ON ADDITIVE FUNCTIONS
HAVING A NON-DECREASING NORMAL ORDER

BY

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1. We shall say that a non-negative additive function $f(n)$ has a non-decreasing normal order if there exists a monotonically non-decreasing function $A(n)$ such that

$$
\frac{f(n)}{A(n)} \to 1 \quad (n \to \infty)
$$

for almost all $n$ (i.e. for all $n$ neglecting a sequence of density 0).

It is obvious that the functions $\log \sigma(n), \log \varphi(n), \log n$ have non-decreasing normal orders. The magnitude at prime places $p$ is $O(\log p)$ for all of them. It would be interesting to characterize the class of the additive functions having a non-decreasing normal order. (P 1004)

Let $H(x)$ be a monotonically non-decreasing positive function and let $f(n)$ be defined as

$$
f(n) = \sum_{p|n} H(p),
$$

where $p$ runs over the prime divisors of $n$. Narkiewicz [3] proved the following assertion:

If $H(x) / \log x$ is a monotonically non-decreasing function and $f(n)$ defined by (1.2) has a non-decreasing normal order, then

$$
H(x) = O((\log x)^{1+\delta})
$$

holds for every positive $\delta$.

He also stated that (1.3) holds without the monotonicity of $H(x) / \log x$. Our purpose is to prove this conjecture (1). Namely, let

$$
t(x) = \frac{H(x)}{\log x},
$$

$$
T(x) = \max_{2 \leq z \leq x} t(z).
$$

(1) Another proof has been supplied by P. D. T. A. Elliott (see this journal 36 (1976), p. 289-294) [Note of Editors].
Theorem 1. Let \( H(x) \) be a positive monotonically non-decreasing function in \([1, \infty]\), and let \( f(n) \) be defined by (1.2). If \( f(n) \) has a non-decreasing normal order, then

\[
T(x') \leq c_1 T(x) \quad (x \geq x_0(v))
\]

for every \( v > 1 \), \( c_1 (c_1 > 1) \) being an absolute constant. Consequently,

\[
H(x) = O \left( (\log x)^{1+\delta} \right) \quad (x \to \infty)
\]

holds for every positive \( \delta \).

We hope to return to this question to give a necessary and sufficient condition in a forthcoming paper.

2. In the sequel, \( c, c_1, c_2, \ldots \) denote absolute positive constants, and \( \delta, \varepsilon, \varepsilon_1, \varepsilon_2, \ldots \) arbitrarily small positive constants, not the same at every occurrence. \( \sigma (\sigma \geq 1) \) is an arbitrary positive integer. \( p \) denotes prime numbers, and \( \omega(n) \) the number of the distinct prime factors of \( n \).

To prove our theorem we need first some lemmas.

Lemma 1. Under the assumption of Theorem 1 we have

\[
f(n) \leq (1 + \varepsilon_1)f(m)
\]

for all \( n \in \left[ \frac{1}{4} x^{r+1}, \frac{3}{4} x^{r+1} \right] \) and \( m \in (\frac{3}{4} x^{r+1}, x^{r+1}) \), apart from \( \varepsilon_2 x^{r+1} \) of \( n \)'s and \( \varepsilon_3 x^{r+1} \) of \( m \)'s if \( x > x_0(\sigma, \varepsilon_1, \varepsilon_2, \varepsilon_3) \).

This is a straightforward consequence of the assumption that \( f(n) \) has a non-decreasing normal order.

Let \( N_s(x) \) denote the number of those integers \( n \) in the interval \( \left( \frac{1}{4} x^{r+1}, \frac{3}{4} x^{r+1} \right] \) the largest prime factors of which are greater than \( x' \).

Lemma 2. We have

\[
N_s(x) = \frac{1}{4} \log \left( 1 + \frac{1}{\sigma} \right) \cdot x^{r+1} \leq c_2 \frac{x^{r+1}}{\log x^{r+1}} \quad \text{for } x \geq 2.
\]

Proof. This is obvious, since

\[
N_s(x) = \sum_{p > x'} \left( \left[ \frac{3}{4} x^{r+1} \right] - \left[ \frac{1}{2} \frac{x^{r+1}}{p} \right] \right) = \frac{1}{4} x^{r+1} - \sum_{x' < p < 3x^{r+1}/4} \frac{1}{p} + O(\pi(x^{r+1})
\]

and

\[
\sum_{u < p < v} \frac{1}{p} = \log \frac{\log v}{\log u} + O(\exp[-\sqrt{\log u}])
\]

(see Prachar [4]).

Let \( F(x', x) \) be the number of those integers smaller than \( x' \) all prime factors of which do not exceed \( x \). Levin and Fainleib [2] proved the asym-
ptotical formula

$$F(x^t, x) = x^t z(t) + O\left(\frac{x}{\log x}\right)$$

as $x \to \infty$ uniformly for all $t$ varying in a bounded interval. In this formula

$$z(t) = \exp\left[-t\left(\log t + \log \log t - 1 - \frac{\log \log t}{\log t}\right) + O\left(\frac{1}{\log t}\right)\right] \quad \text{as } t \to \infty.$$

Let $M_r(x)$ be the number of integers in $(\frac{3}{2}x^{r+1}, x^{r+1}]$ all the prime factors of which do not exceed $x$.

We get immediately

**Lemma 3.** For every fixed large $r$ and for $x > x_0(r)$ we have

(2.4) \quad $M_r(x) \geq c_r C(r)x^{r+1},$

where

(2.5) \quad $C(r) = \exp\left[-(r+1)\left(\log(r+1) + \log \log (r+1) - 1 - \frac{\log \log (r+1)}{\log (r+1)}\right)\right].$

Let $\gamma = \exp(-1) = (2, 71\ldots)^{-1}$, and $w_i = x^i$ ($i = 0, 1, \ldots$).

It is a well-known result of Hardy and Ramanujan [1] that

$$\frac{\omega(n)}{\log \log n} \to 1$$

for almost all $n$. We use the weaker result, namely that

(2.6) \quad $\omega(n) \leq 2\log \log n$

for almost all $n$.

First we estimate the number $N(i, t)$ of those integers $n \leq x^{r+1}$ which have at least $t$ distinct prime divisors in $[w_{i+1}, w_i]$. It is obvious that

$$N(i, t) \leq \sum'\left[\frac{x^{r+1}}{Q_i}\right],$$

where by $\sum'$ we mean the summation extended over the integers $Q_i$ which are the product of $t$ distinct primes $p$ in $[w_{i+1}, w_i]$. From (2.3) we get

$$\sum' \frac{1}{Q_i} \leq \frac{1}{t!} \left(\sum_{w_{i+1} \leq p < w_i} \frac{1}{p}\right)^t < \frac{1 + c_3 \exp\left[-\sqrt{\log w_{i+1}}\right]}{t}$$

assuming only that

(2.7) \quad $t \exp\left[-\sqrt{\log w_{i+1}}\right] \leq 1.$

Consequently,

(2.8) \quad $N(i, t) \leq c_4 \frac{x^{r+1}}{t!}.$
Let
\[(2.9) \quad t_i = c_5 (\nu + 1) (i + 1)^2 \quad (i = 0, 1, \ldots, r_0),\]
\[(2.10) \quad r_0 = \left\lfloor (\log \log x)^{1/3} \right\rfloor,
\]
c5 being a positive integer which we shall specify later.

From (2.8) we get
\[\sum_{i=1}^{r_0} N(i, t_i) \leq c_5 x^{r+1} \sum_{i=0}^{r_0} \frac{1}{t_i!}.
\]

Using the relation
\[t_i! \geq c_7 \left(\frac{t_i}{e}\right)^{t_i} \geq c_7 \exp \left[\frac{c_5}{2} (\nu + 1) (i + 1)^2 \log (\nu + 1)\right]\]
which holds for every large \(c_5\), we get
\[\sum_{i=0}^{r_0} \frac{1}{t_i!} < c_9 \exp \left[\frac{c_5}{2} (\nu + 1) \log (\nu + 1)\right],\]
and so
\[(2.11) \quad \sum_{i=0}^{r_0} N(i, t_i) \leq c_{10} \exp \left[\frac{c_5}{2} (\nu + 1) \log (\nu + 1)\right] x^{r+1} = B_r x^{r+1}.
\]

We see that
\[(2.12) \quad \sum_{i=0}^{r_0} t_i \geq \frac{c_5 (\nu + 1) r_0^2}{6} \geq 2 \log \log x^{r+1}\]
if \(c_5 \geq 1\) and \(x > x_0(\nu)\). Let \(c_5\) be so large that \(B_r < \frac{1}{2} c_1 C(\nu)\) (see (2.11), (2.4) and (2.5)). This holds for \(c_5 = 3\) and for every large \(\nu\).

Hence we get immediately the following assertion.

*There exists at least \(\frac{1}{2} c_1 C(\nu) x^{r+1}\) integers \(m\) in \(\left(\frac{1}{2} x^{r+1}, x^{r+1}\right)\) all prime factors of which are smaller than \(x\) and have at most \(t_i - 1\) prime factors in every interval \([w_{i+1}, w_i)\) \((i = 0, 1, \ldots, r_0)\).*

From (2.6) and (2.12) and by the monotonicity of \(H(p)\) it follows that
\[(2.13) \quad f(m) \leq \sum_{i=0}^{r_0} H(w_i) t_i\]
for all \(m\) but \(\eps x^{r+1}\) of them.
For the set of integers counted in Lemma 2, we get \( H(x^r) \leq f(n) \).
Using Lemma 1 we have

\[
(2.14) \quad H(x^r) \leq (1 + c_1) \sum_{i=0}^{r_0} H(w_i) t_i
\]

for all large \( x (x > x_0(v)) \).
Dividing (2.14) by \( \log x^{r+1} \), we have

\[
2.15 \quad t(x^r) \leq c_{10} \sum_{i=0}^{r_0} \gamma^i (i + 1)^2 t(w_i) \quad \text{for} \quad x > x_1(v).
\]

To prove (1.6) we may assume that \( T(y) \to \infty \) as \( y \to \infty \). So for every \( y \) we have \( u = u(y) \leq y (u(y) \to \infty \) as \( y \to \infty ) \), so that \( T(y) = T(u) = t(u) \).
Now we take \( u = x^r \) and apply (2.15). We get

\[
T(y) = T(x^r) \leq \left( c_{10} \sum_{i=0}^{r_0} \gamma^i (i + 1)^2 \right) T(x) \leq c_1 T(y^{1/v})
\]

if \( y \) is sufficiently large, and \( y \geq y_0(v) \). This proves (1.6).

Now (1.7) follows immediately from (1.6). Let \( v \) be large and fixed.
Take \( x_k = x_0^k, \ x_0 = x_0(v) \). Then

\[
T(x_k) \leq c_k^k T(x_0),
\]

and so

\[
\frac{T(y)}{(\log y)^{c_{11}/\log y}} \to 0 \quad (y \to \infty).
\]

Choosing \( v \) such that \( c_{11}/\log v \leq \delta \), we get (1.7) immediately.

REFERENCES


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