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ON ADDITIVE FUNCTIONS HAVING A NON-DECREASING NORMAL ORDER

BY

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1. We shall say that a non-negative additive function $f(n)$ has a *non-decreasing normal order* if there exists a monotonically non-decreasing function $A(n)$ such that

$$\frac{f(n)}{A(n)} \to 1 \quad (n \to \infty) \tag{1.1}$$

for almost all n (i.e. for all n neglecting a sequence of density 0).

It is obvious that the functions $\log\sigma(n), \log\varphi(n), \log n$ have non-decreasing normal orders. The magnitude at prime places p is $O(\log p)$ for all of them. It would be interesting to characterize the class of the additive functions having a non-decreasing normal order. (**P 1004**)

Let $H(x)$ be a monotonically non-decreasing positive function and let $f(n)$ be defined as

$$f(n) = \sum_{p|n} H(p), \tag{1.2}$$

where p runs over the prime divisors of n. Narkiewicz [3] proved the following assertion:

If $H(x)/\log x$ is a monotonically non-decreasing function and $f(n)$ defined by (1.2) has a non-decreasing normal order, then

$$H(x) = O\big((\log x)^{1+\delta}\big) \tag{1.3}$$

holds for every positive δ.

He also stated that (1.3) holds without the monotonicity of $H(x)/\log x$.

Our purpose is to prove this conjecture ([1]). Namely, let

$$t(x) = \frac{H(x)}{\log x}, \tag{1.4}$$

$$T(x) = \max_{2\leqslant z\leqslant x} t(z). \tag{1.5}$$

([1]) Another proof has been supplied by P. D. T. A. Elliott (see this journal 36 (1976), p. 289-294) [Note of Editors].

THEOREM 1. *Let* $H(x)$ *be a positive monotonically non-decreasing function in* $[1, \infty]$, *and let* $f(n)$ *be defined by* (1.2). *If* $f(n)$ *has a non-decreasing normal order, then*

$$T(x^\nu) \leqslant c_1 T(x) \qquad (x \geqslant x_0(\nu)) \tag{1.6}$$

for every $\nu > 1$, c_1 $(c_1 > 1)$ *being an absolute constant. Consequently,*

$$H(x) = O\big((\log x)^{1+\delta}\big) \qquad (x \to \infty) \tag{1.7}$$

holds for every positive δ.

We hope to return to this question to give a necessary and sufficient condition in a forthcoming paper.

2. In the sequel, $c, c_1, c_2, \ldots$ denote absolute positive constants, and $\delta, \varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ arbitrarily small positive constants, not the same at every occurrence. ν $(\nu \geqslant 1)$ is an arbitrary positive integer. p denotes prime numbers, and $\omega(n)$ the number of the distinct prime factors of n.

To prove our theorem we need first some lemmas.

LEMMA 1. *Under the assumption of Theorem* 1 *we have*

$$f(n) \leqslant (1+\varepsilon_1) f(m) \tag{2.1}$$

for all $n \in [\frac{1}{2}x^{\nu+1}, \frac{3}{4}x^{\nu+1}]$ *and* $m \in (\frac{3}{4}x^{\nu+1}, x^{\nu+1})$, *apart from* $\varepsilon_2 x^{\nu+1}$ *of* n*'s and* $\varepsilon_3 x^{\nu+1}$ *of* m*'s if* $x > x_0(\nu, \varepsilon_1, \varepsilon_2, \varepsilon_3)$.

This is a straightforward consequence of the assumption that $f(n)$ has a non-decreasing normal order.

Let $N_\nu(x)$ denote the number of those integers n in the interval $(\frac{1}{2}x^{\nu+1}, \frac{3}{4}x^{\nu+1}]$ the largest prime factors of which are greater than x^ν.

LEMMA 2. *We have*

$$\left| N_\nu(x) - \frac{1}{4}\log\left(1+\frac{1}{\nu}\right)\cdot x^{\nu+1}\right| \leqslant c_2 \frac{x^{\nu+1}}{\log x^{\nu+1}} \qquad \text{for } x \geqslant 2. \tag{2.2}$$

Proof. This is obvious, since

$$N_\nu(x) = \sum_{p > x^\nu}\left(\left[\frac{3}{4}\frac{x^{\nu+1}}{p}\right] - \left[\frac{1}{2}\frac{x^{\nu+1}}{p}\right]\right) = \frac{1}{4}x^{\nu+1}\sum_{x^\nu < p < 3x^{\nu+1}/4}\frac{1}{p} + O\big(\pi(x^{\nu+1})\big)$$

and

$$\sum_{u<p<v}\frac{1}{p} = \log\frac{\log v}{\log u} + O(\exp[-\sqrt{\log u}]) \tag{2.3}$$

(see Prachar [4]).

Let $F(x^t, x)$ be the number of those integers smaller than x^t all prime factors of which do not exceed x. Levin and Faĭnleĭb [2] proved the asymptotical formula

$$F(x^t, x) = x^t z(t) + O\left(\frac{x}{\log x}\right)$$

as $x \to \infty$ uniformly for all t varying in a bounded interval. In this formula

$$z(t) = \exp\left[-t\left(\log t + \log\log t - 1 - \frac{\log\log t}{\log t}\right) + O\left(\frac{1}{\log t}\right)\right] \quad \text{as } t \to \infty.$$

Let $M_\nu(x)$ be the number of integers in $(\frac{3}{4}x^{\nu+1}, x^{\nu+1}]$ all the prime factors of which do not exceed x.

We get immediately

LEMMA 3. *For every fixed large ν and for $x > x_0(\nu)$ we have*

$$M_\nu(x) \geqslant c_1 C(\nu) x^{\nu+1}, \tag{2.4}$$

where

$$C(\nu) = \exp\left[-(\nu+1)\left(\log(\nu+1) + \log\log(\nu+1) - 1 - \frac{\log\log(\nu+1)}{\log(\nu+1)}\right)\right]. \tag{2.5}$$

Let $\gamma = \exp(-1) = (2,71\ldots)^{-1}$, and $w_i = x^{\gamma^i}$ $(i = 0, 1, \ldots)$.

It is a well-known result of Hardy and Ramanujan [1] that

$$\frac{\omega(n)}{\log\log n} \to 1$$

for almost all n. We use the weaker result, namely that

$$\omega(n) \leqslant 2\log\log n \tag{2.6}$$

for almost all n.

First we estimate the number $N(i, t)$ of those integers $n \leqslant x^{\nu+1}$ which have at least t distinct prime divisors in $[w_{i+1}, w_i]$. It is obvious that

$$N(i, t) \leqslant \sum{}' \left[\frac{x^{\nu+1}}{Q_t}\right],$$

where by $\sum'$ we mean the summation extended over the integers Q_t which are the product of t distinct primes p in $[w_{i+1}, w_i]$. From (2.3) we get

$$\sum{}' \frac{1}{Q_t} \leqslant \frac{1}{t!}\left(\sum_{w_{i+1} \leqslant p < w_i} \frac{1}{p}\right)^t < \frac{1 + c_3 t \exp[-\sqrt{\log w_{i+1}}]}{t}$$

assuming only that

$$t \exp[-\sqrt{\log w_{i+1}}] \leqslant 1. \tag{2.7}$$

Consequently,

$$N(i, t) \leqslant c_4 \frac{x^{\nu+1}}{t!}. \tag{2.8}$$

Let

$$t_i = c_5(\nu+1)(i+1)^2 \quad (i = 0, 1, \ldots, r_0), \tag{2.9}$$

$$r_0 = [(\log\log x)^{1/3}], \tag{2.10}$$

c_5 being a positive integer which we shall specify later.

From (2.8) we get

$$\sum_{i=1}^{r_0} N(i, t_i) \leqslant c_6 x^{\nu+1} \sum_{i=0}^{r_0} \frac{1}{t_i!}.$$

Using the relation

$$t_i! \geqslant c_7\left(\frac{t_i}{e}\right)^{t_i} \geqslant c_8 \exp\left[\frac{c_5}{2}(\nu+1)(i+1)^2 \log(\nu+1)\right]$$

which holds for every large c_5, we get

$$\sum_{i=0}^{r_0} \frac{1}{t_i!} < c_9 \exp\left[-\frac{c_5}{2}(\nu+1)\log(\nu+1)\right],$$

and so

$$\sum_{i=0}^{r_0} N(i, t_i) \leqslant c_{10} \exp\left[-\frac{c_5}{2}(\nu+1)\log(\nu+1)\right] x^{\nu+1} = B_\nu x^{\nu+1}. \tag{2.11}$$

We see that

$$\sum_{i=0}^{r_0} t_i \geqslant \frac{c_5(\nu+1)r_0^3}{6} \geqslant 2\log\log x^{\nu+1} \tag{2.12}$$

if $c_5 \geqslant 1$ and $x > x_0(\nu)$. Let c_5 be so large that $B_\nu < \frac{1}{2}c_1 C(\nu)$ (see (2.11), (2.4) and (2.5)). This holds for $c_5 = 3$ and for every large ν.

Hence we get immediately the following assertion.

There exists at least $\frac{1}{2}c_1 C(\nu)x^{\nu+1}$ integers m in $(\frac{3}{4}x^{\nu+1}, x^{\nu+1})$ all prime factors of which are smaller than x and have at most $t_i - 1$ prime factors in every interval $[w_{i+1}, w_i)$ $(i = 0, 1, \ldots, r_0)$.

From (2.6) and (2.12) and by the monotonicity of $H(p)$ it follows that

$$f(m) \leqslant \sum_{i=0}^{r_0} H(w_i) t_i \tag{2.13}$$

for all m but $\varepsilon_4 x^{\nu+1}$ of them.

For the set of integers counted in Lemma 2, we get $H(x^\nu) \leqslant f(n)$. Using Lemma 1 we have

$$H(x^\nu) \leqslant (1+\varepsilon_1)\sum_{i=0}^{r_0} H(w_i)t_i \tag{2.14}$$

for all large x $(x > x_0(\nu))$.

Dividing (2.14) by $\log x^{\nu+1}$, we have

$$t(x^\nu) \leqslant c_{10}\sum_{i=0}^{r_0} \gamma^i (i+1)^2 t(w_i) \qquad \text{for } x > x_1(\nu). \tag{2.15}$$

To prove (1.6) we may assume that $T(y) \to \infty$ as $y \to \infty$. So for every y we have $u = u(y) \leqslant y$ $(u(y) \to \infty$ as $y \to \infty)$, so that $T(y) = T(u) = t(u)$. Now we take $u = x^\nu$ and apply (2.15). We get

$$T(y) = T(x^\nu) \leqslant \Big(c_{10}\sum_{i=0}^{\infty} \gamma^i (i+1)^2\Big) T(x) \leqslant c_1 T(y^{1/\nu})$$

if y is sufficiently large, and $y \geqslant y_0(\nu)$. This proves (1.6).

Now (1.7) follows immediately from (1.6). Let ν be large and fixed. Take $x_k = x_0^{\nu^k}$, $x_0 = x_0(\nu)$. Then

$$T(x_k) \leqslant c_1^k T(x_0),$$

and so

$$\frac{T(y)}{(\log y)^{c_{11}/\log\nu}} \to 0 \qquad (y \to \infty).$$

Choosing ν such that $c_{11}/\log\nu \leqslant \delta$, we get (1.7) immediately.

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