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A NOISY DUEL UNDER ARBITRARY MOVING. III

1. Introduction. In papers [17]–[21] of the author and in this paper an m versus n bullets noisy duel is considered in which duelists can move at will. The cases $m \leqslant 25$, $n \leqslant 6$ and $n = 1$ for any m are solved. In [21] also an idea is given to determine the optimal (in limit) strategies for any m and n using a computer.

In this paper we shall solve the cases $n = 4$, $m \leqslant n$.

Let us define a game which will be called the *game* (m, n). Two Players I and II fight in a duel. They can move as they like. Maximal speed of Player I is v_1, maximal speed of Player II is v_2 and it is supposed that $v_1 > v_2 \geqslant 0$. Player I has m bullets (or rockets), Player II has n bullets (rockets).

Assume that at the moment $t = 0$ the players are in the distance 1 off and that $v_1 + v_2 = 1$.

Denote by $P(s)$ the probability that Player I (II) achieves a success (destroys the opponent) if he fires in the distance $1-s$. It is assumed that the function $P(s)$ is increasing and continuous in the interval $[0, 1]$ and has a continuous second derivative inside this interval, $P(s) = 0$ for $s \leqslant 0$, $P(1) = 1$.

Player I gains 1 if he only achieves the success, gains -1 if Player II only achieves the success, and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The duel is *noisy* – the player hears the shot of his opponent. Without loss of generality we can also assume that Player II is motionless. Then $v_1 = 1$.

When Player I has fired all his bullets, his motion in the direction of the opponent is unreasonable. Thus we assume that Player I evades with maximal speed after firing all his bullets.

Suppose that Player I has fired all his bullets and he evades. In this case Player II will do the best (if he survives) if he fires all his bullets immediately after the last shot of Player I. If, on the other hand, Player II has fired all his bullets and Player I survives and has yet bullets, the best what he can do is to reach the opponent and to achieve the success surely.

We suppose also that the reader knows the papers [17], [18] of the author and remembers the definitions, notation and results obtained there.

For definitions and notions in games of timing see [5], [22]. For results see [1]–[3], [6], [7], [9]–[11], [13], [23].

2. Duel (1, 4), $\langle a\rangle$. In this section we solve the duel in which Player I has one bullet, Player II has four bullets and the game is beginning when players are in the distance a from each other.

Let us define strategies ξ and η of Players I and II for which we shall prove that they are optimal in limit for some a.

Strategy of Player I. Evade if Player I did not fire. If he fired (say at a'), play optimally the obtained duel (1, 3), $\langle 2, a', a' \wedge c\rangle$.

Strategy of Player II. Fire at $\langle a\rangle$ (at the beginning of the duel) and if Player I did not fire at this moment, play optimally the obtained duel (1, 3), $\langle 2, a, a \wedge c\rangle$.

The duels (m, n), $\langle 1, a\wedge c, a\rangle$ and (m, n), $\langle 2, a, a\wedge c\rangle$ are defined in [18], Section 5.

It is assumed also that between successive shots of the same player the time $\hat{\varepsilon}$ has to pass.

The number $\langle \hat{a}\rangle$ denotes the earliest moment when Player I reaches the point $\hat{a}$.

"Play optimally" means "apply a strategy optimal in limit" (i.e., optimal for $\hat{\varepsilon}\to 0$, see [18] for a precise definition).

We shall prove that *if* $a \leqslant a_{14}$ *when* a_{14} *is the only root of the equation*

$$Q^5(a_{14})-(1+v_{11})Q^3(a_{14})-Q^2(a_{14})+1 = 0, \tag{1}$$

$Q(s) = 1-P(s)$, $v_{11} = 3-2\sqrt{2}$, $Q(a_{14}) \cong 0.851053$, *then the strategies* ξ *and* η *are optimal in limit. The limit value of the game* (1, 4), $\langle a\rangle$ *is*

$$v_{14}^a = \begin{cases} -1+Q^2(a) & \text{if } a \leqslant a_{12}, \\ -1+(1+v_{11})Q^3(a) & \text{if } a_{12} \leqslant a \leqslant a_{14}, \end{cases} \tag{2}$$

$Q(a_{12}) = 0.853553$.

Proof. Suppose that Player II fires at $a' \leqslant a$ and then applies the strategy $\hat{\eta}_0$. For this strategy $(a', \hat{\eta}_0)$ ($\hat{\eta}_0$ may depend on a') we have

$$K(\xi;\, a', \hat{\eta}_0) \geqslant -P(a')+(1-P(a'))\,{}^{2}\hat{v}_{13}^{a}-k(\hat{\varepsilon}),$$

where ${}^{2}\hat{v}_{13}^{a}$ is the limit value of the game (1, 3), $\langle 2, a, a\wedge c\rangle$ and $k(\hat{\varepsilon})\to 0$ if $\hat{\varepsilon}\to 0$. Taking the formulae on ${}^{2}\hat{v}_{13}^{a}$ for particular a into account (see [18]) we obtain

$$K(\xi;\, a', \hat{\eta}_0) \geqslant \begin{cases} -1+Q^2(a')-k(\hat{\varepsilon}) \geqslant -1+Q^2(a)-k(\hat{\varepsilon}) & \text{if } a \leqslant a_{12}, \\ -1+(1+v_{11})Q^3(a')-k(\hat{\varepsilon}) \geqslant -1+(1+v_{11})Q^3(a)-k(\hat{\varepsilon}) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } a_{12} \leqslant a \leqslant a_{14}. \end{cases}$$

Suppose then that Player II does not fire. For such a strategy $\hat{\eta}$,

$$K(\xi;\hat{\eta}) = 0 \geqslant v_{14}^{a}$$

if v_{14}^{a} is given by (2).

On the other hand, assume that Player I does not fire at $\langle a\rangle$. For such a strategy $\hat{\xi}$,

$$K(\hat{\xi};\eta) \leqslant -P(a)+\big(1-P(a)\big)^2 v_{13}^{a}+k(\hat{\varepsilon}) = v_{14}^{a}+k(\hat{\varepsilon})$$

if v_{14}^{a} is given by (2).

If Player I also fires at $\langle a\rangle$, then

$$K(\hat{\xi};\eta) \leqslant -Q^2(a)\big(1-Q^3(a)\big)+k(\hat{\varepsilon}) \leqslant v_{14}^{a}+k(\hat{\varepsilon})$$

if

$$S_1\big(Q(a)\big) = Q^5(a)-2Q^2(a)+1 \leqslant 0$$

for $a \leqslant a_{12}$, and

$$S_2\big(Q(a)\big) = Q^5(a)-(1+v_{11})Q^3(a)-Q^2(a)+1 \leqslant 0$$

for $a_{12} \leqslant a \leqslant a_{14}$.

The function $S_1(Q)$ of the variable Q has its only minimum at the point

$$Q = \sqrt[3]{\tfrac{4}{5}} \cong 0.928318,$$

and

$$S_1\big(Q(a_{12})\big) = -0.004044, \qquad S(1) = 0.$$

Thus the inequality $S_1\big(Q(a)\big) \leqslant 0$ always holds for $a \leqslant a_{12}$.

The function $S_2(Q)$ is a decreasing function of Q and $S_2\big(Q(a_{14})\big) = 0$. Thus also $S_2\big(Q(a)\big) \leqslant 0$ for $a_{12} \leqslant a \leqslant a_{14}$. The proposition is proved.

Let us notice that in the proof it is sufficient to consider only nonrandom strategies $\hat{\xi}$ and $\hat{\eta}$ (and $(a', \hat{\eta}_0)$).

3. Duel (1, 4), $\langle 1, a \wedge c, a\rangle$. Suppose now that Player I can fire a shot beginning from $\langle a\rangle+c$ and Player II can fire a shot beginning from $\langle a\rangle$ (but sometimes not at $\langle a\rangle$, see [18]). Define the strategies ξ and η of Players I and II.

Strategy of Player I. Evade if Player I did not fire. If he fired (say at a'), play optimally the obtained duel (1, 3), $\langle a_1', a_1' \wedge c_1\rangle$, where

$$a_1' = \max(a', a_1), \qquad a_1 = \rangle\langle a\rangle+c\langle.$$

Strategy of Player II. Fire before $\langle a\rangle+c$ and play optimally the obtained duel (1, 3), $\langle 2, a_1, a_1 \wedge c_1\rangle$, where $a_1 = \rangle\langle a\rangle+c\langle$.

The symbol $\rangle t\langle$ denotes the point at which Player I has been at the moment t.

Now the strategies ξ and η are optimal in limit if

$$Q(a) \geqslant Q(a_{13}) \cong 0.814115$$

and

$$(3) \qquad \overset{1}{v}{}^{a}_{14} = \begin{cases} -1+Q^2(a) & \text{if } a \leqslant a_{12}, \\ -1+(1+v_{11})Q^3(a) & \text{if } a_{12} \leqslant a \leqslant a_{13} \end{cases}$$

is the limit value of the game.

To prove this let us notice that, for any strategy $\hat{\xi}$ of Player I,

$$K(\hat{\xi}; \eta) \leqslant -P(a)+Q(a)\overset{2}{v}{}^{a}_{13}+k(\hat{\varepsilon}) = \overset{1}{v}{}^{a}_{14}+k(\hat{\varepsilon})$$

if $a \leqslant a_{13}$ (see [18]).

On the other hand, if Player II fires (at a') and then plays $\hat{\eta}_0$, we have

$$K(\xi; a', \hat{\eta}_0) \geqslant -P(a')+Q(a')\overset{2}{v}{}^{a'}_{13}-k(\hat{\varepsilon}) = v^{a'}_{14}-k(\hat{\varepsilon}) \geqslant \overset{1}{v}{}^{a}_{14}-k(\hat{\varepsilon})$$

if $\overset{1}{v}{}^{a}_{14}$ is given by (3).

If Player II does not fire, we obtain

$$K(\xi; \hat{\eta}) = 0 > \overset{1}{v}{}^{a}_{14}.$$

4. Duel (1, 4), $\langle 2, a, a \wedge c\rangle$. Define ξ and η.

Strategy of Player I. Evade if Player I did not fire. If he fired (say at a'), play optimally the obtained duel (1, 3), $\langle 2, a'_1, a'_1 \wedge c_1\rangle$, where

$$a'_1 = \max(a', a_1), \qquad a_1 = \rangle\langle a\rangle+c\langle.$$

Strategy of Player II. If Player I did not fire before, fire at $\langle a\rangle+c$ and play optimally afterwards.

By definition, a_1 is the point where Player I has been at $\langle a\rangle+c$. The strategy of Player II means that if Player I did not fire before, then Player II should fire a shot at $\langle a\rangle+c$ and should play optimally the obtained duel (1, 3), $\langle 2, a_1, a_1 \wedge c_1\rangle$ (if Player I did not fire also at $\langle a\rangle+c$).

We have

$$K(\xi; \eta) = -1+Q^2(a)+k(\hat{\varepsilon}) \quad \text{if } a \leqslant a_{12}.$$

It is easy to see that Player I always assures in limit the value $-1+Q^2(a)$ if $a < a_{12}$ (compare with the duel (1, 4), $\langle a\rangle$). On the other hand, if Player I fires before $\langle a\rangle+c$, then

$$K(\hat{\xi}; \eta) \leqslant P(a)-Q(a)\big(1-Q^4(a)\big)+k(\hat{\varepsilon})$$
$$= 1-2Q(a)+Q^5(a)+k(\hat{\varepsilon}) \leqslant -1+Q^2(a)+k(\hat{\varepsilon}),$$

which after dividing the difference of both sides by $Q(a)-1$ leads to the inequality

$$Q^4(a)+Q^3(a)+Q^2(a)-2 \geqslant 0. \tag{4}$$

This polynomial has the only root $Q(a) = Q(\check{a}_{14}) \cong 0.871757$. Thus the inequality holds for $a \leqslant \check{a}_{14}$.

The proof in other cases is the same as in the duel (1, 4), $\langle a \rangle$. Thus, if $a < \check{a}_{14}$, the strategies ξ and η are optimal in limit and the limit value of the game is

$$\check{\overset{2}{v}}{}^{a}_{14} = -1+Q^2(a). \tag{5}$$

5. Results for the duel (1, 4). We have

$$\overset{1}{v}{}^{a}_{14} = \begin{cases} -1+Q^2(a) & \text{if } Q(a) \geqslant Q(a_{12}) \cong 0.853553, \\ -1+(1+v_{11})Q^3(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(a_{13}) \cong 0.814115; \end{cases}$$

$$v^{a}_{14} = \begin{cases} -1+Q^2(a) & \text{if } Q(a) \geqslant Q(a_{12}), \\ -1+(1+v_{11})Q^3(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(a_{14}) \cong 0.851053; \end{cases}$$

$$\check{\overset{2}{v}}{}^{a}_{14} = -1+Q^2(a) \quad \text{if } Q(a) \geqslant Q(\check{a}_{14}) \cong 0.871757.$$

6. Duel (2, 4), $\langle a \rangle$.

Case 1. Define strategies ξ and η of Players I and II.

Strategy of Player I. Evade if Player I did not fire a shot. If he fired (say at a'), play optimally the obtained duel (2, 3), $\langle 2, a', a' \wedge c \rangle$.

Strategy of Player II. Do not fire if Player I evades. If he approaches Player II, fire a shot at $\langle a_{24} \rangle$ and play optimally the duel (2, 3). If Player I fired before he reached a_{24} (say at a'), play optimally the obtained duel (1, 4), $\langle 1, a' \wedge c, a' \rangle$.

We prove that these strategies are optimal in limit and

$$v^{a}_{24} = 0 \tag{6}$$

when

$$Q(a) \geqslant Q(a_{24}) \overset{\text{df}}{=} \frac{1}{1+v^{a_1}_{23}} \cong 0.986429, \tag{7}$$

where $v^{a_1}_{23} \cong 0.013757$ is defined in [18].

Suppose that Player II does not fire. For such a strategy $\hat{\eta}$,

$$K(\xi; \hat{\eta}) = 0 = v^{a}_{24}.$$

Suppose that Player II fired at $a' \leqslant a$. For such a strategy $(a', \hat{\eta}_0)$ we have

$$\begin{aligned} K(\xi; a', \hat{\eta}_0) &\geqslant -P(a')+Q(a')v^{a'}_{23}-k(\hat{\varepsilon}) \\ &= -1+(1+v^{a_1}_{23})Q(a')-k(\hat{\varepsilon}) \\ &\geqslant -1+(1+v^{a_1}_{23})Q(a_{24})-k(\hat{\varepsilon}) = -k(\hat{\varepsilon}) = v^{a}_{24}-k(\hat{\varepsilon}) \end{aligned}$$

if the constant a satisfies condition (7).

On the other hand, if Player I does not approach the point a_{24},

$$K(\hat{\xi};\eta) = 0 = v_{24}^{a}.$$

If Player I fires at $a' < a_{24}$, we have

$$\begin{aligned}K(a', \hat{\xi}_0;\eta) &\leqslant P(a') + Q(a')\overset{1}{v}{}_{14}^{a'} + k(\hat{\varepsilon})\\ &= 1-2Q(a')+Q^3(a')+k(\hat{\varepsilon}) \leqslant k(\hat{\varepsilon})\end{aligned}$$

since the polynomial $S(Q(a)) = 1-2Q(a)+Q^3(a)$ is an increasing function of the variable $Q = Q(a)$ for a satisfying (7) and $S(1) = 0$.

If Player I fires at $\langle a_{24}\rangle$,

$$K(\hat{\xi};\eta) \leqslant Q^2(a_{24})v_{13}^{a_{24}} + k(\hat{\varepsilon}) = -Q^2(a_{24})P(a_{24}) + k(\hat{\varepsilon}) < k(\hat{\varepsilon}).$$

If, finally, Player I reaches the point a_{24} without firing a shot, we have

$$\begin{aligned}K(\hat{\xi};\eta) &\leqslant -P(a_{24}) + Q(a_{24})\overset{2}{v}{}_{23}^{a_{24}} + k(\hat{\varepsilon})\\ &= -1+(1+v_{23}^{a_1})Q(a_{24})+k(\hat{\varepsilon}) = k(\hat{\varepsilon}).\end{aligned}$$

The proposition is proved.

Case 2. Define ξ and η.

Strategy of Player I. Evade if Player II did not fire. If he fired a shot (say at a'), play optimally the obtained duel (2, 3), $\langle 2, a', a' \wedge c\rangle$.

Strategy of Player II. Fire at $\langle a\rangle$ and if Player I did not fire at $\langle a\rangle$, play optimally the duel (2, 3), $\langle 2, a, a \wedge c\rangle$. If Player I fired, play optimally the duel (1, 3), $\langle a_1\rangle$, where $a_1 = \rangle\langle a\rangle + \hat{\varepsilon}\langle$.

We remind that $\rangle t\langle$ denotes the point at which Player I has been at the moment t.

The above strategies are optimal in limit when $a_{24} \leqslant a \leqslant \hat{a}_{24}$, where $Q(\hat{a}_{24}) \cong 0.918836$ satisfies the equation

$$Q^3(\hat{a}_{24}) - Q^2(\hat{a}_{24}) - (1+v_{23}^{a_1})Q(\hat{a}_{24}) + 1 = 0. \tag{8}$$

The limit value of the game is

$$v_{24}^{a} = -1+(1+v_{23}^{a_1})Q(a). \tag{9}$$

To prove this suppose that Player II fires when Player I is at the point $a' \leqslant a$. For such a strategy $(a', \hat{\eta}_0)$,

$$\begin{aligned}K(\xi; a', \hat{\eta}_0) &\geqslant -P(a') + Q(a')\overset{2}{v}{}_{23}^{a'} - k(\hat{\varepsilon})\\ &= -1+(1+v_{23}^{a_1})Q(a') - k(\hat{\varepsilon})\\ &\geqslant -1+(1+v_{23}^{a_1})Q(a) - k(\hat{\varepsilon}) = v_{24}^{a} - k(\hat{\varepsilon})\end{aligned}$$

when $a' < a_{23}$, $Q(a_{23}) \cong 0.882709$ (see [18]).

If Player II does not fire,

$$K(\xi;\hat{\eta}) = 0 \geqslant -1+(1+v_{23}^{a_1})Q(a)$$

when $a \geqslant a_{24}$.

Suppose now that Player I does not fire at $\langle a\rangle$. We obtain

$$K(\hat{\xi};\eta) \leqslant -P(a)+Q(a)\overset{2}{v}{}_{23}^{a}+k(\hat{\varepsilon}) = v_{24}^{a}+k(\hat{\varepsilon}).$$

When Player I fires at $\langle a\rangle$,

$$K(\hat{\xi};\eta) = Q^2(a)v_{13}^{a}+k(\hat{\varepsilon}) = -Q^2(a)+Q^3(a)+k(\hat{\varepsilon})$$

if $a \leqslant a_{12}$. Thus we obtain

$$Q^3(a)-Q^2(a)-(1+v_{23}^{a_1})Q(a)+1 \leqslant 0.$$

Since $v_{23}^{a_1} > 0$, this polynomial is a decreasing function of the variable Q and is equal to zero for $Q(a) = Q(\hat{a}_{24})$. Thus the inequality holds for $a \leqslant \hat{a}_{24}$, which completes the proof of the proposition.

Case 3. Define ξ and η.

Strategy of Player I. Fire at $\langle a\rangle$ and play optimally the obtained duel $(1, 4)$, $\langle 1,\ a\wedge c,\ a\rangle$ (or $(1, 3)$, $\langle a_1\rangle$, $a_1 = \rangle\langle a\rangle+\hat{\varepsilon}\langle$).

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally the obtained duel $(2, 3)$, $\langle 2,\ a,\ a\wedge c\rangle$ (or $(1, 3)$, $\langle a_1\rangle$, $a_1 = \rangle\langle a\rangle+\hat{\varepsilon}\langle$).

Now

$$K(\xi;\eta) = Q^2(a)v_{13}^{a}+k(\hat{\varepsilon}) = -Q^2(a)+Q^3(a)+k(\hat{\varepsilon})$$

for $a \leqslant a_{12}$. We now try to prove that for $\hat{a}_{24} \leqslant a \leqslant a_{12}$ the strategies ξ and η are optimal in limit and

$$v_{24}^{a} = -Q^2(a)+Q^3(a). \tag{10}$$

Thus suppose that Player II applying $\hat{\eta}$ fires later than $\langle a\rangle$ or does not fire at all. For $a \leqslant a_{12}$ we obtain

$$\begin{aligned} K(\xi;\hat{\eta}) &\geqslant P(a)+Q(a)\overset{1}{v}{}_{14}^{a}-k(\hat{\varepsilon}) \\ &= 1-2Q(a)+Q^3(a)-k(\hat{\varepsilon}) \geqslant -Q^2(a)+Q^3(a)-k(\hat{\varepsilon}). \end{aligned}$$

From the above it follows that if $a \leqslant a_{12}$, then Player I always assures for himself (in mean) the value v_{24}^{a} given by (10).

On the other hand, if Player I applying $\hat{\xi}$ fires a shot later than $\langle a\rangle$ or does not fire, then

$$\begin{aligned} K(\hat{\xi};\eta) &\leqslant -P(a)+Q(a)\overset{2}{v}{}_{23}^{a}+k(\hat{\varepsilon}) \\ &= \begin{cases} -1+(1+v_{23}^{a_1})Q(a)+k(\hat{\varepsilon}) & \text{if } a \leqslant a_{23}, \\ -1+2Q(a)-2Q^2(a)+Q^3(a)+k(\hat{\varepsilon}) & \text{if } a_{23} \leqslant a \leqslant a_{12}. \end{cases} \end{aligned}$$

In the first case we obtain the inequality

$$Q^3(a)-Q^2(a)-(1+v_{23}^{a_1})Q(a)+1 \geqslant 0$$

which is fulfilled for $a \geqslant \hat{a}_{24}$. In the second case we obtain

$$Q^3(a)-2Q^2(a)+2Q(a)-1 \leqslant Q^3(a)-Q^2(a)$$

which is satisfied for any a. Thus the proposition is proved.

7. **Duel** (2, 4), $\langle 1,\ a\wedge c,\ a\rangle$.

Case 1. It is easy to prove, comparing with the duel (2, 4), $\langle a\rangle$, that if $a \leqslant a_{24}$, then the optimal in limit strategies defined in that duel are also optimal in limit here. Thus we have $\overset{1a}{v}_{24} = 0$ for $a \leqslant a_{24}$.

Case 2. Define ξ and η.

Strategy of Player I. Evade if Player II did not fire. If he fired (say at a'), play optimally the obtained duel (2, 3), $\langle 2,\ a_1',\ a_1'\wedge c_1\rangle$,

$$\text{where } a_1' = \max(a', a_1), \quad a_1 = \rangle\langle a\rangle+c\langle.$$

Strategy of Player II. Fire before $\langle a\rangle+c$ and play optimally the obtained duel (2, 3), $\langle 2,\ a_1,\ a_1\wedge c_1\rangle$, where $a_1 = \rangle\langle a\rangle+c\langle$.

Now

$$\overset{1a}{v}_{24} = -1+(1+v_{23}^{a_1})Q(a) \tag{11}$$

if $a_{23} \leqslant a \leqslant a_{24}$.

The proof that Player I applying ξ assures in limit the value $\overset{1a}{v}_{24}$ given in (11) is the same as in Case 2 of the duel (2, 4), $\langle a\rangle$. The proof that Player II applying η assures in limit this value is obvious.

Case 3. Define ξ and η.

Strategy of Player I. If Player II did not fire before, fire at $\langle a\rangle+c$ and play optimally afterwards. If he fired, play optimally the obtained duel (2, 3), $\langle 2,\ a_1,\ a_1\wedge c_1\rangle$, where $a_1 = \rangle\langle a\rangle+c\langle$.

Strategy of Player II. Fire before $\langle a\rangle+c$ and play optimally the obtained duel (2, 3), $\langle 2,\ a_1,\ a_1\wedge c_1\rangle$, $a_1 = \rangle\langle a\rangle+c\langle$.

Now we show that

$$\overset{2a}{v}_{24} = -1+2Q(a)-2Q^2(a)+Q^3(a) \tag{12}$$

for $a_{12} \leqslant a \leqslant a_{23}$.

Player II always assures in limit that value.

On the other hand, if Player II fires before $\langle a\rangle+c$, we have

$$K(\xi;\hat{\eta}) \geqslant -P(a)+Q(a)\overset{2a}{v}_{23}-k(\hat{\varepsilon}) = \overset{2a}{v}_{24}-k(\hat{\varepsilon})$$

if $\overset{2a}{v}_{24}$ is given by (12).

If Player II fires at $\langle a\rangle+c$, we obtain

$$K(\xi;\hat{\eta}) \geqslant Q^2(a)v^a_{13}-k(\hat{\varepsilon})$$
$$= -Q^2(a)+Q^3(a)-k(\hat{\varepsilon}) \geqslant -1+2Q(a)-2Q^2(a)+Q^3(a)-k(\hat{\varepsilon}).$$

If Player II fires after $\langle a\rangle+c$ or does not fire, then

$$K(\xi;\hat{\eta}) \geqslant P(a)+Q(a)\overset{1a}{v}_{14}-k(\hat{\varepsilon})$$
$$= 1-2Q(a)+Q^3(a)-k(\hat{\varepsilon}) \leqslant -1+2Q(a)-2Q^2(a)+Q^3(a)-k(\hat{\varepsilon}).$$

All the above inequalities hold under the condition $a_{12} \leqslant a \leqslant a_{23}$. The proposition is proved.

Let us notice that now the value of the game $\overset{2a}{v}_{24}$ is different from that for the duel (2, 4), $\langle a\rangle$ in Case 3.

8. Duel (2, 4), $\langle 2, a, a \wedge c\rangle$.

Case 1. Here also the optimal in limit strategies ξ and η are the same as in the duel (2, 4), $\langle a\rangle$ if $a \leqslant a_{24}$. Thus $\overset{2a}{v}_{24}=0$ if $a \leqslant a_{24}$.

Case 2. Define ξ and η.

Strategy of Player I. Evade if Player II did not fire. If he fired (say at a'), play optimally the obtained duel (2, 3), $\langle 2, a_1', a_1' \wedge c\rangle$,

$$a_1' = \max(a', a_1), \qquad a_1 = \rangle\langle a\rangle+c\langle.$$

Strategy of Player II. If Player I did not fire before, fire at $\langle a\rangle+c$ and play optimally the duel (2, 3), $\langle 2, a_1, a_1 \wedge c_1\rangle$, $a_1 = \rangle\langle a\rangle+c\langle$. If he fired before or at $\langle a\rangle+c$, play optimally the obtained duel.

Now

$$\overset{2a}{v}_{24} = -1+(1+v^{a_1}_{23})Q(a) \tag{13}$$

for $a_{24} \leqslant a \leqslant \check{a}_{24}$, where the number $\check{a}_{24}$ is the only root of the equation

$$Q^3(\check{a}_{24})-(3+v^{a_1}_{23})Q(\check{a}_{24})+2=0, \qquad Q(\check{a}_{24}) \cong 0.933827. \tag{14}$$

The proof is the same as in the duel (2, 4), $\langle a\rangle$ with the only exception when Player I fires before $\langle a\rangle+c$. For such a strategy (a', ξ_0), for $a < a_{12}$, we obtain

$$K(a', \xi_0; \eta) \leqslant P(a)+Q(a)\overset{1a}{v}_{14}+k(\hat{\varepsilon})$$
$$= 1-2Q(a)+Q^3(a)+k(\hat{\varepsilon}) \leqslant -1+(1+v^{a_1}_{23})Q(a)+k(\hat{\varepsilon})$$

if

$$Q^3(a)-(3+v^{a_1}_{23})Q(a)+2 \leqslant 0$$

i.e., if $a \leqslant \check{a}_{24}$.

Case 3.

Strategy of Player I. Fire before $\langle a\rangle + c$ and play optimally the obtained duel (1, 4), $(1,\ a_1 \wedge c_1,\ a_1)$, where $a_1 = \rangle\langle a\rangle + c\langle$.

Strategy of Player II. If Player I did not fire before, fire at $\langle a\rangle + c$ and play optimally afterwards. If he fired, play optimally the obtained duel (1, 4), $\langle 1,\ a_1 \wedge c_1,\ a_1\rangle$.

In this case

$$\overset{2}{v}{}^{a}_{24} = 1 - 2Q(a) + Q^3(a) \tag{15}$$

if $\check{a}_{24} \leqslant a \leqslant a_{12}$.

Player I always assures in limit this value.

To prove that Player II does the same assume that Player I fires before $\langle a\rangle + c$. For such a strategy $\hat{\xi}$, if $a \leqslant a_{12}$ we have

$$K(\hat{\xi};\ \eta) \leqslant P(a) + Q(a)\overset{1}{v}{}^{a}_{14} + k(\hat{\varepsilon}) = \overset{2}{v}{}^{a}_{24} + k(\hat{\varepsilon}).$$

If Player I fires at $\langle a\rangle + c$, we have (for $a \leqslant a_{12}$)

$$K(\hat{\xi};\ \eta) \leqslant Q^2(a)v^{a}_{13} + k(\hat{\varepsilon}) = -Q^2(a) + Q^3(a) + k(\hat{\varepsilon})$$
$$\leqslant 1 - 2Q(a) + Q^3(a) + k(\hat{\varepsilon}) = \overset{2}{v}{}^{a}_{24} + k(\hat{\varepsilon}).$$

If Player I has the intention to fire after $\langle a\rangle + c$ or not to fire at all, we obtain (for $a < a_{12}$)

$$K(\hat{\xi};\ \eta) \leqslant -P(a) + Q(a)\overset{2}{v}{}^{a}_{23} + k(\hat{\varepsilon})$$
$$= \begin{cases} -1 + (1 + v^{a_1}_{23})Q(a) + k(\hat{\varepsilon}) & \text{if } a \leqslant a_{23}, \\ -1 + 2Q(a) - 2Q^2(a) + Q^3(a) + k(\hat{\varepsilon}) & \text{if } a_{23} \leqslant a \leqslant a_{12}. \end{cases}$$

In the first case we obtain the inequality

$$Q^3(a) - (3 + v^{a_1}_{23})Q(a) + 2 \geqslant 0,$$

which is satisfied for $a \geqslant \check{a}_{24}$ (see equation (14)).

In the second case we obtain

$$-1 + 2Q(a) - 2Q^2(a) + Q^3(a) \leqslant 1 - 2Q(a) + Q^3(a),$$

which is satisfied for any a. Then the strategies ξ and η are optimal in limit for $\check{a}_{24} \leqslant a \leqslant a_{12}$.

9. Results for the duel (2, 4). We have

$$\overset{1}{v}{}^{a}_{24} = \begin{cases} 0 & \text{if } Q(a) \geqslant Q(a_{24}) \cong 0.986429, \\ -1 + (1 + v^{a_1}_{23})Q(a) & \text{if } Q(a_{24}) \geqslant Q(a) \geqslant Q(a_{23}) \cong 0.882709, \\ -1 + 2Q(a) - 2Q^2(a) + Q^3(a) & \text{if } Q(a_{23}) \geqslant Q(a) \geqslant Q(a_{12}) \cong 0.853553; \end{cases}$$

$$v_{24}^{a} = \begin{cases} 0 & \text{if } Q(a) \geqslant Q(a_{24}), \\ -1+(1+v_{23}^{a_1})Q(a) & \text{if } Q(a_{24}) \geqslant Q(a) \geqslant Q(\hat{a}_{24}) \cong 0.918836, \\ -Q^2(a)+Q^3(a) & \text{if } Q(\hat{a}_{24}) \geqslant Q(a) \geqslant Q(a_{12}); \end{cases}$$

$$\overset{2}{v}{}_{24}^{a} = \begin{cases} 0 & \text{if } Q(a) \geqslant Q(a_{24}), \\ -1+(1+v_{23}^{a_1})Q(a) & \text{if } Q(a_{24}) \geqslant Q(a) \geqslant Q(\check{a}_{24}) \cong 0.933827, \\ 1-2Q(a)+Q^3(a) & \text{if } Q(\check{a}_{24}) \geqslant Q(a) \geqslant Q(a_{12}). \end{cases}$$

10. Duel (3, 4). Let us consider the duel (3, 4), $\langle a\rangle$. Define ξ and η.

Strategy of Player I. Reach the point a_{34} and if Player II did not fire before, fire a shot at a_{34}^{ε} and play optimally the duel (2, 4), $\langle 1, \rangle a_{34}^{\varepsilon}\langle \wedge c, \rangle a_{34}^{\varepsilon}\langle\rangle$. If Player II fired, play optimally the duel (3, 3).

Strategy of Player II. If Player I did not fire before, fire at $\langle a_{34}\rangle$ and play optimally the duel (3, 3). If he fired (say at a'), play optimally the obtained duel (2, 4), $\langle 1, a' \wedge c, a'\rangle$. If Player I did not reach the point a_{34}, do not fire.

The number a_{34} is determined from the equation

$$v_{34}^{a} = P(a_{34})+Q(a_{34})\overset{1}{v}{}_{24}^{a_{34}} = -P(a_{34})+Q(a_{34})v_{33} \overset{\mathrm{df}}{=} v_{34}^{a_1}, \tag{16}$$

where a_{mn} denotes a random moment, $\langle a_{mn}\rangle \leqslant a_{mn}^{\varepsilon} \leqslant \langle a_{mn}\rangle+\alpha(\varepsilon)$, distributed according to an absolute continuous probability distribution in the above interval, $\alpha(\varepsilon)\to 0$ if $\varepsilon\to 0$.

Assume that

$$(0.986429 \cong)\ Q(a_{24}) \geqslant Q(a_{34}) \geqslant Q(a_{23})\ (\cong 0.882709).$$

From (16) we obtain

$$(1+v_{23}^{a_1})Q^2(a_{34})-(3+v_{33})Q(a_{34})+2 = 0. \tag{17}$$

Since

$$v_{23}^{a_1} \cong 0.013757, \qquad v_{33} \cong 0.129435,$$

we have

$$Q(a_{34}) \cong 0.903576, \qquad v_{34}^{a_1} \cong 0.020530. \tag{18}$$

To prove that ξ and η are optimal in limit for this a_{34} let us assume that Player II fired before $\langle a_{34}\rangle$, $a' < a_{34}$. In this case we have

$$\begin{aligned} K(\xi;\hat{\eta}) &\geqslant -P(a')+Q(a')v_{33}-k(\hat{\varepsilon}) \\ &\geqslant -P(a_{34})+Q(a_{34})v_{33}-k(\hat{\varepsilon}) = v_{34}^{a_1}-k(\hat{\varepsilon}). \end{aligned}$$

Suppose that Player II fired after $\langle a_{34}\rangle+\alpha(\varepsilon)$. Then under the assumed connection between numbers ε and $\hat{\varepsilon}$ ([18], formula (7)) we obtain

$$K(\xi;\hat{\eta}) \geqslant P(a_{34})+Q(a_{34})\overset{1}{v}{}_{24}^{a_{34}}-k(\hat{\varepsilon}) = v_{34}^{a_1}-k(\hat{\varepsilon}).$$

From the above two inequalities it follows that

$$K(\xi;\hat{\eta}) \geqslant v_{34}^{a_1} - k(\hat{\varepsilon})$$

for any strategy $\hat{\eta}$ of Player II if the function $k(\hat{\varepsilon})$ is chosen properly.

On the other hand, if Player I fired before a_{23}, $a' < a_{23}$, then

$$K(a', \xi_0; \eta) \leqslant P(a') + Q(a')v_{24}^{1a'} + k(\hat{\varepsilon})$$

$$= \begin{cases} 1 - Q(a') + k(\hat{\varepsilon}) & \text{if } a \leqslant a_{24}, \\ 1 - 2Q(a') + (1 + v_{23}^{a_1})Q^2(a') + k(\hat{\varepsilon}) & \text{if } a_{24} \leqslant a \leqslant a_{23}. \end{cases}$$

Both functions are increasing ones of the variable a'. Then

$$K(a', \xi; \eta) \leqslant 1 - 2Q(a_{34}) + (1 + v_{23}^{a_1})Q^2(a_{34}) + k(\hat{\varepsilon}) = v_{34}^{a_1} + k(\hat{\varepsilon}),$$

which can be seen from (16).

Suppose that Player I fires at $\langle a_{34}\rangle$. For such a strategy $\hat{\xi}$ we have

$$K(\hat{\xi}; \eta) \leqslant Q^2(a_{34})v_{23}^{a_{34}} + k(\hat{\varepsilon}) = Q^2(a_{34})v_{23}^{a_1} + k(\hat{\varepsilon})$$

$$\cong 0.012431 + k(\hat{\varepsilon}) < v_{34}^{a_1} + k(\hat{\varepsilon}).$$

If Player I did not fire a shot before or at $\langle a_{34}\rangle$ but reaches this point, then

$$K(\hat{\xi}; \eta) \leqslant -P(a_{34}) + Q(a_{34})v_{33} + k(\hat{\varepsilon}) = v_{34}^{a_1} + k(\hat{\varepsilon}).$$

If, finally, Player I never reaches the point a_{34} and never fires, then

$$K(\hat{\xi}; \eta) = 0 < v_{34}^{a_1}.$$

Thus the strategies ξ and η are optimal in limit for $a \leqslant a_{34}$.

Let us notice that these strategies are also optimal in limit for the duels (3, 4), $\langle a, a \wedge c\rangle$ and (3, 4), $\langle 2, a, a \wedge c\rangle$ if $a \leqslant a_{34}$.

11. Duel (4, 4). Let us consider the duel (4, 4), $\langle a\rangle$. Define ξ and η.

Strategy of Player I. Reach the point a_{44} and if Player I did not fire before, fire a shot at a_{44}^{ε} and play optimally the duel (3, 4), $\langle 1, \rangle a_{44}^{\varepsilon} \langle \wedge c, \rangle a_{44}^{\varepsilon} \langle\rangle$. If Player II fired, play optimally the duel (4, 3).

Strategy of Player II. If Player I did not fire before, fire at $\langle a_{44}\rangle$ and play optimally the duel (4, 3). If he fired, play optimally the duel (3, 4). If Player I did not reach the point a_{34}, do not fire.

We assume that

$$v_{44} = P(a_{44}) + Q(a_{44})v_{34}^{a_{44}} = -P(a_{44}) + Q(a_{44})v_{43}. \tag{19}$$

Since $v_{43} \cong 0.230895$ (see [21]), if $a_{44} < a_{34}$ we obtain

$$Q(a_{44}) = \frac{2}{2 + v_{43} - v_{34}^{a_1}} \cong 0.904828. \tag{20}$$

Then $a_{44} < a_{34}$ and the strategies ξ and η are well defined. Moreover,

$$v_{44} = -1+(1+v_{43})Q(a_{44}) \cong 0.113748. \tag{21}$$

To prove that the strategies ξ and η are optimal in limit suppose that Player I fires before a_{44}. Then

$$K(\xi;\, a',\, \hat{\eta}_0) \geqslant -P(a')+Q(a')v_{43}-k(\hat{\varepsilon}) \geqslant -P(a_{44})+Q(a_{44})v_{43}-k(\hat{\varepsilon}).$$

If Player II fires after $\langle a_{44}\rangle+\alpha(\varepsilon)$, then

$$K(\xi;\, \hat{\eta}) \geqslant P(a_{44})+Q(a_{44})v_{34}^{a_{44}}-k(\hat{\varepsilon}).$$

Since equations (19) hold, Player I assures in limit the value v_{44}.

On the other hand, when Player I fires before a_{44}, $a' < a_{44}$, we have

$$K(a',\, \hat{\xi}_0;\, \eta) \leqslant P(a')+Q(a')\hat{v}_{34}^{1a'}+k(\hat{\varepsilon}) = P(a')+Q(a')v_{34}^{a_1}+k(\hat{\varepsilon})$$
$$\leqslant P(a_{44})+Q(a_{44})\, v_{34}^{a_{44}}+k(\hat{\varepsilon}) = v_{44}+k(\hat{\varepsilon}).$$

When he fires at $\langle a_{44}\rangle$, we obtain

$$K(\hat{\xi};\, \eta) \leqslant Q^2(a_{44})v_{33}+k(\hat{\varepsilon}) \cong 0.105970+k(\hat{\varepsilon}) < v_{44}+k(\hat{\varepsilon}).$$

When he intends to fire after $\langle a_{44}\rangle$ or not to fire at all, we get

$$K(\hat{\xi};\, \eta) \leqslant -P(a_{44})+Q(a_{44})v_{43}+k(\hat{\varepsilon}) = v_{44}+k(\hat{\varepsilon}).$$

Finally, when Player I never reaches the point a_{44} and never fires, we have

$$K(\hat{\xi};\, \eta) = 0 \leqslant v_{44}.$$

The optimality in limit of the strategies ξ and η is proved.

It is easy to see that these strategies are optimal in limit also for the duels (4, 4), $\langle 1,\, a \wedge c,\, a\rangle$ and (4, 4), $\langle 2,\, a,\, a \wedge c\rangle$. Each of these duels will be also denoted by (4, 4).

Duels $(m, 4)$, $4 \leqslant m \leqslant 25$ (and some others), are solved by the author in [21].

Noisy duels with retreat after the shots are considered by the author in [14]–[16].

For other noisy duels see [4], [8], [12], [24].

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