SOME MODEL THEORY OF SIMPLE ALGEBRAIC GROUPS
OVER ALGEBRAICALLY CLOSED FIELDS

BY

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Let $G$ be a simple algebraic group over an algebraically closed field $K$. It was proved in [5] that the group-language theory of $G$ is categorical in uncountable powers, i.e. for every uncountable power $\mu$ there is a unique, up to isomorphisms, group $G'$ of cardinality $\mu$, which satisfies the same group-theoretical conditions as $G$. It is worth noting that for the proof no special information on the structure of $G$ is needed.

One can easily observe that the theorem implies that every $G'$, satisfying the same group-theoretical axioms as $G$, can be equipped with the algebraic-geometrical structure of the same kind as that of $G$. In some sense this means that the algebraic-geometrical structure on such groups is indeed defined by the abstract group structure.

Another, more exact version of the last fact is given by the known theorem of Borel and Tits [1], which states, in particular, that every (abstract) automorphism of $G$ is a composition of a rational automorphism of $G$ and of an automorphism induced by an automorphism of the field $K$.

The known proofs of the theorem use a deep structural theory of simple algebraic groups. Since the analogous theorem of [5] was proved by different and easier methods, the same kind of proof can be expected for the theorem cited above. Such a proof of the Borel–Tits theorem is given in the paper.

One of our purposes is also to show how model-theoretical methods can work in the theory of algebraic groups and, more generally, in algebraic geometry. There are two basic facts which link the general algebraic geometry with model theory. First, since the theory of algebraically closed fields of a given characteristic is categorical in uncountable powers, all structures definable in an algebraically closed field, e.g., structures of algebraic geometry, are $\omega$-stable of finite Morley rank. Moreover, the Morley rank is a good analogue of the algebraic-geometrical dimension and in most cases coincides with it. The second fact is the theorem of A. Tarski, which states that every relation definable in an algebraically closed field is a Boolean combination of polynomial equations.
We suppose the reader is familiar with main facts of the theory of categoricity (see, e.g., [4]). We also use some standard facts of the theory of algebraic groups, which can be found in [2].

1. Connected solvable groups of finite Morley rank. In this section, $B$ is a group whose theory is $\omega$-stable and of finite Morley rank. "Definable" always means "definable with parameters".

A group $B$ is called $m$-connected (model connected) if $B$ does not contain any definable subgroup $H$ of finite index in $B$.

Let $H$ be an arbitrary subgroup of $B$. The minimal definable subgroup $\hat{H} \supseteq H$ is called the $m$-closure (model-closure) of $H$ in $B$.

**Lemma 1.** If $H$ is soluble (nilpotent) of class $n$, then so is $\hat{H}$.

**Proof.** For $n = 1$ this was proved in [4], Lemma 7. Proceeding by induction and using the same method we get the lemma for all $n$.

**Lemma 2.** Let $B$ be an $m$-connected solvable group with trivial center. Let $V$ be a minimal normal definable subgroup of $B$ and let the factor group $T = B/C(V)$ be commutative ($C(V)$ denotes the centralizer of $V$ in $B$). Then there are definable binary operations $+$ and $\cdot$ on $V$ such that $\langle V, +, \cdot \rangle$ is an algebraically closed field.

**Proof.** Since $B$ is soluble, $V$ is abelian. Since $B$ is $m$-connected and $V$ is not central, $V$ is infinite.

Every element of $T$ acts by conjugation on $V$ and defines an automorphism of $V$. In what follows we shall denote the group operation in $V$ by $+$ and the action by conjugation of $t \in T$ on $v \in V$ by $tv$.

Thus, for any $t \in T$, $v_1, v_2 \in V$, we have

$$t(v_1 + v_2) = tv_1 + tv_2, \quad t0 = 0.$$ 

Therefore, we interpret elements of $T$ as additive operators on $V$. Let $-t$ denote the operator defined by $(-t)v = -tv$ and

$$T' = T \cup \{-t: t \in T\}.$$ 

For any $v \in V$ we also set

$$T' \cdot v = \{tv: t \in T'\}.$$ 

Since $T$ is $m$-connected, $T' \cdot v$ is infinite provided $v$ is not central, i.e. $v \neq 0$. Using the finiteness of Morley rank of $V$, it is easy to prove that for every $v_0 \in V$, $v_0 \neq 0$, the subgroup $V_0$ of $V$ generated by $T' \cdot v_0$ is definable and there is a natural number $N$ such that every $v \in V_0$ is of the form

$$v = t_1 v_0 + \ldots + t_N v_0, \quad t_1, \ldots, t_N \in T'$$ 

(for a detailed proof see [5], Theorem 3.3). It is evident that $V_0$ is normal in $B$, so $V = V_0$ by the minimality of $V$. 

Now we prove that if \( t_1 v_0 + \cdots + t_k v_0 = 0 \) for some \( t_1, \ldots, t_k \in T' \), then \( t_1 v + \cdots + t_k v = 0 \) for all \( v \in V \). For this we observe that for every \( t \in T' \)
\[
t \cdot (t_1 v_0 + \cdots + t_k v_0) = t_1 \cdot t v_0 + \cdots + t_k \cdot t v_0.
\]
Thus, if \( t_1 v_0 + \cdots + t_k v_0 = 0 \), then \( t_1 v + \cdots + t_k v = 0 \) for all \( v \in T' \cdot v_0 \) and, since \( V \) is generated by \( T' \cdot v_0 \), for all \( v \in V \).

For any \( t_1, \ldots, t_k \in T' \) we denote by \( t_1 + \cdots + t_k \) the additive operator on \( V \) defined by \( (t_1 + \cdots + t_k) v = t_1 v + \cdots + t_k v \) for all \( v \in V \). Denote by \( F \) the ring of all operators of the form \( t_1 + \cdots + t_k \) for any \( t_1, \ldots, t_k \in T' \).

It follows from the fact proved above that if \( f_1 v_0 = f_2 v_0 \) for some \( f_1, f_2 \in F \), \( v_0 \in V \), \( v_0 \neq 0 \), then \( f_1 v = f_2 v \) for any \( v \in V \). Thus we get \( f_1 = f_2 \).

So there is a one-to-one correspondence between elements \( f \) of \( F \) and pairs \( \langle v_0, v \rangle \), namely \( v = f v_0 \), where \( v_0 \neq 0 \), \( v_0 \in V \), is fixed and \( v \) is an arbitrary element of \( V \). Moreover, every non-zero element of \( F \) is invertible. Indeed, if \( f_1 v_0 = v_0 \), \( v_0 \neq 0 \), then there is a \( g \in F \) such that \( g v_0 = v_1 \). Thus \( g \cdot f \cdot v = v_1 \), which implies \( g \cdot f = 1 \).

Finally, we summarize the above-proved facts:

- \( F \) is an infinite field, interpretable in \( B \) by elements \( v \in V \), provided \( v_0 \in V \), \( v_0 \neq 0 \), is fixed. \( F \) is algebraically closed since \( F \) is \( \omega \)-stable (see [3]).

**Corollary.** If \( B \) is an m-connected solvable group which is not nilpotent, then an algebraically closed field is definable in \( B \).

**Proof.** There exists an infinite group \( B' \) which is definable in \( B \), m-connected, solvable, has a trivial center, and is of minimal Morley rank among all the groups satisfying these conditions. This group satisfies the assumptions of Lemma 2.

2. Algebraic groups over algebraically closed fields. In this section, \( G \) is an algebraic group over an algebraically closed field \( K \). Every such group is definable in \( K \) (for details see, e.g., [5], Section 2), and therefore \( G \) has an \( \omega \)-stable theory of finite Morley rank.

Now we recall the following well-known fact (see [4], Theorem 13.3):

**Tarski's theorem.** Every definable subset \( S \) of the set \( K^n \) of \( n \)-tuples of elements of \( K \) is a Boolean combination of subsets of the form

\[
\{ \langle x_1, \ldots, x_n \rangle \in K^n: p(x_1, \ldots, x_n) = 0 \},
\]

where \( p \) is an \( n \)-variable polynomial with coefficients from \( K \). In other words, \( S = F \setminus E \), where \( F, E \subseteq K^n \) are closed in Zariski topology.

**Corollary.** Every definable subgroup \( H \) of \( G \) is an algebraic subgroup of the algebraic group \( G \).

The corollary follows from the fact that every open subgroup of \( G \) is algebraic.

In the sequel, let \( G \) be a simple algebraic group and let \( B \) be a Borel
subgroup of $G$, i.e., a maximal solvable algebraic subgroup of $G$. The $m$-closure $\tilde{B}$ of $B$ in $G$ is a solvable algebraic group, and thus $\tilde{B} = B$.

Take a minimal normal subgroup $V$ definable in $B$, which lies in the center of the unipotent part of $B$. Then $T = B/C(V)$ is a torus, and so $T$ is abelian.

Thus $B$ satisfies all the assumptions of Lemma 2. Consequently, as was shown in the proof of Lemma 2, $T$ is embeddable in the field $F$ of automorphisms of the unipotent subgroup $V$. The characteristic of $F$ coincides with that of $K$ because it is defined by the exponent of $V$. As was already noted, $T$ is a torus, i.e., a product of a finite number of copies of the multiplicative group $K^*$ of the field $K$. On the other hand, $T$ is a subgroup of the multiplicative group $F^*$ of $F$. Comparing the fusion parts of $T$, $F^*$, and $K^*$ we infer that $T$ is a one-dimensional torus. Since any torus $T$ acting on an abelian unipotent group has a one-dimensional invariant subgroup on which $T$ acts transitively, by the minimality of $V$ we get the following special version of Lemma 2.

**Proposition 1.** Assume that $V$ is a one-dimensional unipotent group. Let $kv$ be the action of $k \in K \cong T$ on $v \in V$. Then $V$ is a one-dimensional vector space over $K$ with respect to the multiplication $kv$ and the group operation on $V$ is the addition. Fixing an arbitrary non-zero element $v_0 \in V$ we get a birational isomorphism $j: K \rightarrow \langle V, +, \cdot \rangle$ such that $j(k) = kv_0$ for any $k \in K$.

Remark. It is worth noting that $V$ is just the root subgroup $U_d$ corresponding to the maximal root $d$ of the root system $\mathfrak{R}$ of $G$.

As a consequence of Proposition 1 we get the following statement:

**Proposition 2.** Let $G_1$ be a linear (not necessarily algebraic) group over an algebraically closed field $K_1$ such that there is an abstract isomorphism $s: G \rightarrow G_1$. Then the subgroups $B_1 = s(B)$ and $V_1 = s(V)$ of $G_1$ satisfy the assertions of Proposition 1 and there is a subfield $K'_1$ of $K_1$ such that

$s: \langle V, +, \cdot \rangle \rightarrow \langle V_1, +, \cdot \rangle$, \hspace{1cm} $j_1: K_1 \rightarrow \langle V_1, +, \cdot \rangle$,

$j_1^{-1} \circ s \circ j: K \rightarrow K'_1 \subseteq K_1$

are isomorphisms of the fields.

Proof. Since $B_1$ is a solvable linear group, by the Kolchin–Mal’cev theorem it contains a triangularizable normal subgroup $B_1^0$ of finite index. The intersection $V_1 \cap B_1^0$ is non-trivial since $B_1^0$ is of finite index in $B_1$. Thus $V_1 \subseteq B_1^0$ by minimality. It is easy to see that $V_1$ lies in the commutant of $B_1^0$, which is a unipotent subgroup. Moreover, $V_1$ lies in the center of the unipotent part of $B_1$ since every non-trivial normal subgroup of a nilpotent group intersects non-trivially with center.

Now, considerations analogous to those from the proof of Proposition 1 and the fact that $T_1 = B_1/C(V_1)$ acts transitively on $V_1$ show that $V_1$ is a one-dimensional algebraic unipotent group and $T_1$ is an algebraic torus.
over a field $K' \subseteq K_1$, which is birationally isomorphic by $j_1$ to $\langle V_1, +, \cdot \rangle$.

Given a field embedding $u: K \rightarrow K_1$, we denote by $u^*$ the group embedding

$$u^* : GL(n, K) \rightarrow GL(n, K_1)$$

induced by $u$.

**Lemma 3.** Let $u$ be a field isomorphism and $j$ a birational isomorphism of the additive group of $K$ onto a unipotent group $V$. Then there exists a birational isomorphism $j_0$ such that the following diagram is commutative:

$$\begin{array}{c}
K \\ \\
\downarrow u \\
K_1 \cong K'_1 \\
\downarrow j_0 \\
V_1 \\
\end{array}$$

$u^*(kv_0) = (u(k))v_1$,

where $v_1 = u^*(v_0)$. Therefore, $j_0(u(k)) = (u(k))v_1$, and so $j_0$ is birational.

**Lemma 4.** Under the assumptions of Proposition 2 define $u = j_1^{-1} \circ s \circ j$, $G_2 = u^*(G)$, and $V_2 = u^*(V)$. Then there is a group isomorphism $r: G_2 \rightarrow G_1$ such that $s = r \circ u^*$ and $r$ is birational on $V_2$.

**Proof.** By the definition we have $j \circ u = s \circ j$, and by Lemma 3 we obtain $j_0 \circ u = u^* \circ j$ for some birational isomorphism $j_0: K'_1 \rightarrow V_2$. Thus we have $j_1 \circ j_0^{-1} \circ u^* \circ j = s \circ j$. Putting $r = s(u^*)^{-1}$, we get $r|V_2 = j_1 \circ j_0^{-1}$, and thus $r$ is a birational isomorphism on $V_2$.

Without loss of generality we can assume that $K_1$ is algebraically closed.

The groups $G_1$ and $G_2$ are subgroup of $GL(n, K_1)$, and $\bar{G}_2$ is an algebraic subgroup of $GL(n, K'_1)$. Denote by $\bar{G}_1$ and $\bar{G}_2$, respectively, closures of these groups in $GL(n, K_1)$ in Zariski topology. Obviously, $\bar{G}_2$ is simple since $G_2$ is algebraic and simple.

**Lemma 5.** The mapping $r: G_2 \rightarrow G_1$ of Lemma 4 can be extended to a mapping $\bar{r}: \bar{G}_2 \rightarrow \bar{G}_1$ which is definable in $K_1$ (in the field-language) and is a group isomorphism. $G_1$ is a simple algebraic group over $K_1$.

**Proof.** The least normal subgroup of $G_2$ containing $V_2$ coincides with $G_2$. Thus every element of $G_2$ is of the form $v_1^{g_1} \ldots v_k^{g_k}$ for some $v_1, \ldots, v_k \in V_2$ and $g_1, \ldots, g_k \in G_2$. Using the finiteness of Morley rank of $G_2$, we get a natural number $N$ and some fixed $g_1, \ldots, g_N \in G_2$ such that every element of $G_2$ is of the form $v_1^{g_1} \ldots v_N^{g_N}$ for some $v_1, \ldots, v_N \in V_2$.

Define $\bar{r}$ on $\bar{V}_2$ as a birational isomorphism extending $r$. Put $h_i = \ldots$
$r(g_i), \ i \in \{1, \ldots, N\}$, and consider the following binary relation between
$x \in \tilde{G}_2 \subseteq K_{1'}$ and $y \in \tilde{G}_1 \subseteq K_{1''}$:

$$R(x, y) \equiv (\forall v_1, \ldots, v_N \in \tilde{V}_2)(x = v_1^{\#1} \ldots v_N^{\#N} \leftrightarrow y = \tilde{r}(v_1)^{\#1} \ldots \tilde{r}(v_N)^{\#N}).$$

It is easy to see that $R(x, y)$ is equivalent to $r(x) = y$ for $x \in G_2$. Since $K_{1'} \subseteq K_1$ is an elementary extension, $R$ defines the graph of a group isomorphism of $\tilde{G}_2$ onto some subgroup of $GL(n, K_1)$ containing $G_1$ and $\tilde{V}_1$. Since $\tilde{r}|\tilde{V}_2$ is a mapping definable in $K_1$ and $\tilde{V}_2$ is a definable subset of $K_{1''}$, $R$ is definable in $K_1$.

The last fact implies, in particular, that $\tilde{r}(\tilde{G}_2)$ is definable in $K_1$. Thus, by the Corollary to Tarski’s theorem, $\tilde{r}(\tilde{G}_2)$ is an algebraic group over $K_1$, and hence $\tilde{r}(\tilde{G}_2) = \tilde{G}_1$.

For $\text{char}(K_1) = p \neq 0$, denote by $\text{Fr}_k$ the Frobenius automorphism $\text{Fr}_k: y \mapsto y^{p^k}$. Set $\text{Fr}_k = \text{id}$ if $\text{char}(K_1) = 0$.

**Proposition 3.** Let $H$ be a definable subset of $K_{1''}$ and $t$ a definable mapping $H \to K_1$, where $K_1$ is an algebraically closed field. Then there are an open subset $\tilde{H}$ of $H$, a natural number $k$, and a rational function $\hat{t}: \tilde{H} \to K_1$ such that, for every $x \in \tilde{H}$, $y \in K_1$,

$$t(x) = y \iff \hat{t}(x) = \text{Fr}_k y.$$

**Proof.** By Tarski’s theorem we have

$$t(x) = y \equiv P_0(x, y) \lor \ldots \lor P_m(x, y),$$

where every $P_i(x, y), \ i \in \{0, \ldots, m\}$, is a conjunction of polynomial equations and inequalities. Moreover,

$$H = H_0 \cup \ldots \cup H_m, \quad \text{where } H_i = \{x \in H: (\exists y) P_i(x, y)\}, \ i \in \{0, \ldots, m\}.$$

Again by Tarski’s theorem, $H_i$ are of the form $F_i \setminus E_i$, where $F_i$ and $E_i$ are closed in $H$. Hence for some $i \in \{0, \ldots, m\}$, say $i = 0$, $H_i$ is an open subset of $H$. We also assume without loss of generality that the closure of $H_0$ is irreducible.

It is obvious that $t(x) = y \equiv P_0(x, y)$ for $x \in H_0$. Let

$$P_0(x, y) = (\bigwedge_{1 \leq i \leq p} f_i(x, y) = 0 \land \bigwedge_{1 \leq j \leq q} g_j(x, y) \neq 0).$$

Note that $p > 0$, for otherwise $P_0(x, y)$ is not a graph of a function.

Now, consider $f_i$ ($1 \leq i \leq p$) and $g_j$ ($1 \leq j \leq q$) as polynomials with one variable $y$ and with rational functions on $H_0$ as coefficients. Since a ring of one-variable polynomials is a principal ideal domain, we obtain

$$P_0(x, y) \equiv (f(x, y) = 0 \land \bigwedge_{1 \leq j \leq q} g_j(x, y) \neq 0)$$

for some polynomial $f(x, y)$ of the ring.
We can assume that \( f(x, y) \) is irreducible over the field of rational functions on \( H_0 \), for otherwise we could decompose \( P_0(x, y) \) into a disjunction of expressions of the same kind.

Since \( f(x, y) \) is irreducible, \( g_j(x, y) \) \((1 \leq j \leq q)\) and \( f(x, y) \) have no common solution in \( K_1 \) for every \( x \) belonging to an open subset \( \bar{H} \) of \( H_0 \). Thus for \( x \in \bar{H} \) we get \( P_0(x, y) \equiv (f(x, y) = 0) \) and \( f(x, y) \) has a unique solution in \( K_1 \) for every \( x \in \bar{H} \). Consequently,

\[
f(x, y) = a(x)(\text{Fr}_k(y) - \hat{\ell}(x))
\]

for some rational functions \( a(x) \), \( \hat{\ell}(x) \) and a natural number \( k \).

**Lemma 6.** The isomorphism \( \bar{r}: \bar{G}_2 \rightarrow \bar{G}_1 \) of Lemma 5 is birational.

**Proof.** The isomorphism \( \bar{r} \) is an \( n \)-tuple \( \langle r_1, \ldots, r_n \rangle \), where \( r_i: K_1^* \rightarrow K_1 \) are definable functions for all \( i \in \{1, \ldots, n\} \). It follows from Proposition 3 that for every \( x \) from an open subset of \( \bar{G}_2 \) we have \( r_i(x) = \text{Fr}_k(\hat{\ell}_i(x)) \), where \( \hat{\ell}_i(x) \) is a rational function and \( k_i \) an integer, \( i \in \{1, \ldots, n\} \). We assume that if \( \text{char}(K_1) = 0 \), then \( k_i = 0 \), and if \( \text{char}(K_1) \neq 0 \), then \( k_i \) is maximal. Thus \( \hat{\ell}_i(x) \) is separable, i.e., \( d\hat{\ell}_i(x) \neq 0 \), where \( d\hat{\ell}_i \) is the differential of \( \hat{\ell}_i \).

Let \( k \) be the minimal natural number such that \( k + k_i \geq 0 \) for all \( i \in \{1, \ldots, n\} \). Then \( \text{Fr}_k \circ \bar{r} \) is rational on an open subset of \( G_2 \), and thus on \( G_2 \). Note that \( d(\text{Fr}_k \circ \bar{r}) \neq 0 \) since \( d(\text{Fr}_k \circ r_i) \neq 0 \) for some \( i \in \{1, \ldots, n\} \). We shall prove that \( k = 0 \), i.e., \( \text{Fr}_k = \text{id} \).

Suppose that \( k > 0 \). Then \( d(\text{Fr}_k \circ \bar{r}) \) is a non-zero homomorphism of Lie algebras \( L(G_2) \rightarrow L(G_1) \). But \( d(\text{Fr}_k \circ \bar{r}) \) is a zero homomorphism on \( L(\bar{V}_2) \) since \( \bar{r} \) is rational on \( \bar{V}_2 \) and \( k > 0 \). Thus, the kernel of \( d(\text{Fr}_k \circ \bar{r}) \) is non-trivial. This is a contradiction since \( L(G_2) \) is a simple Lie algebra.

Thus we have proved that \( k_i \geq 0 \) for all \( i \in \{1, \ldots, n\} \), and so \( \bar{r} \) is rational. By symmetry, \( \bar{r}^{-1} \) is also rational, which completes the proof.

**Theorem (Borel and Tits [1]).** Let \( s \) be an isomorphism of a simple algebraic group \( G \) over an algebraically closed field \( K \) onto a linear group \( G_1 \) over an algebraically closed field \( K_1 \). Then there exist:

- an embedding of fields \( u: K \rightarrow K_1 \);
- algebraic groups \( \bar{G}_1 \) and \( \bar{G}_2 \) over \( K_1 \), \( G_1 \subseteq \bar{G}_1 \);
- a group embedding \( u^*: G \rightarrow \bar{G}_2 \) induced by \( u \) and a rational isomorphism \( r: \bar{G}_2 \rightarrow \bar{G}_1 \) such that \( s = r \circ u^* \).

This theorem follows immediately from Proposition 2 and Lemmas 4–6.

**References**


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