < \beta < \alpha_2$. It is easy to see that $y' \in F\big(v, B, \beta(v, B)\big)$. Since

$$p_\beta^{-1} p_\beta\big(F\big((v, B), \beta(v, B)\big)\big) = F\big(v, B, \beta(v, B)\big) \quad \text{and} \quad y'|\beta = y|\beta,$$

we have also $y \in F\big(v, B, \beta(v, B)\big)$.

Proof of Claim (ii). Let U be an open subset of Y such that $Y' \subset U$. There is an open covering $\mathscr{H}$ of Y' in Y such that $H \subset U$ for every $H \in \mathscr{H}$ and

$$H = \big(\mathop{\mathbf{P}}_{\alpha < \omega_1} H_\alpha\big) \cap Y.$$

The family $\mathscr{H}$ has a countable subcover of Y', so there is $\beta < \omega_1$ such that $p_\beta^{-1} p_\beta(Y') \subseteq U$. To complete the proof of the claim it is enough to observe that if $a \in p_\alpha(Z) \setminus p_\alpha(Y')$ for $\alpha < \omega_1$, then $\{z \in Z\colon z|\alpha = a\}$ is countable (see the definition of A_α, (b) and (c)), and to apply (a).

Remark 2. We do not know whether X from the Example belongs to $\mathscr{L}$. **(P 1333)** We think that the answer to this question would be much more interesting than the results of our paper[1].

REFERENCES

[1] K. Alster, *A class of spaces whose Cartesian product with every hereditarily Lindelöf space is Lindelöf*, Fund. Math. 114 (1981), pp. 173–181.
[2] — *On Michael's problem concerning the Lindelöf property in the Cartesian products*, ibidem 121 (1984), pp. 149–167.
[3] — *A note on Michael's problem concerning the Lindelöf property in the Cartesian products*, Trans. Amer. Math. Soc. 278 (1983), pp. 369–375.

(1) The answer is negative (see *Problèmes*, p. 339).

[4] T. J. Jech, *Lectures in set theory*, Lecture Notes in Math. 217 (1971).
[5] E. Michael, *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, Compositio Math. 23 (1971), pp. 199–214.
[6] N. Noble, *Products with closed projection. II*, Trans. Amer. Math. Soc. 160 (1971), pp. 169–183.

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