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ADDITIVE FUNCTIONS AND SINGULAR MEASURES

BY

MAREK KANTER (SYDNEY)

0. Introduction. Let μ be a probability measure on R, the real line. Let f: $R \to R$ be an additive function which is μ-measurable (i.e. f is measurable with respect to the μ-completion of Borel subsets of R). We show (in Theorem 1) that there exists a constant c such that $f(x) = cx$ a.e. (μ) if μ is a symmetric measure which satisfies the following

CONDITION A. There exists a sequence of probability measures μ_n, with $\mu_n(Q) = 1$ for all n, where Q stands for the rationals, such that we can write μ as the infinite convolution

$$\mu = \mu_1 * \ldots * \mu_n * \ldots$$

If is easy to see that Lebesgue measure on R is equivalent to a symmetric probability measure satisfying A so, in particular, we have reproved the theorem due to Fréchet [2] which states that any Lebesgue measurable additive f is in fact linear. (Note that $D = \{x \mid f(x) = cx\}$ is an additive subgroup of R. Theorem 1 shows that the Lebesgue measure of D^c is 0. It is then easy to argue that $D = R$.) Our theorem is more general than Fréchet's, since there exist many singular symmetric probability measures on R satisfying Condition A.

In the second section of this paper we construct a Borel measurable rational subspace W of R on which a sequence A_n of additive "coordinate" functions is defined. We then apply this construction to show that

(1) There exists a continuous probability measure μ satisfying Condition A and a μ-measurable additive function f such that $f(x) = b$ a.e. (μ) with $b \neq 0$.

(This shows that the symmetry hypothesis was necessary in the preceding result.)

(2) There exists a symmetric, infinitely divisible probability measure μ on R such that the measures μ^t for $t > 0$ are all mutually singular.

The existence of a measure μ as in (2) is perhaps new. Its construction is, in any case, novel and involves the following idea: Let S'' stand for the set of all integer-valued functions z on $[0, \infty]$ with $z(0) = 0$ and z

continuous from the right. Let

$$S' = \{z \mid z \in S'' \text{ and } \sup_n \sup_m |B_m(z)| \text{ is finite}\},$$

where

$$m = 2^n + k \text{ with } 0 \leqslant k < 2^n \quad \text{and} \quad B_m(z) = z\left(\frac{(k+1)n}{2^n}\right) - z\left(\frac{kn}{2^n}\right).$$

Let S be the vector space spanned by rational linear combinations of elements of S'. We show that there exists a 1-1 additive Borel measurable function $F: S \to R$.

The idea of using function space techniques for constructing measures on the real line is not new. Kaufman in [3] has a similar point of view.

1. Additive functions on R.

THEOREM 1. *If μ is a probability measure on R satisfying Condition A and f is an additive μ-measurable function, then $b, c \in R$ are such that*

$$f(x) = b + cx \text{ a.e. } (\mu).$$

Proof. $\mu = \mu_1 * \ldots * \mu_n * \ldots$, where $\mu_n(Q) = 1$. Let $(\Omega, \mathscr{B}, P)$ be a probability triple, where $\mathscr{B}$ is a σ-field of subsets of Ω, and P is a probability measure on $\mathscr{B}$. Let Z_n be a sequence of independent random variables on $(\Omega, \mathscr{B}, P)$ such that μ_n is the distribution of Z_n. Since f is additive, it follows that there exists a constant c such that $f(r) = cr$ for every $r \in Q$. By the additivity of f we can write

$$f\Big(\sum_{k=1}^{\infty} Z_k(w)\Big) - c\Big(\sum_{k=1}^{\infty} Z_k(w)\Big)$$
$$= \Big[f\Big(\sum_{k=1}^{n} Z_k(w)\Big) - c\sum_{k=1}^{n} Z_k(w)\Big] + \Big[f\Big(\sum_{k=n+1}^{\infty} Z_k(w)\Big) - c\sum_{k=n+1}^{\infty} Z_k(w)\Big]$$

for all points w in our probability space for which $\sum_{k=1}^{\infty} Z_k(w)$ converges. (This is a set of probability 1.) Now the first bracketed term is zero a.s., while the second is P-measurable with respect to the σ-field generated by $(Z_{n+1}, Z_{n+2}, \ldots)$. We conclude, by a modification of the Kolmogorov 0-1 law (cf. [4]), that

$$f\Big(\sum_{k=1}^{\infty} Z_k\Big) - c\Big(\sum_{k=1}^{\infty} Z_k\Big)$$

is a.s. equal to some constant b. In other words, there exists $b \in R$ such that $f(x) = b + cx$ a.e. (μ).

COROLLARY 1. *If in Theorem 1 the measure μ is symmetric (i.e. $\mu(A) = \mu(-A)$ for any Borel set A), then $f(x) = cx$ a.e. (μ).*

Proof. Since $f(-x) = -f(x)$ for $x \in R$, we conclude that the random variable f on (R, μ) must have a symmetric distribution, which is impossible unless $b = 0$.

2. Construction of W and A_n. We start by defining certain sets W_k of real numbers for any positive integer k.

Definition. W_k stands for the set of all x in R which can be written in the form

$$x = \sum_{n=1}^{\infty}\left(\sum_{r=-k}^{k} a_r^n \left(\frac{1}{2}\right)^{r+2^n}\right),$$

where $a_r^n = \pm 1$ or 0 for all n and all $r \in [-k, k]$.

LEMMA 1. *Let $x \in W_k$. Suppose that*

$$x = \sum_{n=1}^{\infty}\left(\sum_{r=-k}^{k} a_r^n \left(\frac{1}{2}\right)^{r+2^n}\right) \quad \textit{and} \quad x = \sum_{n=1}^{\infty}\left(\sum_{r=-k}^{k} b_r^n \left(\frac{1}{2}\right)^{r+2^n}\right)$$

are two different representations of x, where b_r^n as well as a_r^n takes values in $\{1, 0, -1\}$. We claim that the sum

$$A_k^n(x) \equiv \sum_{r=-k}^{k} a_r^n \left(\frac{1}{2}\right)^{r+2^n}$$

and the sum

$$B_k^n(x) \equiv \sum_{r=-k}^{k} b_r^n \left(\frac{1}{2}\right)^{r+2^n}$$

are equal for $n > k+1$ (and thus the numbers $A_k^n(x)$ are well-defined functions of x in W_k for $n > k+1$).

Proof. Let us write A_k^n for $A_k^n(x)$ and B_k^n for $B_k^n(x)$, where the dependence on x is suppressed, since x is supposed fixed. Suppose now that $A_k^N \neq B_k^N$ and that $N > k+1$.

We write

$$S_N = \sum_{n=1}^{N} A_k^n \quad \text{and} \quad T_N = \sum_{n=1}^{N} B_k^n.$$

Case 1. $S_N \neq T_N$.

Noting that $(1/2)^{k+2^N}$ divides S_N and T_N and also that

$$S_N + \sum_{n=N+1}^{\infty} A_k^n = T_N + \sum_{n=N+1}^{\infty} B_k^n = x,$$

we conclude that

$$\text{(a)} \qquad \left(\frac{1}{2}\right)^{k+2^N} \leqslant |S_N - T_N| = \left|\sum_{n=N+1}^{\infty} (A_k^n - B_k^n)\right|.$$

However, the right-hand side of inequality (a) is not greater than $4k(2^k)(4/3)(1/4)^{2^N}$. It follows that

$$3\left(\frac{1}{4}\right)^k (4k)^{-1} \leqslant 4\left(\frac{1}{2}\right)^{2^N}$$

which cannot occur, since $N > k > 0$.

Case 2. If $S_N = T_N$, then, since $A_k^N \neq B_k^N$, we must have $S_{N-1} \neq T_{N-1}$. Reasoning as in Case 1 we conclude that

$$3\left(\frac{1}{4}\right)^k (4k)^{-1} \leqslant \left(\frac{1}{2}\right)^{2(N-1)}$$

which cannot occur, since $N-1 > k > 0$.

COROLLARY 2. *If $x \in W_k$ and $j > k$, then $A_j^n(x) = A_k^n(x)$ for $n > j+1$.*

Proof. This follows since $A_j^n(x)$ is well defined for $n > j+1$.

We note the following facts:

FACT 1. *If $x \in W_k$ and m is a positive integer, then $mx \in W_{mk}$.*

FACT 2. *If $x \in W_k$ and $y \in W_j$, then $x+y \in W_{k+j}$.*

FACT 3. *If m is a positive integer with $x \in W_k$, then*

$$A_j^n(mx) = mA_j^n(x) \quad \textit{for } n > j+1 \geqslant mk+1.$$

(This follows from Lemma 1.)

FACT 4. *If $x \in W_k$ and $y \in W_r$, then*

$$A_j^n(x+y) = A_j^n(x)+A_j^n(y) \quad \textit{for } n > j+1 \geqslant k+r+1.$$

(This follows from Lemma 1.)

We now write

$$W' = \bigcup_{k=1}^{\infty} W_k \quad \text{and} \quad W = \bigcup_{r \in Q} rW'.$$

By Facts 1 and 2 it is clear that W is a linear subspace of R if R is treated as a vector space over the rationals.

Definition. For $x \in W$ let $p(x)$ be the least positive integer p such that $px \in W_k$ for some positive integer k. Let $k(x)$ be the least $k \geqslant p(x)$ such that $xp(x) \in W_k$. We define, for $n > k(x)+1$,

$$A_n(x) = \big(p(x)\big)^{(-1)} A_{k(x)}^n\big(xp(x)\big).$$

LEMMA 2. *We claim that, for each $x \in W$ and $y \in W$, there exists a positive integer $k(x, y)$ such that*

$$A_n(x+y) = A_n(x)+A_n(y) \quad \textit{for } n > k(x, y)+1.$$

Proof. Let $k(x, y) = 2k(x)k(y)k(x+y)$. Now $xp(x) \in W_{k(x)}$ and $yp(y) \in W_{k(y)}$. It follows that $xp(y)p(x)+yp(y)p(x) \in W_{2k(x)k(y)}$ by Facts 1 and 2.

We claim that, for $n > k(x, y)+1$,

$$\big(p(x+y)\big)^{(-1)} A^n_{k(x,y)}\big((x+y)p(x+y)\big)$$
$$= \big(p(x)\big)^{(-1)} A^n_{k(x,y)}\big(xp(x)\big) + \big(p(y)\big)^{(-1)} A^n_{k(x,y)}\big(yp(y)\big).$$

However,

$$\text{(1)} \quad A^n_{k(x,y)}\big((x+y)p(x+y)p(x)p(y)\big) = p(x)p(y)A^n_{k(x,y)}\big((x+y)p(x+y)\big)$$

for $n > k(x, y)+1$ by Facts 1 and 3, since

$$k(x, y) \geqslant k(x+y)p(x)p(y).$$

Also

$$\text{(2)} \quad A^n_{k(x,y)}\big((x+y)p(x+y)p(x)p(y)\big) = p(x+y)A^n_{k(x,y)}\big((x+y)p(x)p(y)\big)$$

for $n > k(x, y)+1$ by Facts 1 and 3, since

$$(x+y)p(x)p(y) = xp(y)p(x)+yp(y)p(x) \in W_{2k(x)k(y)},$$

while

$$k(x, y) \geqslant 2p(x+y)k(x)k(y).$$

Finally, we have

$$\text{(3)} \quad A^n_{k(x,y)}\big(xp(x)p(y)+yp(x)p(y)\big) = A^n_{k(x,y)}\big(xp(x)p(y)\big)+A^n_{k(x,y)}\big(yp(x)p(y)\big)$$

for $n > k(x, y)+1$ by Fact 4, and we can then write

$$\text{(4)} \quad A^n_{k(x,y)}\big(xp(x)p(y)\big) = p(y)A^n_{k(x,y)}\big(xp(x)\big),$$

$$\text{(5)} \quad A^n_{k(x,y)}\big(yp(x)p(y)\big) = p(x)A^n_{k(x,y)}\big(yp(y)\big)$$

for $n > k(x, y)+1$ by Fact 3.

If we put equalities (1) through (5) together we see that our claim is proved.

The following lemma can be proved in the same way as Lemma 2.

LEMMA 2′. *For $r, s \in Q$ and $x, y \in W$, there exists $k(r, s, x, y)$ such that*

$$A_n(rx+sy) = rA_n(x)+sA_n(y) \quad \textit{for } n > k(s, r, x, y)+1.$$

3. Symmetry is necessary in Theorem 1. Let $(X_n \mid n \geqslant 1)$ be a sequence of independent random variables with

$$\mathrm{P}[X_n = 1] = \mathrm{P}[X_n = 0] = \frac{1}{2}.$$

Let

$$X = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2^n} X_n$$

and let μ be the probability measure on R induced by the distribution of X. It is clear that $\mu(W) = 1$. For $x \in W$ we define

$$f^*(x) = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=k(x)+2}^{N} 2^{2^n}A_n(x)\right)$$

if this limit exists. Since, with probability 1, $2^{2^n}A_n(X) = X_n$, it follows from the strong law of large numbers that the set W^* of x in W, where the above limit exists, is Borel measurable and has μ measure 1.

LEMMA 3. *W^* is a rational linear subspace of W and f^* is additive on W^*.*

Proof. Let $x, y \in W^*$. Suppose that

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=k(x)+2}^{N} 2^{2^n}A_n(x)\right) = f^*(x)$$

and that

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=k(y)+2}^{N} 2^{2^n}A_n(y)\right) = f^*(y).$$

Now $x+y \in W$ and

$$\begin{aligned} f^*(x+y) &= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=k(x,y)+1}^{N} 2^{2^n}A_n(x+y)\right) \\ &= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=k(x,y)+1}^{N} 2^{2^n}A_n(x)\right) + \lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=k(x,y)+1}^{N} 2^{2^n}A_n(x)\right) \\ &= f^*(x)+f^*(y). \end{aligned}$$

The proof that, for any rationals r, s, $rx+sy \in W^*$ and $f(rx+sy) = rf(x)+sf(y)$ is equally easy.

We can extend f^* to an additive function f defined on all of R by using a Hamel basis well-ordering type argument. Since f will agree with f^* on W^*, it follows that f is μ-measurable. Also, it is clear, by the strong law of large numbers, that $f^* = 1/2$ a.e. (μ), i.e., $f = 1/2$ a.e. (μ).

The following lemma fills out the picture:

LEMMA 4. *Let μ be as above; we claim that μ is continuous (i.e., for any $x \in R$, we have $\mu(\{x\}) = 0$).*

Proof. We have

$$\mathrm{P}\left[\sum_{n=1}^{N}\left(\frac{1}{2}\right)^{2^n} X_n = x\right] \leqslant \left(\frac{1}{2}\right)^N,$$

since there exists at most one combination of $\varepsilon_n = 0$ or 1 such that

$$\sum_{n=1}^{N} \varepsilon_n \left(\frac{1}{2}\right)^{2^n} = x.$$

If we now let μ_N stand for the measure induced on R by

$$\sum_{n=1}^{N} \left(\frac{1}{2}\right)^{2^n} X_n$$

while $\hat{\mu}_N$ stands for the measure induced on R by

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{2}\right)^{2^n} X_n,$$

then we have

$$\mu_N(\{x\}) \leqslant \left(\frac{1}{2}\right)^N \quad \text{for all } x,$$

and

$$\mu(\{x\}) = \int_R \mu_N(\{x-y\})\, d\hat{\mu}_N(y).$$

It follows that $\mu(\{x\}) \leqslant (1/2)^N$ for any N.

4. An additive function from S to R. We start by constructing a function F from S' to W' as follows. For $z \in S'$ and $m = 2^n + k$, where n is a non-negative integer and $0 < k \leqslant 2^n$, we write

$$B_m(z) = z\left(\frac{n(k+1)}{2^n}\right) - z\left(\frac{nk}{2^n}\right).$$

We can assume that $|B_m(z)| \leqslant K(z)$ for some constant $K(z)$ for all $m \geqslant 1$.

We write

$$F(z) = \sum_{n=1}^{\infty} B_n(z)\left(\frac{1}{2}\right)^{2^n}.$$

It is clear that

$$B_m(a_1 z_1 + \ldots + a_n z_n) = a_1 B_m(z_1) + \ldots + a_n B_m(z_n)$$

for $z_1, \ldots, z_n \in S'$ and $a_1, \ldots, a_n$ any integers. Hence, from Lemma 1 it follows that

$$(*) \qquad F(a_1 z_1 + \ldots + a_n z_n) = a_1 F(z_1) + \ldots + a_n F(z_n).$$

We extend F to S by defining

$$F(z) = r_1 F(z_1) + \ldots + r_n F(z_n) \quad \text{for } z = r_1 z_1 + \ldots + r_n z_n,$$

where $z_1, \ldots, z_n \in S'$ and $r_1, \ldots, r_n \in Q$. To check that F is well defined suppose that $0 = r_1 z_1 + \ldots + r_n z_n$, where $z_1, \ldots, z_n \in S'$ and $r_1, \ldots, r_n \in Q$. We can write $r_i = p_i/q$ for some integers p_i ($i \in [1, \ldots, n]$) and q. It follows that

$$0 = p_1 z_1 + \ldots + p_n z_n$$

and hence, by (*),

$$p_1 F(z_1) + \ldots + p_n F(z_n) = 0$$

which implies that F is well defined.

To check that F is 1-1 we notice that if $z \neq 0$, then the sequence $B_m(z)$ will have infinitely many non-zero terms and hence, by Lemma 1, $F(z)$ will not be zero.

We endow S with the smallest σ-field $\mathscr{B}_S$ for which all evaluation functionals e_t are measurable. (For $t \in [0, \infty)$ we write $e_t(z) = z(t)$ for $z \in S$.) It is then clear that the functions B_m are all measurable with respect to $\mathscr{B}_S$, and hence F is measurable with respect to $\mathscr{B}_S$.

We now show that $(S, \mathscr{B}_S)$ supports a Poisson process.

LEMMA 5. *Let $(X(t) | t \geqslant 0)$ be a stochastic process with $X(0) = 0$ a.s. such that $X(t)$ has independent increments $X(t) - X(s)$ which are Poisson of mean $t - s$ for $s < t$. Then*

$$\mathrm{P}[\sup_n \sup_{2^n \leqslant m < 2^{n+1}} |B_m(X)| < \infty] = 1,$$

where

$$B_m(X) = X\left(\frac{(k+1)n}{2^n}\right) - X\left(\frac{kn}{2^n}\right) \quad \text{for } m = 2^n + k,\ 0 \leqslant k < 2^n.$$

Proof. Let

$$a_n = \mathrm{P}[\sup_{2^n \leqslant m < 2^{n+1}} |B_m(X)| \geqslant 2].$$

We show that $\sum a_n < \infty$, hence, by the Borel-Cantelli Lemma, we have

$$\sup_{2^n \leqslant m < 2^{n+1}} |B_m(X)| \leqslant 1$$

for n sufficiently large. We write

$$a_n = 1 - \prod_{m=0}^{2^n - 1} \mathrm{P}[B_m(X) \leqslant 1] = 1 - \left[\exp\left(-\frac{n}{2^n}\right)\left(1 + \frac{n}{2^n}\right)\right]^{2^n}$$
$$\sim 1 - \left[\left(1 - \frac{1}{2^n}\right)^n \left(1 + \frac{n}{2^n}\right)\right]^{2^n} \sim 1 - \left(1 - \frac{1}{4^n}\right)^{2^n} \sim \frac{1}{2^n}.$$

Notation. Let μ be any measure on R. For $a \in R$ we denote by $T_a\mu$ the measure defined by $T_a\mu(B) = \mu(a^{-1}B)$ for any Borel set B.

THEOREM 2. *There exists a symmetric infinitely divisible probability measure μ on R such that the measures*

$$T_{a_1}\mu^{t_1} * \ldots * T_{a_n}\mu^{t_n} \quad \text{and} \quad T_{b_1}\mu^{s_1} * \ldots * T_{b_m}\mu^{s_m}$$

are mutually disjoint unless $n = m$ and there exists a permutation σ of $[1, \ldots, n]$ such that $b_i = a_{\sigma(i)}$ and $s_i = t_{\sigma(i)}$ ($a_1, \ldots, a_n$ are positive rational numbers no two of which are equal and the same holds for $b_1, \ldots, b_m$; the numbers $t_1, \ldots, t_n$ as well as $s_1, \ldots, s_m$ are points in $(0, \infty)$; the measures $(\mu^t \mid t \in [0, \infty))$ denote the convolution semigroup of measures in which μ is imbedded as $\mu^1 = \mu$).

Proof. Let $(Z(t) \mid t \in [0, \infty))$ be a stochastic process with $Z(0) = 0$ a.s. and assume that $Z(t)$ has independent time homogeneous increments with

$$\mathrm{E}\{\exp[iv(Z(t) - Z(s))]\} = \exp[|t-s|(\cos(v) - 1)] \qquad \text{for } s, t \in [0, \infty), v \in R.$$

(This is the "symmetrized" Poisson process.) It is a well-known fact (see, e.g., Breiman [1], Problem 13, p. 316) that there exists a probability measure P on $(S, \mathscr{B}_S)$ such that the evaluation functionals $(e_t \mid t \in [0, \infty))$ have the same finite-dimensional distributions as $(Z(t) \mid t \in [0, \infty))$. We define μ to be $P \cdot F^{(-1)}$, i.e., the measure induced on R by the measurable function F: $S \to R$. Now, for any n, there exist independent processes $Z_1, \ldots, Z_n$ on $[0, \infty)$ such that $Z_k(0) = 0$ a.s. and

$$\mathrm{E}\{\exp[iv(Z_k(t) - Z_k(s))]\} = \exp\left[\frac{1}{n}|t-s|(\cos(v) - 1)\right] \quad \text{for } k \in [1, \ldots, n].$$

It follows that $P = (P_n)^n$ for some probability measure P_n on $(S, \mathscr{B}_S)$. Letting $\mu_{1/n}$ denote $P_n \cdot F^{(-1)}$ we have $\mu = (\mu_{1/n})^n$ and this implies that μ is infinitely divisible. It is clearly a symmetric measure, since P is symmetric and since $F(-z) = -F(z)$ for all $z \in S$.

Now, if $Z_1^{t_1}, \ldots, Z_n^{t_n}$ denote independent processes on $[0, \infty)$ with $Z_k^{t_k}(0) = 0$ a.s. and

$$F\{\exp[iv(Z_k^{t_k}(t) - Z_k^{t_k}(s))]\} = \exp[t_k|t-s|(\cos(v) - 1)] \quad \text{for } k \in [1, \ldots, n],$$

then whenever P' is the measure induced on S by $a_1 Z_1^{t_1} + \ldots + a_n Z_n^{t_n}$ we have

$$P' \cdot F^{(-1)} = T_{a_1}\mu^{t_1} * \ldots * T_{a_n}\mu^{t_n}$$

using the additivity of F.

Letting $Z_1^{s_1}, \ldots, Z_m^{s_m}$ be processes corresponding to $T_{b_1}\mu^{s_1} * \ldots * T_{b_m}\mu^{s_m}$, in the same way we remark that if P'' corresponds to the measure induced on S by the process $b_1 Z_1^{s_1} + \ldots + b_m Z_m^{s_m}$, then it is easy to check that the measures P' and P'' are mutually singular unless the conditions of the theorem are met. (Just check that the distributions of $(e_{n+1}-e_n \mid n \geqslant 0)$ are different under P' and P'', and then use the Glivenko-Cantelli Theorem (see Loève [4], p. 20).)

Since the mapping F is 1-1, it follows that the measures

$$T_{a_1}\mu^{t_1} * \ldots * T_{a_n}\mu^{t_n} \quad \text{and} \quad T^{v}{}_{1}\mu^{s_1} * \ldots * T_{b_m}\mu^{s_m}$$

are mutually singular unless the conditions of the theorem are met.

As a particular application of Theorem 2 we have

COROLLARY 3. *There exists a symmetric infinitely divisible probability measure μ such that the measures $(\mu^t \mid t > 0)$ are all mutually singular.*

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