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Basic concepts of the difference geometry

by Czesław Woźniak (Warszawa)

Abstract. The difference geometry discussed in the paper is based on the concept of difference structure prescribed on the finite or countable set D and on the concept of the connexion in the bundle of vector spaces over the set D endowed with the difference structure. These concepts were previously introduced in the discretized formulation of the mechanics of deformable bodies. The difference structure on the set D enables to assign to an arbitrary real valued function $\varphi\colon S \to R$, $S \subset D$, the set of real valued functions $\Delta_A\varphi\colon S_A \to R$, $S_A \subset S$, $A = 1, 2, \ldots, m$ (m is an order of the difference structure) which are called the first differences of φ. In this contex the concept of the difference structure on D leads to a generalization of the known finite difference calculus. Defining the concept of connexion in the bundle of vector spaces over the set D, on which the difference structure is prescribed, we introduce the connexion matrices and curvature tensors in such bundle. The transformation formulas for these objects have the form similar to the known formulas of the differential geometry. Using this approach we can formulate the absolute difference calculus which is another generalization of the finite difference calculus.

Introducton. The difference geometry has been formulated in the connexion with the mechanics of discretized bodies, [5]. The difference geometry is based on the concept of what is called the *difference manifold* (cf. Section 1) and on the concept of the connexion in a bundle of vector spaces over a difference manifold (cf. Section 2). This last concept gives a position to introduce also the notion of curvature in the difference geometry (cf. Section 3). In this paper we also deal with the problem of a bundle of subspaces over a difference manifold and we introduce the concept of projections in such bundles (cf. Section 4).

Notations. The indices $A, \Phi, \ldots$ take the values $I, II, \ldots, m$, the indices $a, b, \ldots$ run over the sequence $1, 2, \ldots, n$ and the indices $\alpha, \beta, \ldots$ have the range $1, 2, \ldots, \mathrm{n}$; $\mathrm{n} = n$. Summation convention holds. Symbols $f_A d, f_{-A} d, \ldots$ stand for $f_A(d), f_{-A}(d), \ldots$, respectively, and $f_{-A} \stackrel{\mathrm{df}}{=} f_A^{-1}$.

1. Difference manifold. Before we pass to the basic concept of a difference manifold, we have to introduce the auxiliary notion of a difference structure defined on a finite or countable set.

Definition 1. Let D be a finite or countable set of elements d, $d \in D$, and let m, $m+1 \leqslant \overline{\overline{D}}$, be a positive integer. We say that a difference structure of order m is *prescribed* on D if there is given a sequence of m one-to-one mappings $f_\Lambda\colon D_\Lambda \to D_{-\Lambda}$, $\Lambda = I, II, \ldots, m$, $D_\Lambda \subseteq D$, $D_{-\Lambda} \subseteq D$, such that:

1° $\bigcup_{\Lambda=1}^{m} (D_\Lambda \cup D_{-\Lambda}) = D$,

2° $\left(\bigcap_{\Lambda=I}^{m} D_\Lambda \neq \emptyset\right) \vee \left(\bigcap_{\Lambda=I}^{m} D_{-\Lambda} \neq \emptyset\right)$,

3° $\bigwedge_{\Lambda} \bigwedge_{d \in D_\Lambda} (f_\Lambda d \neq d)$,

4° $\bigwedge_{\Lambda} \bigwedge_{\Phi} \bigwedge_{d \in D_\Lambda \cap D_\Phi} [(f_\Lambda d = f_\Phi d) \Rightarrow (\Lambda = \Phi)]$.

If a difference structure of order m is prescribed on D, then we can define, for each $d \in D$, the sets $R_d \subseteq \{I, II, \ldots, m\}$, $L_d \subseteq \{I, II, \ldots, m\}$, putting $R_d\colon = \{\Lambda | d \in D_\Lambda\}$, $L_d\colon = \{\Lambda | d \in D_{-\Lambda}\}$. For an arbitrary subset $S \subseteq D$ we can also define the subsets $S_\Lambda \subseteq S$, $S_{-\Lambda} \subseteq S$, putting $S_\Lambda\colon = \{d | f_\Lambda d \in S\}$, $S_{-\Lambda}\colon = \{d | f_{-\Lambda} d \in S\}$; some of the subsets $S_\Lambda, S_{-\Lambda}$, or even all of them, may be empty. In the same way we are able to define the subsets $S_{\Lambda_1, \Lambda_2, \ldots, \Lambda_s} \subseteq S_{\Lambda_1, \Lambda_2, \ldots, \Lambda_{s-1}}$, using the formula $S_{\Lambda_1, \Lambda_2, \ldots, \Lambda_s}\colon = \{d | f_{\Lambda_s} d \in S_{\Lambda_1, \Lambda_2, \ldots, \Lambda_{s-1}}\}$, where $\Lambda_i = \pm I, \pm II, \ldots, \pm m$, $i = 1, 2, \ldots, s$, and s is an arbitrary positive integer. For example, we can write

$$S_{\Lambda,\Phi}\colon = \{d | f_\Phi d \in S_\Lambda\}, \qquad S_{-\Lambda,\Phi}\colon = \{d | f_\Phi d \in S_{-\Lambda}\},$$
$$S_{\Lambda,-\Phi}\colon = \{d | f_{-\Phi} d \in S_\Lambda\}, \qquad S_{-\Lambda,-\Phi}\colon = \{d | f_{-\Phi} d \in S_{-\Lambda}\}.$$

Let $\varphi\colon S \to R$ be an arbitrary real-valued function and let a difference structure of order m on D be given. It follows that there exist $2m$ functions[1] $\Delta_\Lambda \varphi\colon S_\Lambda \to R$, $\bar{\Delta}_\Lambda \varphi\colon S_{-\Lambda} \to R$, defined by

$$\Delta_\Lambda \varphi(d) \stackrel{\mathrm{df}}{=} \varphi(f_\Lambda d) - \varphi(d) \quad \text{and} \quad \bar{\Delta}_\Lambda \varphi(d) \stackrel{\mathrm{df}}{=} \varphi(d) - \varphi(f_{-\Lambda} d), \quad \text{respectively.}$$

These functions are said to be the right-hand side and left-hand side differences of the function $\varphi\colon S \to R$, respectively. Since all the first differences $\Delta_\Lambda \varphi$, $\bar{\Delta}_\Lambda \varphi$ of an arbitrary real-valued function φ are also real-valued functions, then we can also define the second and higher ordered differences of the function φ. The second differences will be defined as the first differences of the first differences:

$$\begin{aligned}
&\Delta_\Phi \Delta_\Lambda \varphi\colon S_{\Lambda,\Phi} \to R, && \Delta_\Phi \Delta_\Lambda \varphi(d) \stackrel{\mathrm{df}}{=} \Delta_\Phi\big(\Delta_\Lambda \varphi(d)\big),\\
&\bar{\Delta}_\Phi \Delta_\Lambda \varphi\colon S_{\Lambda,-\Phi} \to R, && \bar{\Delta}_\Phi \Delta_\Lambda \varphi(d) \stackrel{\mathrm{df}}{=} \bar{\Delta}_\Phi\big(\Delta_\Lambda \varphi(d)\big),\\
&\Delta_\Phi \bar{\Delta}_\Lambda \varphi\colon S_{-\Lambda,\Phi} \to R, && \Delta_\Phi \bar{\Delta}_\Lambda \varphi(d) \stackrel{\mathrm{df}}{=} \Delta_\Phi\big(\bar{\Delta}_\Lambda \varphi(d)\big),\\
&\bar{\Delta}_\Phi \bar{\Delta}_\Lambda \varphi\colon S_{-\Lambda,-\Phi} \to R, && \bar{\Delta}_\Phi \bar{\Delta}_\Lambda \varphi(d) \stackrel{\mathrm{df}}{=} \bar{\Delta}_\Phi\big(\bar{\Delta}_\Lambda \varphi(d)\big).
\end{aligned} \tag{1.1}$$

(1) Some of them may be defined on empty sets.

In the same way we can define the higher order differences of the function $\varphi: S \to R$, putting $\Delta_{\Lambda_s}\Delta_{\Lambda_{s-1}}\dots\Delta_{\Lambda_1}\varphi \overset{\mathrm{df}}{=} \Delta_{\Lambda_s}(\Delta_{\Lambda_{s-1}}\dots\Delta_{\Lambda_1}\varphi)$ for $\Lambda_i = I, II, \dots, m$; $i = 1, 2, \dots, s$, where s is an arbitrary positive integer.

In what follows we always assume that a difference structure of order m is given on a fixed set D. Let us denote by $\psi: D \to R$ an arbitrary real-valued function. It follows from Definition 1 that for each $d \in D$ there exist $\overline{\overline{R}}_d$ differences $\Delta_\Lambda\psi(d)$ and $\overline{\overline{L}}_d$ differences $\overline{\Delta}_\Lambda\psi(d)$, where $\overline{\overline{R}}_d + \overline{\overline{L}}_d > 0$. It also follows that there always exists a point $d \in D$ for which we have either exactly m differences $\Delta_\Lambda\psi(d)$ or m differences $\overline{\Delta}_\Lambda\psi(d)$; it means that $m = \max(\overline{\overline{R}}_d, \overline{\overline{L}}_d)$, $d \in D$. Using Definition 1 it is easy to prove that the differences $\Delta_\Lambda\varphi(d)$, $\overline{\Delta}_\Lambda\varphi(d)$ always depend on the function φ and that they are uniquely defined.

A difference structure on D is said to be *regular* if and only if

$$(1.2) \qquad f_\Lambda f_\Phi d = f_\Phi f_\Lambda d \quad \text{for } (\forall\Phi)(\forall\Lambda), (\forall d \in [D_{\Lambda,\Phi} \cap D_{\Phi,\Lambda}]).$$

If a difference structure on D is regular, we can prove that

$$(1.3) \quad \Delta_\Lambda\Delta_\Phi\psi(d) = \Delta_\Phi\Delta_\Lambda\psi(d) \quad \text{for } (\forall\Phi)(\forall\Lambda)(\forall\varphi), (\forall d \in [D_{\Lambda,\Phi} \cap D_{\Phi,\Lambda}]).$$

The proof of (1.3) is obtained with use of (1.1) and (1.2).

Now let S be a given subset of D, $S \subset D$, and put $S_0 = \bigcap_{\Lambda=I}^{m} (S_\Lambda \cap S_{-\Lambda})$. If $S_0 \neq \emptyset$, then the subset $S_0 \subseteq S$ is said to be the *interior of* S, and $\partial S = S - S_0$ will be called the *boundary of* S. An arbitrary function $\varphi: S \to R$ has for each $d \in S_0$ exactly m right-hand differences $\Delta_\Lambda\varphi(d)$ and m left-hand side differences $\overline{\Delta}_\Lambda\varphi(d)$, because $\overline{\overline{R}}_d = \overline{\overline{L}}_d = m$ for each $d \in S_0$. For each $d \in \partial S$ we have either $\overline{\overline{R}}_d < m$ or $\overline{\overline{L}}_d < m$; it follows that on the boundary ∂S not all the differences of the function $\varphi: S \to R$ exist.

DEFINITION 2. A pair $(D, \mathscr{E})$, where D is a finite or countable set and $\mathscr{E}$ is a given covering of D, is said to be a *difference manifold* if and only if there exists a difference structure on D which determines the covering $\mathscr{E}$ by the formula $\mathscr{E} = (E_d)_{d \in D_*}$, where $D_*: = \{d | \overline{\overline{R}}_d \neq 0\}$, and each E_d is a set of $1 + \overline{\overline{R}}_d > 1$ points $d, f_\Lambda d$, $\Lambda \in R_d$.

The concept of a difference manifold has its origin in the mechanics of discretized bodies, [5], where each $d \in D$ is a particle of a discretized body and those particles are interacting only in subsets $E \in \mathscr{E}$, which are called *discrete elements*. The set of $1 + \overline{\overline{R}}_d > 1$ numbers $\psi(d)$, $\Delta_\Lambda\psi(d)$ determines the function $\psi: D \to R$ in the discrete element $E = E_d$. Analogously, the $\overline{\overline{L}}_d$ numbers $\overline{\Delta}_\Lambda\psi(d)$ describe the change of the function ψ if we pass form a discrete element E_d to the discrete elements $E_{f_{-\Lambda}d}$, provided $d \in \bigcup_{A=1}^{m} D_{-A}$. Hence we see that the difference calculus given above can be called the discrete element calculus and it constitutes a generalization of the well-known finite difference calculus, where one deals with functions defined on lattices in the affine space, [1].

Let us denote by $\mathscr{E}_d$ the set of all discrete elements which contain the point d. It is easy to prove that $\overline{\overline{\mathscr{E}}}_d$ is a finite number.

Thus we see that to each difference manifold $(D, \mathscr{E})$ we can uniquely assign the positive integer $m \overset{\mathrm{df}}{=} \max(\overline{\overline{E}}, \overline{\overline{\mathscr{E}}}_d)-1$, $E \in \mathscr{E}$, $d \in D$.

Definition 3. Let $(D, \mathscr{E})$ be a difference manifold. Each difference structure on D of order $m = \max(\overline{\overline{E}}, \overline{\overline{\mathscr{E}}}_d)-1$, $E \in \mathscr{E}$, $d \in D$, which determines the covering $\mathscr{E}$ (in the sense of Definition 2) is said to be an *admissible difference structure on* $(D, \mathscr{E})$.

Suppose that an admissible difference structure on $(D, \mathscr{E})$, regarded as a difference structure on D, defines the interior D_0 of D, i.e. $D_0 = \bigcap_{A=1}^{m} (D_A \cap D_{-A}) \neq \emptyset$. Then for each $d \in D_0$ we have $\overline{\overline{\mathscr{E}}}_d = m+1$. It follows that each admissible difference structure on $(D, \mathscr{E})$ defines the same interior D_0 and the same boundary $\partial D = D - D_0$ of D. The transformation formulas for left-hand side and right-hand side differences of an arbitrary real-valued function $\psi\colon D \to R$, under a transition from one admissible difference structure on $(D, \mathscr{E})$ to another, will be analysed in a separate paper.

2. Bundles of vector spaces. In what follows we assume that a difference manifold $(D, \mathscr{E})$ is given and that an admissible difference structure (of order m) on $(D, \mathscr{E})$ is prescribed. Let us assign to each $d \in D$ the n-dimensional vector space V_d^n, and let us denote by V_d^{*n} the dual vector space. In the mechanics of discretized bodies the space V_d^n is the configuration space of the particle d, and the space V_d^{*n} is the space of generalized forces acting an the particle d. The spaces $V = \bigcup_{d \in D} V_d^n$ and $V^* = \bigcup_{d \in D} V_d^{*n}$ are bundle spaces in the bundle of vector spaces V_d^n, $d \in D$, and in the bundle of vector spaces V_d^{*n}, $d \in D$, respectively. The set D is the base space in these bundles, each V_d^n or V_d^{*n} is the fibre over the point $d \in D$, and the linear transformation group L^n is the group of the two introduced bundles. Such vector bundles are L^n-equivalent to a product bundle, [4].

Let us denote by D_d the subset of D which is covered by $\mathscr{E}_d$ (cf. Section 1). The concept of connexion in a bundle of vector spaces will be introduced for each bundle $V_{d'}^n$, $d' \in D_d$, independently.

Definition 4. A vector $v(f_A d)$ from the space $V_{f_A d}^n$ with components $v^\beta(f_A d)$ is said to be *shifted from* $V_{f_A d}^n$ *to* V_n^d, if its components in V_d^n are given by $v_{(A)}^\alpha(d) = [\delta_\beta^\alpha + G_{A\beta}^{\ \alpha}(d)] v^\beta(f_A d)$, where $(G_{A\beta}^{\ \alpha}(d))$ is, for each $A \in R_d$, a given $n\times n$ matrix, which satisfies the condition

$$\det\left(\delta^a_\beta + G_{A\beta}{}^a(d)\right) \neq 0 \tag{2.1}$$

and is called the *connexion matrix.*

Let us denote by $v\colon D\to V$ an arbitrary cross-section over D in the bundle of vector spaces. Any such cross-section will be called the *vector field on D.*

Definition 5. The vector field $\delta_A v\colon D_A\to V$ with components given by $\delta_A v^\beta(d) = v^\beta_{(A)}(d) - v^\beta(d) = \Delta_A v^\beta(d) + G_{A\alpha}{}^\beta(d)v^\alpha(f_A d)$ is called the *right-hand side absolute difference of the vector field $v\colon D\to V$* with components $v^\beta(d)$, $d\in D$. If $\delta_A v^\beta(d) = 0$, then the vectors $v(d)$ and $v(f_A d)$ are said to be *parallel.*

Analogously, in order to define the left-hand side absolute difference of the vector field $v\colon D\to V$ at the point $d\in D_{-A}$, let us denote by $v^\alpha_{(-A)}(d)$ the components of the vector $v(f_{-A}d)$ after a parallel transport (i.e. after shifting) from $V^n_{f_{-A}d}$ to V^n_d. Let us put $v^\alpha_{(-A)}(d) = [\delta^\alpha_\beta - G^\alpha_{\beta A}(d)]v^\beta(f_{-A}d)$, where $G^\alpha_{\beta A}(d)$ are, for each $A\in L_d$, the elements of an $n\times n$ matrix satisfying the condition

$$\det\left(\delta^\alpha_\beta - G^\alpha_{\beta A}(d)\right) \neq 0. \tag{2.2}$$

Further, let us assume that if $v^\alpha_{(-A)}(d) = v^\alpha(d)$, then the vectors with components $v^\alpha(f_{-A}d)$ and $v^\alpha(d)$ are parallel. It follows that

$$\left(\delta^\beta_\gamma + G_{A\gamma}{}^\beta(f_{-A}d)\right)G^\gamma_{\alpha A}(d) = G_{A\alpha}{}^\beta(f_{-A}d) \tag{2.3}$$

holds for each $d\in D_{-A,A}$. From (2.3) we conclude that for each $d\in D_{-A,A}$ and each $A\in R_d\cap L_d$ the matrix $\left(G^\gamma_{\alpha A}(d)\right)$ is uniquely determined by the connexion matrix $\left(G_{A\alpha}{}^\beta(f_{-A}d)\right)$. The vector with components $\bar\delta_A v^\alpha(d) = v^\alpha(d) - v^\alpha_{(-A)}(d) = \bar\Delta_A v^\alpha(d) + G^\alpha_{\beta A}(d)v^\beta(f_{-A}d)$ is called the *left-hand side absolute difference of the vector field $v\colon D\to V$ at the point $d\in D_{-A}$.* All vectors $\bar\delta_A v(d)$, $d\in D_{-A}$, form a vector field on D_{-A}.

Let $A^a_\alpha(d)$ be the elements of any non-singular $n\times n$ matrix, defined for each $d\in D$, and let $A^\alpha_a(d)$ be the elements of the inverse matrix, $A^a_\alpha(d)A^\alpha_\beta(d) = \delta^a_\beta$. We shall transform the components of the vector $\delta_A v(d)$ according to the well-known transformation formula

$$\begin{aligned}\delta_A v^a(d) &= A^a_\alpha(d)\,\delta_A v^\alpha(d) = A^a_\alpha(d)[\Delta_A v^\alpha(d) + G_{Ab}{}^\alpha(d)v^b(f_A d)]\\ &= A^a_\alpha(d)[A^\alpha_\beta(f_A d)v^\beta(f_A d) - A^\alpha_a(d)v^a(d) + G_{Ab}{}^\alpha(d)A^b_\beta(f_A d)v^\beta(f_A d)]\\ &= \Delta_A v^a(d) + [A^a_\alpha(d)\Delta_A A^\alpha_\beta(d) + A^a_\alpha(d)A^b_\beta(f_A d)G_{Ab}{}^\alpha(d)]v^\beta(f_A d).\end{aligned}$$

Thus we arrive at the following theorem.

THEOREM 1. *The transformation formula for the connexion matrix has the form*

$$G_{A\beta}^{\alpha}(d) = A_a^{\alpha}(d)\Delta_A A_\beta^a(d) + A_a^{\alpha}(d)A_\beta^b(f_A d)G_{Ab}^a(d) \tag{2.4}$$

for each $d \in D_A$ *and for a given* $A \in R_d$.

In the same way we can prove that the following formula holds

$$G_{\beta A}^{\alpha}(d) = A_a^{\alpha}(d)\overline{\Delta}_A A_\beta^a(d) + A_a^{\alpha}(d)A_\beta^b(f_{-A} d)G_{bA}^a(d), \tag{2.5}$$

where $d \in D_{-A}$. Let us note that (2.4) and (2.5) are given in a fixed admissible difference structure on $(D, \mathcal{E})$.

All preceding considerations can be applied if we are to introduce the connexion in the bundle of the dual vector spaces V_d^{*n}, $d \in D$, with respect to the same difference structure on D. For an arbitrary covector field $u: D \to V^*$ (which is a cross section in a bundle of dual vector spaces over D) we shall define the right-hand side absolute differences $\underline{\delta}_A u: D_A \to V^*$ and the left-hand side absolute differences $\overline{\delta}_A u: D_{-A} \to V^*$, putting $\overline{\delta}_A u_a(d) = \Delta_A u_a(d) + \overset{*}{G}{}_{Aa}^{\beta}(d)u_\beta(f_A d)$ and $\overline{\delta}_A u_a(d) = \overline{\Delta}_A u_a(d) + \overset{*}{G}{}_{aA}^{\beta}(d)u_\beta(f_{-A} d)$, respectively, where $\overset{*}{G}{}_{Aa}^{\beta}(d)$, $\overset{*}{G}{}_{aA}^{\beta}(d)$ satisfy, for each A, the conditions $\det\big(\delta_a^\beta + \overset{*}{G}{}_{Aa}^{\beta}(d)\big) \neq 0$ and $\det\big(\delta_a^\beta - \overset{*}{G}{}_{aA}^{\beta}(d)\big) \neq 0$. Moreover, two covectors with components $u_a(d)$ and $u_a(f_A d)$ are said to be *parallel*, if and only if $\delta_A u_a(d) = 0$ and $\overline{\delta}_A u_a(f_A d) = 0$. From the last conditions follows that the following relation

$$\big(\delta_\gamma^\beta + \overset{*}{G}{}_{A\gamma}^{\beta}(f_{-A} d)\big)\overset{*}{G}{}_{aA}^{\gamma}(d) = \overset{*}{G}{}_{Aa}^{\beta}(f_{-A} d), \tag{2.6}$$

holds for each $d \in D_{-A,A}$. In virtue of $\delta_A u_a(d) = A_a^{\alpha}(d)\,\delta_A u_\alpha(d)$, $u_a(d) = A_a^{\alpha}(d)u_\alpha(d)$ and $A_a^{\alpha}(d)A_\beta^a(d) = \delta_\beta^\alpha$, after simple calculations, we arrive at

$$\begin{aligned} \overset{*}{G}{}_{A\alpha}^{\beta}(d) &= A_a^{\alpha}(d)\,\Delta_A A_a^\beta(d) + A_a^{\alpha}(d)A_b^\beta(f_A d)\overset{*}{G}{}_{Aa}^{b}(d); \qquad d \in D_A, \\ \overset{*}{G}{}_{\alpha A}^{\beta}(d) &= A_a^{\alpha}(d)\,\overline{\Delta}_A A_a^\beta(d) + A_a^{\alpha}(d)A_b^\beta(f_{-A} d)\overset{*}{G}{}_{aA}^{b}(d); \qquad d \in D_{-A}. \end{aligned} \tag{2.7}$$

The $n \times n$ matrices with components $\overset{*}{G}{}_{Aa}^{\beta}(d)$, given for each A and each $d \in D_A$, are said to be the *dual connexion matrices*. According to (2.6), the matrices $\big(\overset{*}{G}{}_{aA}^{\beta}(d)\big)$ are uniquely determined by the dual connexion matrices $\big(\overset{*}{G}{}_{Aa}^{\beta}(d)\big)$. Equations (2.7) represent the transformation formulas for all those matrices.

The components of an arbitrary vector in V_d^n can be represented in the form $v^a(d) = e_{(\beta)}^a(d)\lambda^{(\beta)}$, where $\lambda^{(\beta)}$ are real numbers and $e_{(\beta)}^a(d)$ are components of the vector basis in V_d^n. We can also write $u_a(d)$

$= e_{\alpha}^{(\beta)}(d)\lambda_{(\beta)}$, where $\lambda_{(\beta)}$ are real numbers and $e_{\alpha}^{(\beta)}(d)$ satisfy the conditions $e_{\alpha}^{(\beta)}(d)e_{(\beta)}^{\gamma}(d) = \delta_{\alpha}^{\gamma}$. We conclude that the following identities hold:

$$\begin{aligned} G_{A\gamma}^{\beta}(d) &= -e_{\gamma}^{(\alpha)}(f_A d)\Delta_A e_{(\alpha)}^{\beta}(d) = e_{(\alpha)}^{\beta}(d)\Delta_A e_{\gamma}^{(\alpha)}(d), \\ \overset{*}{G}{}_{A\beta}^{\gamma}(d) &= -e_{(\alpha)}^{\gamma}(f_A d)\Delta_A e_{\beta}^{(\alpha)}(d) = e_{\beta}^{(\alpha)}(d)\Delta_A e_{(\alpha)}^{\gamma}(d), \qquad d\in D_A, \end{aligned} \tag{2.8}$$

where $\delta_A e_{(\alpha)}^{\gamma}(d) = 0$, i.e. the vector basis in V_d^n and $V_{f_A d}^n$ are assumed to be parallel. In virtue of (2.8) we arrive at the following theorem.

THEOREM 2. *Between the connexion matrix $(G_{A\beta}^{\alpha}(d))$ and the dual connexion matrix $(\overset{*}{G}{}_{A\beta}^{\alpha}(d))$ there exists the relation*

$$\big(\delta_{\beta}^{\alpha} + G_{A\beta}^{\alpha}(d)\big)\overset{*}{G}{}_{A\alpha}^{\gamma}(d) = -G_{A\beta}^{\gamma}(d), \qquad d\in D_A, \tag{2.9}$$

which uniquely determines the dual connexion matrix by the connexion matrix and vice versa.

The proof of the second part of this theorem follows directly form (2.1). Thus we can see, that the connexion in the bundle of vector spaces, as well as the connexion in the bundle of dual vector spaces, is determined by the connexion matrices $(G_{A\beta}^{\alpha}(d))$, $d\in D_A$.

Now let us denote

$$\{\alpha\} = \alpha_1\alpha_2\ldots\alpha_T, \quad e_{\{(\beta)\}}^{\{\alpha\}} = e_{(\beta_1)}^{\alpha_1}e_{(\beta_2)}^{\alpha_2}\ldots e_{(\beta_T)}^{\alpha_T}, \quad e_{\{\alpha\}}^{\{(\beta)\}} = e_{\alpha_1}^{(\beta_1)}e_{\alpha_2}^{(\beta_2)}\ldots e_{\alpha_T}^{(\beta_T)}, \tag{2.10}$$

where T is a given positive integer. Let us introduce in the usual way the components of covariant and contravariant tensors assigned to the point $d\in D$, and let us denote them by $u_{\{\alpha\}}(d)$ and $v^{\{\alpha\}}(d)$, respectively. In the same way as in the preceding considerations we arrive at the following formulas:

$$\begin{aligned} \delta_A v^{\{\alpha\}}(d) &= \Delta_A v^{\{\alpha\}}(d) + G_{A\{\gamma\}}^{\{\alpha\}}(d)v^{\{\gamma\}}(f_A d), \\ \delta_A u_{\{\alpha\}}(d) &= \Delta_A u_{\{\alpha\}}(d) + \overset{*}{G}{}_{A\{\alpha\}}^{\{\gamma\}}(d)u_{\{\gamma\}}(f_A d), \qquad d\in D_A, \end{aligned} \tag{2.11}$$

where we wrote:

$$\begin{aligned} G_{A\{\gamma\}}^{\{\alpha\}}(d) &= e_{\{(\beta)\}}^{\{\alpha\}}(d)\,\Delta_A e_{\{\gamma\}}^{\{(\beta)\}}(d), \quad & \Delta_A e_{\gamma}^{(\beta)}(d) &= G_{A\gamma}^{\alpha}(d)e_{\alpha}^{(\beta)}(f_A d), \\ \overset{*}{G}{}_{A\{\alpha\}}^{\{\gamma\}}(d) &= e_{\{\alpha\}}^{\{(\beta)\}}(d)\,\Delta_A e_{\{(\beta)\}}^{\{\gamma\}}(d), \quad & \Delta_A e_{(\beta)}^{\gamma}(d) &= \overset{*}{G}{}_{A\alpha}^{\gamma}(d)e_{(\beta)}^{\alpha}(f_A d); \quad d\in D_A, \end{aligned} \tag{2.12}$$

and where the vector basis: $e_{(\beta)}(d)$ in V_d^n and $e_{(\beta)}(f_A d)$ in $V_{f_A d}^n$ are assumed to be parallel. The tensors with components $\delta_A v^{\{\alpha\}}(d)$, $\delta_A u_{\{\alpha\}}(d)$, are called the absolute differences of the tensors with components $v^{\{\alpha\}}(f_A d)$, $v^{\{\alpha\}}(d)$ and $u_{\{\alpha\}}(f_A d)$, $u_{\{\alpha\}}(d)$, respectively. For $T = 2$ we obtain from (2.12)

$$G_{A\gamma_1\gamma_2}^{\alpha_1\alpha_2}(d) = G_{A\gamma_1}^{\alpha_1}(d)\,\delta_{\gamma_2}^{\alpha_2} + \delta_{\gamma_1}^{\alpha_1}G_{A\gamma_2}^{\alpha_2}(d) + G_{A\gamma_1}^{\alpha_1}(d)G_{A\gamma_2}^{\alpha_2}(d).$$

In the more general case we obtain the formula

$$\delta_A w^{\{\alpha\}}{}_{\{\beta\}}(d) = \Delta_A w^{\{\alpha\}}{}_{\{\beta\}}(d) + G_A{}^{\{\alpha\}}_{\{\gamma\}}(d) w^{\{\gamma\}}{}_{\{\beta\}}(f_A d) + \overset{*}{G}_A{}^{\{\delta\}}_{\{\beta\}}(d) w^{\{\alpha\}}{}_{\{\delta\}}(f_A d), \qquad d \in D_A,$$

where $\{\alpha\} = \alpha_1 \alpha_2 \ldots \alpha_T$, $\{\gamma\} = \gamma_1 \gamma_2 \ldots \gamma_T$, $\{\beta\} = \beta_1 \beta_2 \ldots \beta_S$, $\{\delta\} = \delta_1 \delta_2 \ldots \delta_S$, and where T, S are given positive integers.

In an Euclidean space the concept of the absolute differences of tensors can be introduced by means of the notion of shifters, [2], which are related to the connexion matrices by the formula $g^a_\beta = \delta^a_a(\delta^a_\beta + G_{A\beta}^{\ a}(d))$; where g^a_β are components of a shifter and where $d, f_A d$, are a pair of points in an Euclidean space. Of course, it is a very special and simple case, in which each vector space V^n_d is the same Euclidean space. The same problem but in a somewhat changed form was introduced in the book [3], where instead of shifters, the differences between unit tensors and shifters were used.

3. Curvature tensors. Let now there be given a bundle of vector spaces V^n_d, $d \in D$, and a connexion in this bundle with respect to the difference structure on D. Moreover, let the difference structure on D be regular (cf. Section 1). We denote by $\boldsymbol{v}_{(\Phi A)}(d)$, $d \in D_{A,\Phi} \cap D_{\Phi,A}$, the vector in V^n_d obtained from the vector $\boldsymbol{v}(f_A f_\Phi d)$, by shifting (of a parallel transport) from $V^n_{f_A f_\Phi d}$ to $V^n_{f_\Phi d}$ and then from $V^n_{f_\Phi d}$ to V^n_d. Applying the results of the previous Section, we get

$$\begin{aligned} v^a_{(\Phi A)}(d) &= [\delta^a_\beta + G_{\Phi\beta}^{\ a}(d)][\delta^\beta_\gamma + G_{A\gamma}^{\ \beta}(f_\Phi d)] v^\gamma(f_A f_\Phi d) \\ &= v^a(f_A f_\Phi d) + G_{\Phi A\gamma}^{\ \ a}(d) v^\gamma(f_A f_\Phi d), \end{aligned} \tag{3.1}$$

where we wrote:

$$\begin{aligned} G_{\Phi A\gamma}^{\ \ a}(d) = G_{A\gamma}^{\ a}(d) + G_{\Phi\gamma}^{\ a}(d) + \Delta_\Phi G_{A\gamma}^{\ a}(d) + G_{\Phi\beta}^{\ a}(d) G_{A\gamma}^{\ \beta}(d) + \\ + G_{\Phi\beta}^{\ a}(d) \Delta_\Phi G_{A\gamma}^{\ \beta}(d). \end{aligned} \tag{3.2}$$

Let us assume that the parallel transport of an arbitrary vector from $V^n_{f_A f_\Phi d}$ to V^n_d is independent of the way of transport. In this case the relation $G_{\Phi A\gamma}^{\ \ a}(d) = G_{A\Phi\gamma}{}^a(d)$ holds. Denoting

$$R_{A\Phi\beta}{}^a(d) \overset{\mathrm{df}}{=} 2G_{[\Phi A]\beta}^{\ \ \ a}(d) = 2[\Delta_{[\Phi} G_{A]\beta}^{\ \ a}(d) + G_{[\Phi|\gamma|}^{\ \ \ a}(d) G_{A]\beta}^{\ \ \gamma}(f_{[\Phi} d)], \tag{3.3}$$

we can write

$$R_{A\Phi\beta}{}^a(d) = 0, \qquad d \in D_{A,\Phi} \cap D_{\Phi,A}. \tag{3.4}$$

Equations (3.4) represent the necessary conditions for the independence of the parallel transport of vectors from $V^n_{f_A f_\Phi d}$ to V^n_d, of the way of transport. In general (3.4) are not sufficient conditions, because the vector from $V^n_{f_A f_\Phi d}$ to V^n_d can be shifted in different ways (not only through the spaces $V^n_{f_A d}$ or $V^n_{f_\Phi d}$).

Let us now consider the special situation in which in the vector spaces V_d^n, $d \in D$, there are introduced vector bases $e^a_{(\mu)}(d)$, such that the relations $G_{Aa}{}^b(d) = 0$ hold for each a, b and each $d \in D_A$. In view of (2.4) we obtain $G_{\Phi\beta}^{\gamma}(d) = A_a^{\gamma}(d) \Delta_\Phi A_\beta^a(d)$. It follows that

$$(3.5) \qquad \Delta_\Phi A_\beta^a(d) = A_a^a(d) G_{\Phi\beta}^a(d), \qquad d \in D_\Phi.$$

After simple calculations, in virtue of (3.3) and (3.5), we arrive at

$$(3.6) \quad \Delta_\Phi \Delta_A A_a^b(d) - \Delta_A \Delta_\Phi A_a^b(d) = A_\gamma^b(d) R_{A\Phi a}{}^\gamma(d), \qquad d \in D_{A,\Phi} \cap D_{\Phi,A}.$$

The left-hand sides of (3.6) are equal zero (we consider in this section only regular difference structures) and, since $\det A_\gamma^b(d) \neq 0$, we finally obtain conditions (3.4). Thus we see that equations (3.4) can be called the *compatibility conditions* for equations (3.5); they represent the necessary conditions for the existence of connexion matrices $G_{Aa}{}^b(d)$ equal to zero at each $d \in D_A$.

By means of direct calculations we can prove that the following relations hold:

$$(3.7) \qquad \delta_{[A}\delta_{\Phi]} v^a(d) = \tfrac{1}{2} R_{A\Phi\beta}{}^a(d) v^\beta(f_A f_\Phi d), \qquad d \in D_{A,\Phi} \cap D_{\Phi,A}.$$

From (3.7) it follows that the transformation formula for the $n \times n$ matrix with components $R_{A\Phi\beta}{}^a(d)$ (for each fixed A and Φ) has the form

$$(3.8) \qquad R_{A\Phi\beta}{}^a(d) = A_a^a(d) A_\beta^b(f_A f_\Phi d) R_{A\Phi b}{}^a(d).$$

Hence we see that the matrices with elements $R_{A\Phi\beta}{}^a(d)$ represent the components of the two-point tensor, [2], given for each fixed A, Φ and for $d \in D_{A,\Phi} \cap D_{\Phi,A}$. The upper index a in $R_{A\Phi\beta}{}^a(d)$ is assigned to the space V_d^n, and the lower index β in $R_{A\Phi\beta}{}^a(d)$ is assigned to the space $V^n_{f_A f_\Phi d}$.

DEFINITION 6. The two-point tensor with components $R_{A\Phi\beta}{}^a(d)$, given by equations (3.3) for each A, Φ and $d \in D_{A,\Phi} \cap D_{\Phi,A}$, is called the *curvature tensor at d* in the bundle of vector spaces over D and with respect to the given regular difference structure on D.

All concepts previously introduced in this section have their duals in the bundle of the dual vector spaces V_p^{*n}, $d \in D$. Defining the $n \times n$ matrices with elements

$$(3.9) \qquad \overset{*}{R}_{A\Phi\beta}{}^a(d) \overset{\mathrm{df}}{=} 2\big[\Delta_{[\Phi} \overset{*}{G}_{A]\beta}^a(d) + \overset{*}{G}_{[\Phi|\beta|}^{\gamma}(d) \overset{*}{G}_{A]\gamma}^a(f_{[\Phi} d)\big],$$

given for each A, Φ and each $d \in D_{A,\Phi} \cap D_{\Phi,A}$, we can easily prove that the following relation

$$(3.10) \qquad \delta_{[A}\delta_{\Phi]} u_a(d) = \tfrac{1}{2} \overset{*}{R}_{A\Phi a}{}^\beta(d) u_\beta(f_A f_\Phi d), \qquad d \in D_{A,\Phi} \cap D_{\Phi,A},$$

holds for an arbitrary covector field on D, with components $u_a(d)$, $d \in D$.

The transformation formula for the object defined by (3.9) has the form

$$\overset{*}{R}_{\Lambda\Phi\beta}{}^{a}(d) = A^{a}_{a}(f_{\Lambda}f_{\Phi}d)A^{b}_{\beta}(d)\overset{*}{R}_{\Lambda\Phi b}{}^{a}(d). \tag{3.11}$$

From (3.11) we conclude that, for each fixed Λ and Φ, the $n\times n$ matrix with components $\overset{*}{R}_{\Lambda\Phi\beta}{}^{a}(d)$ represents the components of the two-point tensor; the upper index a in (3.11) belongs to the space $V^{n}_{f_{\Lambda}f_{\Phi}d}$ while the lower index β in (3.11) belongs to the space V^{n}_{d}. This tensor is called the *dual curvature tensor* at $d\in D_{\Lambda,\Phi}\cap D_{\Phi,\Lambda}$, in the bundle of vector spaces over D and with respect to the given regular difference structure on D. The dual of formula (3.5) has the form

$$\Delta_{\Phi}A^{\beta}_{a}(d) = A^{a}_{a}(d)\overset{*}{G}_{\Phi a}{}^{\beta}(d), \qquad d\in D_{\Phi}, \tag{3.12}$$

and the duals of the compatibility conditions (3.6) can be written as follows:

$$\overset{*}{R}_{\Lambda\Phi\beta}{}^{a}(d) = 0, \qquad d\in D_{\Lambda,\Phi}\cap D_{\Phi,\Lambda}; \tag{3.13}$$

these conditions are necessary for the existence of the dual connexion matrices $\overset{*}{G}_{\Lambda a}{}^{b}(d)$ equal zero. According to (2.9) the corresponding connexion matrices are also equal zero.

4. Subspaces and projections. Let there be given a connexion in the bundle of n-dimensional vector spaces over D and with respect to a certain difference structure on D. Moreover, let N be a given positive integer such that $N<n$. For each $d\in D$ we shall introduce the N-dimensional vector space V^{N}_{d}, which is assumed to be the subspace of V^{n}_{d}. In the same way as previously, we denote by V^{*N}_{d} the vector space dual to V^{N}_{d}. Further, we define the basis in each V^{N}_{d}, $d\in D$, as the set of N vectors with components $C^{a}_{1}(d), C^{a}_{2}(d), \ldots, C^{a}_{N}(d)$. At the same time we introduce, for each $d\in D$, the $n\times N$ matrix with elements $C^{K}_{a}(d)$, satisfying the conditions $C^{K}_{a}(d)C^{a}_{L}(d) = \delta^{K}_{L}$, $K, L = 1, 2, \ldots, N$.

DEFINITION 7. By the projection of an arbitrary vector $\boldsymbol{v}(d)\in V^{n}_{d}$ onto V^{N}_{a} we mean the vector $'\boldsymbol{v}(d)$ with components $'v^{K}(d) = C^{K}_{a}(d)v^{a}(d)$. Analogously, by the projection of an arbitrary covector $\boldsymbol{u}(d)\in V^{*}_{d}$ onto V^{*N}_{d} we mean the covector $'\boldsymbol{u}(d)$ with components $'u_{K}(d) = C^{a}_{K}(d)u_{a}(d)$(1).

In what follows we assume that all vectors and covectors belong exclusively to the subspaces V^{N}_{d} or V^{*N}_{d}, $d\in D$. The absolute differences of any vector field (i.e. the cross-section in the bundle of spaces V^{N}_{d}, $d\in D$) are defined as the projections

$$'\delta_{\Lambda}v^{L}(d) = C^{L}_{a}(d)\,\delta_{\Lambda}v^{a}(d), \qquad d\in D_{\Lambda}. \tag{4.1}$$

(1) The indices K, L, M take the values $1, 2, \ldots, N$. Summation convention holds.

From (4.1) we obtain

$$(4.2)\qquad {}'\delta_\Lambda v^L(d) = \Delta_\Lambda v^L(d) + G_{\Lambda M}^{\ L}(d) v^M(f_\Lambda d), \qquad d \in D_\Lambda,$$

where

$$(4.3)\qquad G_{\Lambda M}^{\ L}(d) = C_\alpha^L(d)\Delta_\Lambda C_M^\alpha(d) + C_\alpha^L(d) C_M^\beta(f_\Lambda d) G_{\Lambda\beta}^{\ \alpha}(d), \qquad d \in D_\Lambda .$$

Now let there be given any covector field which is a cross-section in the bundle of dual vector subspaces V_d^{*N}, $d \in D$. The absolute difference of this field will be defined by

$$(4.4)\qquad {}'\delta_\Lambda u_L(d) = \Delta_\Lambda u_L(d) + \overset{*}{G}{}_{\Lambda L}^{\ M}(d) u_M(f_\Lambda d), \qquad d \in D_\Lambda,$$

where

$$(4.5)\qquad \overset{*}{G}{}_{\Lambda L}^{\ M}(d) = C_L^\alpha(d)\Delta_\Lambda C_\alpha^M(d) + C_L^\alpha(d) C_\beta^M(f_\Lambda d)\overset{*}{G}{}_{\Lambda\alpha}^{\ \beta}(d), \qquad d \in D_\Lambda .$$

The $N \times N$ matrices with components $G_{\Lambda M}^{\ L}(d)$, given for each Λ and each $d \in D_\Lambda$, are called the *connexion matrices* in the bundle of subspaces V_d^N, $d \in D$, with respect to the differences structure prescribed on D. The $N \times N$ matrices $\big(\overset{*}{G}{}_{\Lambda L}^{\ M}(d)\big)$, also given for each Λ and each $d \in D_\Lambda$, are called the *dual connexion matrices* in the bundle of subspaces V_d^{*N}, $d \in D$, with respect to the same difference structure. It is easy to prove that the transformation formulas for $G_{\Lambda L}^{\ M}(d)$ and $\overset{*}{G}{}_{\Lambda L}^{\ M}(d)$ have the form

$$(4.6)\qquad \begin{gathered} G_{\Lambda L'}^{\ M'}(d) = A_M^{M'}(d)\Delta_\Lambda A_{L'}^M(d) + A_M^{M'}(d) A_{L'}^L(f_\Lambda d) G_{\Lambda L}^{\ M}(d), \\ \overset{*}{G}{}_{\Lambda L'}^{\ M'}(d) = A_{L'}^L(d)\Delta_\Lambda A_L^{M'}(d) + A_{L'}^L(d) A_M^{M'}(f_\Lambda d)\overset{*}{G}{}_{\Lambda L}^{\ M}(d), \qquad d \in D_\Lambda; \\ \det A_M^{M'}(d) \neq 0, \qquad A_M^{M'}(d) A_{L'}^M(d) = \delta_{L'}^{M'} . \end{gathered}$$

If the difference structure on D is regular, then simple calculations show that the following relations

$$(4.7)\qquad \begin{gathered} {}'\delta_{[\Lambda}{}'\delta_{\Phi]} v^K(d) = \tfrac{1}{2}{}'R_{\Lambda\Phi L}{}^K(d) v^L(f_\Lambda f_\Phi d), \\ {}'\delta_{[\Lambda}{}'\delta_{\Phi]} u_K(d) = \tfrac{1}{2}{}'\overset{*}{R}_{\Lambda\Phi K}{}^L(d) u_L(f_\Lambda f_\Phi d); \qquad d \in D_{\Lambda,\Phi} \cap D_{\Phi,\Lambda}, \end{gathered}$$

hold for any fields $\boldsymbol{v}(d)$, $d \in D$, and $\boldsymbol{u}(d)$, $d \in D$, where

$$(4.8)\qquad \begin{gathered} {}'R_{\Lambda\Phi L}{}^K(d) = 2\big[\Delta_{[\Phi} G_{\Lambda]L}^{\ \ K}(d) + G_{[\Phi|M|}^{\ \ K}(d) G_{\Lambda]L}^{\ \ M}(f_{[\Phi} d)\big], \\ {}'\overset{*}{R}_{\Lambda\Phi K}{}^L(d) = 2\big[\Delta_{[\Phi}\overset{*}{G}{}_{\Lambda]K}^{\ \ L}(d) + \overset{*}{G}{}_{[\Phi|K|}^{\ \ M}(d)\overset{*}{G}{}_{\Lambda]M}^{\ \ L}(f_{[\Phi} d)\big], \qquad d \in D_{\Lambda,\Phi} \cap D_{\Phi,\Lambda}. \end{gathered}$$

From (4.7) we conclude that the objects with components ${}'R_{\Lambda\Phi L}{}^K(d)$, ${}'\overset{*}{R}_{\Lambda\Phi K}{}^L(d)$ are, for each Λ, Φ, d, the two-point tensors, [2]. They are called the *curvature tensors* in the bundle of vector subspaces over D and with respect to the difference structure given on D.

Now we are going to study the special case in which $n = N + \cdot$, and where V_d^N is, for each $d \in D$, the N-dimensional hyperplane $v^n = 0$

in the space V_d^n ("n" is fixed). In that case we assume that the indices $K, L, \ldots$ stand for $\alpha, \beta, \ldots$ if $\alpha, \beta, \ldots$ take the values $1, 2, \ldots, N$. Let a connexion in the bundle of vector spaces V_d^n, $d \in D$, be given with respect to the fixed difference structure on D. Let us denote

$$(4.9)\quad \begin{aligned} &b_A^K(d) \stackrel{*}{=} G_{An}^{K}(d), \quad b_{AK}(d) \stackrel{*}{=} G_{AK}^{n}(d), \quad b_A(d) \stackrel{*}{=} G_{An}^{n}(d),\\ &h_A^K(d) \stackrel{*}{=} \overset{*}{G}{}_{An}^{K}(d), \quad h_{AK}(d) \stackrel{*}{=} \overset{*}{G}{}_{AK}^{n}(d), \quad h_A(d) \stackrel{*}{=} \overset{*}{G}{}_{An}^{n}(d); \; d \in D_A, \end{aligned}$$

and let us introduce the following operators:

$$(4.10)\quad \begin{aligned} &\beta_A^K \varphi(d) = b_A^K(d)\varphi(f_A d), \quad \beta_{AK}\varphi(d) = b_{AK}(d)\varphi(f_A d),\\ &\beta_A \varphi(d) = b_A(d)\varphi(f_A d),\\ &\eta_A^K \varphi(d) = h_A^K(d)\varphi(f_A d), \quad \eta_{AK}\varphi(d) = h_{AK}(d)\varphi(f_A d),\\ &\eta_A \varphi(d) = h_A(d)\varphi(f_A d), \end{aligned}$$

where $\varphi(d)$, $d \in D$, is an arbitrary real-valued function. An arbitrary vector field can be now represented in the form $v^a(d) = \delta_K^a v^K(d) + \delta_n^a v(d)$, $d \in D$, where $a = 1, 2, \ldots, N, n$, since $N = n-1$. Analogously, we can write $u_a(d) = \delta_a^K u_K(d) + \delta_a^n u(d)$, $d \in D$, $a = 1, 2, \ldots, N, n$. By (4.10) we get

$$(4.11)\quad \begin{aligned} &\delta_A v^K(d) = {}'\delta_A v^K(d) + \beta_A^K v(d), \quad \delta_A v^n(d) = \Delta_A v(d) + \beta_{AK} v^K(d) + \beta_A v(d);\\ &\delta_A u_K(d) = {}'\delta_A u_K(d) + \eta_{AK} u(d),\\ &\delta_A u_n(d) = \Delta_A u(d) + \eta_A^K u_K(d) + \eta_A u(d), \quad d \in D_A, \end{aligned}$$

where "n" is a "dead" index. At the same time we obtain

$$(4.12)\quad \begin{aligned} &\bar\delta_A v^K(d) = {}'\bar\delta_A v^K(d) + \bar\beta_A^K v(d), \quad \bar\delta_A v^n(d) = \bar\Delta_A v(d) + \bar\beta_{AK} v^K(d) + \bar\beta_A v(d);\\ &\delta_A u_K(d) = {}'\bar\delta_A u_K(d) + \bar\eta_{AK} u(d),\\ &\bar\delta_A u_n(d) = \bar\Delta_A u(d) + \bar\eta_A^K u_K(d) + \bar\eta_A u(d), \quad d \in D_{-A}, \end{aligned}$$

where

$$(4.13)\quad \begin{aligned} &{}'\bar\delta_A v^K(d) = \bar\Delta_A v^K(d) + G_{LA}^K(d) v^L(f_{-A} d), \quad \bar\beta_A^K v(d) \stackrel{*}{=} G_{nA}^K(d) v(f_A d),\\ &\bar\beta_{AK} v^K(d) \stackrel{*}{=} G_{KA}^n(d) v^K(f_{-A} d),\\ &\bar\beta_A v(d) \stackrel{*}{=} G_{nA}^n(d) v(f_{-A} d), \quad \bar\delta_A u_K(d) = \bar\Delta_A u_K(d) + \overset{*}{G}{}_{KA}^L(d)\, u_L(f_{-A} d),\\ &\bar\eta_{AK} u(d) \stackrel{*}{=} \overset{*}{G}{}_{KA}^n(d) u(f_{-A} d), \quad \bar\eta_A^K u_K(d) \stackrel{*}{=} \overset{*}{G}{}_{nA}^K(d) u_K(f_{-A} d),\\ &\bar\eta_A u(d) \stackrel{*}{=} \overset{*}{G}{}_{nA}^n(d) u(f_{-A} d). \end{aligned}$$

Moreover, let us assume that the difference structure on D is regular and that the relations $R_{A\Phi\alpha}{}^{\beta}(d) = 0$ hold for each $d \in D_{A,\Phi} \cap D_{\Phi,A}$. Instead of $R_{A\Phi K}{}^{L}(d) = 0$, $R_{A\Phi n}{}^{L}(d) = 0$, $R_{A\Phi n}{}^{n}(d) = 0$, we can write

$$(4.14)\qquad 'R_{A\Phi K}{}^{L}(d) = \beta^{L}_{[A} b_{\Phi]K}(d), \quad '\delta_{[\Phi} b^{L}_{A]}(d) = \beta^{L}_{[A} b_{\Phi]}(d),$$
$$\beta^{M}_{[A} b_{\Phi]M}(d) = '\delta_{[\Phi} b_{A]}(d) + \beta_{[\Phi} b_{A]}(d),$$

where $d \in D_{A,\Phi} \cap D_{\Phi,A}$. In analogy with (4.14) we can also write

$$(4.15)\qquad '\overset{*}{R}_{A\Phi K}{}^{L}(d) = \eta_{[A|K|} h^{L}_{\Phi]}(d), \quad '\delta_{[\Phi} h_{A]K}(d) = \eta_{[A|K|} h_{\Phi]}(d),$$
$$\eta^{M}_{[A} h_{\Phi]M}(d) = '\delta_{[\Phi} h_{A]}(d) + \eta_{[\Phi} h_{A]}(d), \qquad d \in D_{A,\Phi} \cap D_{\Phi,A},$$

and the above equations can be obtained directly from $\overset{*}{R}_{A\Phi K}{}^{L}(d) = 0$, $\overset{*}{R}_{A\Phi K}{}^{n}(d) = 0$, $\overset{*}{R}_{A\Phi n}{}^{n}(d) = 0$. In the special case in which each vector space V^{n}_{d}, $d \in D$, is the same three-dimensional Euclidean space, the preceding formulas reduce to the form given in [3].

It is easy to observe that nearly all relations given in Sections 2–4 have the form similar to the well-known formulas of the differential geometry. In particular, equations $(4.14)_{1,2}$ have a form similar to the familar Gauss and Mainardi–Codazzi equations and equation $(4.14)_{3}$ corresponds to the known symmetry condition of the third fundamental tensor which describes an N-dimensional smooth surface imbedded in the $(N+1)$-dimensional Euclidean space. Thus we see that there is a formal correspondence between formulas of the difference and differential geometries.

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